Section 10. Ordinary and regular singular points of differential equations

In this section we consider a *linear, second order, homogeneous* ordinary differential equation (ODE) for an unknown function $y = y(x)$:

$$M(x)y'' + P(x)y' + Q(x)y = 0. \tag{14}$$

Clearly (14) can also be written in the form

$$y'' + p(x)y' + q(x)y = 0, \tag{15}$$

with

$$p(x) = \frac{P(x)}{M(x)}, \qquad q(x) = \frac{Q(x)}{M(x)}. \tag{16}$$

We say that a point x_0 is an *ordinary point* for the differential equation (14) (or equivalently for (15)) if both $p(x)$ and $q(x)$ are analytic at x_0. If x_0 is not an ordinary point then it is a *singular point*. x_0 is a *regular singular point* (RSP) if it is a singular point (so that one, or both, of $p(x)$ and $q(x)$ are not analytic at x_0) and both of the functions

$$(x - x_0)\, p(x) \qquad \text{and} \qquad (x - x_0)^2\, q(x) \tag{17}$$

are analytic at x_0. Otherwise, x_0 is an *irregular singular point*.

Example 10: (a) Consider the Legendre equation

$$(1 - x^2)y'' - 2xy' + \lambda y = 0 \qquad \Leftrightarrow \qquad y'' - \frac{2x}{1 - x^2}y' + \frac{\lambda}{1 - x^2}y = 0, \tag{18}$$

where λ is some real number (a *parameter*). Here $p(x) = -2x/(1 - x^2)$ and $q(x) = \lambda/(1 - x^2)$. If x_0 is neither 1 nor -1 then $p(x)$ and $q(x)$ are analytic at x_0, by the Principle of the Radius of Convergence on page 10, so that such a point x_0 is an ordinary point. But $x_0 = 1$ and $x_0 = -1$ are singular points, since the functions $p(x)$ and $q(x)$ are not analytic at ± 1: they are undefined there and in fact blow up when $x \to \pm 1$. On the other hand, taking $x_0 = 1$ and considering (17) we have

$$\begin{aligned} (x - 1)\, p(x) &= -(x - 1)\frac{2x}{1 - x^2} = \frac{2x}{1 + x}, \\ (x - 1)^2\, q(x) &= (x - 1)^2\frac{\lambda}{1 - x^2} = \frac{\lambda(1 - x)}{1 + x}. \end{aligned} \tag{19}$$

Both the functions in (19) are analytic at $x = 1$, so $x_0 = 1$ is a regular singular point. One finds similarly that $x_0 = -1$ is also a regular singular point.

(b) Consider the equation $x^3y'' + y = 0$, with $p(x) = 0$ and $q(x) = 1/x^3$. Here it is easy to see that if $x_0 \neq 0$ then x_0 is an ordinary point of the equation, just as above. But $x_0 = 0$ is a singular point; moreover, $(x - 0)^2q(x) = x^2q(x) = 1/x$ is not analytic at $x = 0$, so this is *not* a regular singular point.

Now we would like to find solutions of (14) which are valid near some point x_0, in the form of a series involving powers of $x - x_0$. When x_0 is an ordinary point of the equation we can do this with a standard power series, as discussed in the next section. When x_0 is a regular singular point we can use a modified power series; this is called the *Method of Frobenius*. When x_0 is an irregular singular point we have no way to find such a solution.

Section 11. Power series solutions centered at an ordinary point

When x_0 is an ordinary point for the ODE (14) we can find a solution in the form of a power series with center x_0:

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n. \tag{20}$$

We will first work this out in an example, and then draw from this some general conclusions.

Example 11: We consider again the Legendre equation (18) and take $x_0 = 0$, so we are looking for a solution of the form $y(x) = \sum_n a_n x^n$. To find out whether or not there is such a solution we must plug the proposed form into the differential equation and try to choose the coefficients a_n so that the equation is satisfied. Computing the derivatives as

$$y = \sum_{n=0}^{\infty} a_n x^n, \qquad y' = \sum_{n=0}^{\infty} n a_n x^{n-1}, \qquad y''(x) = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2}. \tag{21}$$

and inserting these into the left hand side of (18) yields

$$\begin{aligned}
&(1 - x^2)y'' - 2xy' + \lambda y \\
&\quad = (1 - x^2) \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} - 2x \sum_{n=0}^{\infty} n a_n x^{n-1} + \lambda \sum_{n=0}^{\infty} a_n x^n && \text{(22a)} \\
&\quad = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=0}^{\infty} n(n-1) a_n x^n - \sum_{n=0}^{\infty} 2n a_n x^n + \sum_{n=0}^{\infty} \lambda a_n x^n && \text{(22b)} \\
&\quad = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=0}^{\infty} \big[n(n-1) + 2n - \lambda\big] a_n x^n && \text{(22c)} \\
&\quad = \sum_{k=0}^{\infty} (k+2)(k+1) a_{k+2} x^k - \sum_{k=0}^{\infty} \big[k(k-1) + 2k - \lambda\big] a_k x^k && \text{(22d)} \\
&\quad = \sum_{k=0}^{\infty} \Big[(k+2)(k+1) a_{k+2} - \big[k(k+1) - \lambda\big] a_k\Big] x^k. && \text{(22e)}
\end{aligned}$$

Most of these steps are straightforward, but several need special comment. In (22c) we were able to change the lower summation limit in the first sum from $n = 0$ to $n = 2$ because the $n = 0$ and $n = 1$ terms of the sum were equal to zero, due to the factor $n(n-1)$ in the summand. In passing to (22d) we changed the summation index from n to k, but did this in two different ways, setting $k = n - 2$ in the first sum and $k = n$ in the second; this is legitimate because these summation indices just keep track of the terms and have no intrinsic meaning.

Once we have obtained (22e) the Legendre equation (18) becomes

$$\sum_{k=0}^{\infty}\Big[(k+2)(k+1)a_{k+2} - \big[k(k+1) - \lambda\big]a_k\Big]x^k = 0.$$

What this means is that the right left hand side must be zero for all x, and this is possible only if each term in the power series is itself zero. Thus we are led to the condition that

$$(k+2)(k+1)a_{k+2} - \big[k(k+1) - \lambda\big]a_k = 0, \qquad k = 0, 1, 2, \ldots, \tag{23a}$$

or equivalently that

$$a_{k+2} = \frac{k(k+1) - \lambda}{(k+2)(k+1)}a_k, \qquad k = 0, 1, \ldots\ . \tag{23b}$$

Either of the equations (23a) or (23b) is called the *recursion relation* for the coefficients a_k. We can systematically solve (23b) for a_2, a_3, etc.:

$$\begin{aligned}
k=0: &\qquad a_2 = -\frac{\lambda}{2}a_0 = \frac{(-\lambda)}{2!}a_0\\
k=1: &\qquad a_3 = \frac{2-\lambda}{3\cdot 2}a_1 = \frac{2-\lambda}{3!}a_1\\
k=2: &\qquad a_4 = \frac{6-\lambda}{4\cdot 3}a_2 = \frac{(6-\lambda)(-\lambda)}{4!}a_0\\
k=3: &\qquad a_5 = \frac{12-\lambda}{5\cdot 4}a_3 = \frac{(12-\lambda)(2-\lambda)}{5!}a_1\\
k=4: &\qquad a_6 = \frac{20-\lambda}{6\cdot 5}a_4 = \frac{(20-\lambda)(6-\lambda)(-\lambda)}{6!}a_0\\
k=5: &\qquad a_7 = \frac{30-\lambda}{7\cdot 6}a_4 = \frac{(30-\lambda)(12-\lambda)(2-\lambda)}{7!}a_1,\\
&\qquad \vdots
\end{aligned} \tag{24}$$

Putting the values from (24) back into $y(x) = \sum_{n=0}^{\infty} a_n x^n$ yields

$$\begin{aligned}
y(x) = a_0 + a_1 x + \frac{(-\lambda)}{2!}a_0x^2 + \frac{2-\lambda}{3!}a_1x^3 + \frac{(6-\lambda)(-\lambda)}{4!}a_0x^4&\\
+ \frac{(12-\lambda)(2-\lambda)}{5!}a_1x^5 + \frac{(20-\lambda)(6-\lambda)(-\lambda)}{6!}a_0x^6 + \cdots.&
\end{aligned} \tag{25}$$

If we group separately the terms with coefficient a_0 and those with coefficient a_1 we find

$$y(x) = a_0y_0(x) + a_1y_1(x);$$

here a_0 and a_1 are arbitrary coefficients and

$$\begin{aligned} y_0(x) &= 1 + \frac{(-\lambda)}{2!}x^2 + \frac{(6-\lambda)(-\lambda)}{4!}x^4 + \frac{(20-\lambda)(6-\lambda)(-\lambda)}{6!}x^6 + \cdots \\ y_1(x) &= x + \frac{2-\lambda}{3!}x^3 \frac{(12-\lambda)(2-\lambda)}{5!}x^5 + \frac{(30-\lambda)(12-\lambda)(2-\lambda)}{7!}x^7 + \cdots\cdots. \end{aligned} \tag{26}$$

Thus we have indeed found a power series solution of the Legendre equation with center $x_0 = 0$.

Remark 12: We make several further comments about the general problem of series solutions at an ordinary point and about this example.

(a) The procedure we used above, in Example 11, is completely general. When x_0 is an ordinary point for the ODE (14) we can look for a solution in the form (20). The recursion relation for the coefficients always has the form

$$(n+2)(n+1)a_{n+2} = (\text{some linear combination of } a_0, a_1, \ldots, a_{n+1}),$$

so that we can solve for a_2, a_3, etc. in terms of lower coefficients and eventually in terms of a_0 and a_1. This means that there is a solution

$$y(x) = a_0 y_0(x) + a_1 y_1(x) \tag{27}$$

such that $y_0(x)$ and $y_1(x)$ are as in (26):

$$\begin{aligned} y_0(x) &= 1 + b_2(x-x_0)^2 + b_3(x-x_0)^3 + \cdots, \\ y_1(x) &= (x-x_0) + c_2(x-x_0)^2 + c_3(x-x_0)^3 + \cdots, \end{aligned} \tag{28}$$

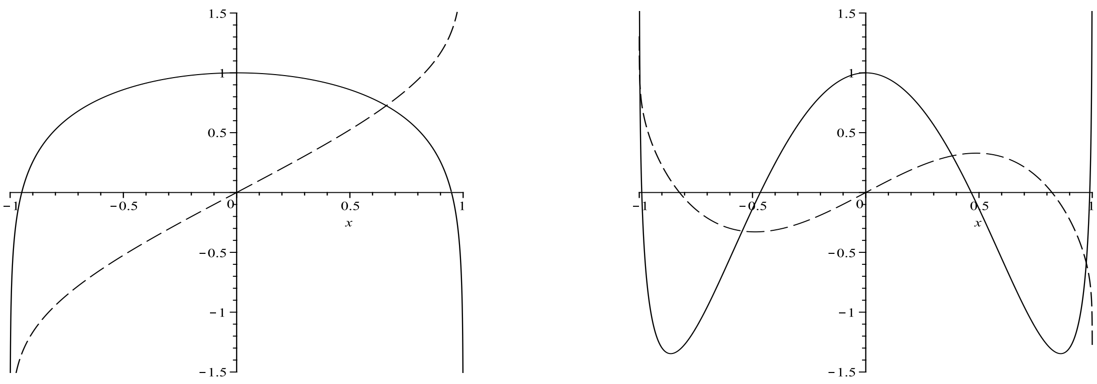
with $b_2, b_3, \ldots$ and $c_2, c_3, \ldots$ *computable* expressions (not unknown coefficients) like those in (26). Note that (28) implies that

$$y_0(x_0) = 1, \quad y_0'(x_0) = 0, \qquad y_1(x_0) = 0, \quad y_1'(x_0) = 1, \tag{29}$$

so that y_0 and y_1 are just the solutions of (14) satisfying the initial conditions (29). One should remember at this point that the general solution of a linear second order homogeneous ODE always has the form (27), with y_0 and y_1 two linearly independent solutions; here the initial conditions (29) guarantee that the y_0 and y_1 that we have found are in fact linearly independent.

(b) We continue with the discussion of a general equation (14), with x_0 an ordinary point. Then by definition the functions $p(x)$ and $q(x)$ given in (16) are analytic at x_0; suppose that power series for these functions, with center x_0, have radii of convergence R_1 and R_2 respectively. Then one can prove that the solution series for y_0 and y_1 given in (28) have radii of convergence at least as large as the smaller of R_1 and R_2, that is, that the series for the solution converges everywhere that we might reasonably expect it to.

(c) Let us again consider our series solutions $y_0(x)$ and $y_1(x)$ (see (26)) of the Legendre equation with center $x_0 = 0$. The power series for $p(x) = -2x/(1-x^2)$ and $q(x) =$



$\lambda = 1$ $\lambda = 10$

Figure 1: Solutions $y_0(x)$ (solid) and $y_1(x)$ (dashed) of the Legendre equation.

$\lambda/(1-x^2)$ with center 0 will have radii of convergence $R = 1$, by the Principle of the Radius of Convergence on page 10, and so (b) above guarantees that y_0 and y_1 will have radii of convergence at least 1. For most values of the parameter λ the radii are exactly 1, and the functions approach $\pm\infty$ as x approaches 1; plots (from Maple) of $y_0(x)$ and $y_1(x)$ for $\lambda = 1$ and $\lambda = 10$ are shown in Figure 1.

(d) Suppose, however, that $\lambda = n(n+1)$ for some nonnegative integer n, that is, λ has one of the values $0, 2, 6, 12, 20, \ldots$. Then from (23b) we see that $a_{n+2} = 0$, and since (again from (23b)) $n+4$ is proportional to a_{n+2} we must have $a_{n+4} = 0$, and continuing we see that $a_{n+2k} = 0$ for all $k \ge 1$. Suppose, for example, that $n = 6$; then $a_8 = a_{10} = a_{12} = \cdots = 0$ and so y_0 is a polynomial: $y_0(x) = a_0 + a_2x^2 + a_4x^4 + a_6x^6$. In general the argument above shows that:

If $\lambda = n(n+1)$ with $n = 0, 2, 4, 6, \ldots$ then $y_0(x)$ is a polynomial of degree n;

If $\lambda = n(n+1)$ with $n = 1, 3, 5, 7, \ldots$ then $y_1(x)$ is a polynomial of degree n.

The *Legendre polynomials* $P_n(x)$ are defined by $P_n(x) = c_n y_0(x)$ when n is even, $P_n(x) = c_n y_1(x)$ when n is odd, where y_0 and y_1 are the solutions of the Legendre equation with $\lambda = n(n+1)$. The constants c_n are chosen so that $P_n(1) = 1$; then

$$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = \frac{3x^2-1}{2}, \quad P_3(x) = \frac{5x^3-3x}{2}, \quad \ldots.$$

More details about these polynomials are given in Section 4.4 of our text.

(e) For the Legendre equation with general λ one can see from (24) that

$$a_n(x) = \begin{cases} \dfrac{\big((n-1)(n-2)-\lambda\big)\big((n-3)(n-4)-\lambda\big)\cdots(6-\lambda)(-\lambda)}{n!}\,a_0; & \text{if } n \text{ is even,} \\[2ex] \dfrac{\big((n-1)(n-2)-\lambda\big)\big((n-3)(n-4)-\lambda\big)\cdots(12-\lambda)(2-\lambda)}{n!}\,a_1, & \text{if } n \text{ is odd.} \end{cases}$$

Finding a general formula of this sort is frequently possible; it is basically a matter of seeing the pattern in the first few coefficients.

Section 12. Series solutions centered at a regular singular point

We now suppose that x_0 is a regular singular point for the ODE (14); again for notational simplicity we will often take $x_0 = 0$, but it should be remembered that there is nothing special about this choice. We first consider a very special ODE, the *Euler*, or *Cauchy-Euler*, or *equidimensional* equation (see Section 3.6.1 of our text):

$$x^2y'' + p_0xy' + q_0y = 0, \tag{30}$$

where p_0 and q_0 are constants. One checks easily that 0 is indeed an RSP for this equation. The equation always has at least one solution of the form $y(x) = x^r$, for if we plug this form into (30) we find that it is a solution if and only if r is a root of the *indicial equation*

$$r(r-1) + p_0r + q_0 = 0. \tag{31}$$

The equation (31) is just a quadratic equation and so will usually have two distinct roots r_1 and r_2; in this case we have two independent solutions of (30), $y_1(x) = x^{r_1}$ and $y_2(x) = x^{r_2}$. When (31) has a double root, $r_1 = r_2 = r$, the two solutions are $y_1(x) = x^r$ and $y_2(x) = x^r \ln x$. For later convenience we will call the left hand side of (32) $\gamma(r)$:

$$\gamma(r) = r(r-1) + p_0r + q_0. \tag{32}$$

Now consider the general equation (15), that is,

$$y'' + p(x)y' + q(x)y = 0, \tag{33}$$

and suppose that $x_0 = 0$ is a regular singular point. Thus by definition we have power series for $xp(x)$ and $x^2q(x)$ with positive radii of convergence R_1 and R_2:

$$xp(x) = \sum_{n=0}^{\infty} p_nx^n, \quad |x| < R_1; \qquad x^2q(x) = \sum_{n=0}^{\infty} q_nx^n \quad |x| < R_2. \tag{34}$$

Notice that when x is small we have $xp(x) \approx p_0$, $x^2q(x) \approx q_0$. If we multiply (33) by x^2 and use (34) with these approximations then our ODE becomes

$$x^2y'' + x[xp(x)]y' + [x^2q(x)]y = 0 \quad \approx \quad x^2y'' + xp_0y' + q_0y = 0 \text{ when } x \text{ is small.}$$

That is, the equidimensional equation (30) is a good approximation to (33) when x is small. We hope that the form of the solutions of the equidimensional equation will provide a guide to the form of the solutions of (33), and will thus look for solutions of (33) in the form

$$y(x) = x^r\sum_{n=0}^{\infty} a_nx^n = \sum_{n=0}^{\infty} a_nx^{n+r}, \qquad a_0 \neq 0, \tag{35}$$

with r an unknown quantity, to be determined. This is the key to the *Method of Frobenius* for finding a series representation, valid in the neighborhood of a regular singular point, of the solution of an ODE.

There are two comments to make about the form (35). First, why did we require that $a_0 \neq 0$? Suppose to the contrary that $a_0 = 0$, then we can bring a factor of x out of the sum in (35) to obtain $y(x) = x^r \sum_{n=1}^{\infty} a_n x^n = x^{r+1} \sum_{n=0}^{\infty} a_{n+1} x^n$, that is, we would replace r by $r+1$. Moreover, we could continue doing this until we obtained the form (35) with nonzero constant term. Thus the assumption $a_0 \neq 0$ just means that we want to remove all factors of x from the series before writing (35).

The second comment involves the fact that r may not be an integer, and if r is not an integer and $x < 0$, x^r may not be defined, at least as a real number; for example, $(-1)^{1/2}$ is not real. For this reason when $x < 0$ we should write $y(x) = \sum_n a_n |x|^{n+r}$. We will usually ignore this difficulty, tacitly assuming that we are obtaining the solution for $x > 0$; if a solution for $x < 0$ is needed one simply replaces x by $|x|$ throughout.

We now illustrate the method of Frobenius in a concrete example.

Example 13: Consider the ODE

$$x^2 y'' + 3xy' - 3(1+x^2)y = 0. \tag{36}$$

Since $p(x) = 3x/x^2 = 3/x$ and $q(x) = -3(1+x^2)/x^2$ are not analytic (i.e., are *singular*) at $x = 0$, $x = 0$ is a singular point; since

$$xp(x) = x\left(\frac{3}{x}\right) = 3 \qquad \text{and} \qquad x^2 q(x) = x^2\left(-\frac{3(1+x^2)}{x^2}\right) = -3(1+x^2) \tag{37}$$

are analytic at $x = 0$, this is a regular singular point. Note also that in (37) we have directly the power series expansions of $xp(x)$ and $x^2 q(x)$, since these are just polynomials, and so by comparison with (34) we have $p_0 = 3$ and $q_0 = -3$; then (32) gives that for this example

$$\gamma(r) = r(r-1) + 3r - 3 = r^2 + 2r - 3. \tag{38}$$

For $y(x)$ of the form (35) we have

$$\begin{aligned} y(x) &= \sum_{n=0}^{\infty} a_n x^{n+r}, \\ y'(x) &= \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}, \\ y''(x) &= \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}, \end{aligned}$$

with which (36) becomes

$$\begin{aligned} x^2 y'' + 3xy' - 3(1+x^2)y = x^2 \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} + 3x \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} \\ - 3(1+x^2) \sum_{n=0}^{\infty} a_n x^{n+r} \end{aligned}$$

$$
\begin{aligned}
&= \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r} + \sum_{n=0}^{\infty} 3(n+r)a_n x^{n+r} \\
&\qquad\qquad - \sum_{n=0}^{\infty} 3a_n x^{n+r} - \sum_{n=0}^{\infty} 3a_n x^{n+r+2} \\
&= \sum_{n=0}^{\infty}\bigl[(n+r)(n+r-1)+3(n+r)-3\bigr]a_n x^{n+r} - \sum_{n=0}^{\infty} 3a_n x^{n+r+2} \\
&= \sum_{k=0}^{\infty}\bigl[(k+r)^2+2(k+r)-3\bigr]a_k x^{k+r} - \sum_{k=2}^{\infty} 3a_{k-2} x^{k+r}. \qquad (39)
\end{aligned}
$$

All of this is much as what we did for Example 11, and the same comments apply—for example, at the last step we wrote $k = n$ in the first sum and $k = n+2$ in the second. We would like to combine the two terms in (39) into a single sum; they have the same power of x, which is a good start, but the summation limits are different, since the second sum starts at $k = 2$ rather than $k = 0$. We fix this by a trick: we agree that we will always define

$$a_{-1} = a_{-2} = a_{-3} = \cdots = 0. \qquad (40)$$

Then if we extend the limits in the second sum to include $k = 0$ and $k = 1$ we are just adding two expressions which are zero, and this will not change anything. With (39) and this remark our differential equation (36) becomes

$$\sum_{k=0}^{\infty}\Bigl[\bigl[(k+r)^2+2(k+r)-3\bigr]a_k - 3a_{k-2}\Bigr]x^{k+r} = 0, \qquad (41a)$$

or, if we use (38) to make the equation look simpler,

$$\sum_{k=0}^{\infty}\bigl[\gamma(k+r)a_k - 3a_{k-2}\bigr]x^{k+r} = 0. \qquad (41b)$$

From an argument as in Example 11, that (41) can hold only if the coefficient of each power of x vanishes, we find the recursion

$$\bigl[(k+r)^2+2(k+r)-3\bigr]a_k = 3a_{k-2} \quad\Leftrightarrow\quad \gamma(k+r)a_k = 3a_{k-2}, \quad k = 0,1,2,\ldots. \qquad (42)$$

We consider this recursion first for $k = 0$, when with (40) it becomes $\gamma(r)a_0 = 3a_{-2} = 0$. Since $\gamma(r)a_0 = 0$ we must have either $\gamma(r) = 0$ or $a_0 = 0$. But we agreed (see (35)) that $a_0 \neq 0$, so necessarily

$$\gamma(r) = r^2 + 2r - 3 = 0. \qquad (43)$$

This is again the *indicial equation.* From (43) it follows that is r **must be one of the two roots of this quadratic equation, either** $r = r_1 = 1$ **or** $r = r_2 = -3$**.** We must deal with these two cases separately.

Before we do so we note that, as pointed out above (38), here $p_0 = 3$ and $q_0 = -3$, and that the indicial equation (43) is just $r(r-1) + p_0 r + q_0 = 0$, as it would have been for the equidimensional equation (30). This means that we could have written down (43), and thus found the two possible values of r, without going through the series computations above. We could then look directly for two solutions $y_1 = x^{r_1} \sum a_n x^n$ and $y_2 = x^{r_2} \sum b_n x^n$. However, it is actually advantageous to have obtained the recursion (42) for general r, since we can now proceed by simply substituting $r = r_1$ and $r = r_2$ into that recursion.

Case 1: $r = r_1 = 1$. We solve the recursion (42) for $k = 1, 2, \ldots$; note that $\gamma(k + r_1) = (k+1)^2 + 2(k+1) - 3 = k(k+4)$ so that $a_k = 3a_{k-2}/k(k+4)$:

$$\begin{aligned}
k = 1: &\qquad a_1 = \frac{3}{5 \cdot 1} a_{-1} = 0 \\
k = 2: &\qquad a_2 = \frac{3}{6 \cdot 2} a_0 \\
k = 3: &\qquad a_3 = \frac{3}{7 \cdot 3} a_1 = 0 \\
k = 4: &\qquad a_4 = \frac{3}{8 \cdot 4} a_2 = \frac{3^2}{(8 \cdot 6)(4 \cdot 2)} a_0 \\
k = 5: &\qquad a_5 = \frac{3}{9 \cdot 5} a_3 = 0 \\
k = 6: &\qquad a_6 = \frac{3}{10 \cdot 6} a_4 = \frac{3^3}{(10 \cdot 8 \cdot 6)(6 \cdot 4 \cdot 2)} a_0 \\
&\qquad \vdots
\end{aligned} \tag{44}$$

Clearly all odd coefficients a_{2k+1}, $k = 0, 1, \ldots$, are zero. With a bit more work one can also guess the general form of the even coefficients:

$$\begin{aligned}
a_{2k} &= \frac{3^k}{[(2k+4)(2k+2)\cdots 6][(2k)(2k-2)\cdots 2]} a_0 \\
&= \frac{3^k}{2^{2k}[(k+2)(k+1)\cdots 3][k(k-1)\cdots 1]} a_0 = \frac{3^k}{2^{2k-1}\,(k+2)!\,k!} a_0,
\end{aligned}$$

so that our equation has a solution

$$y_1(x) = x \sum_{k=0}^{\infty} \frac{3^k}{2^{2k-1}\,(k+2)!\,k!}\, x^{2k}. \tag{45}$$

In writing down (45) we have chosen $a_0 = 1$; this is legitimate because once we have found a second solution $y_2(x)$ we will write the general solution as

$$y(x) = \alpha_1 y_1(x) + \alpha_2 y_2(x) \tag{46}$$

and the coefficient a_0 could have been absorbed into α_1.

Case 2: $r = r_2 = -3$**.** Now we are looking for a solution $y_2(x) = x^{-3}\sum_{n=0}^{\infty} b_n x^n$, with $b_0 \neq 0$; here we write the coefficients as b_n to distinguish them from the coefficients of y_1. The recursion (42) is now $\gamma(k-3)b_k = 3b_{k-2}$, and we try to solve this for $k = 1, 2, \ldots$. Noting that $\gamma(k-3) = (k-3)^2 + 2(k-3) - 3 = k(k-4)$ we have:

$$\begin{aligned}
k=1: &\qquad (1)(-3)b_1 = 3b_{-1} &&\implies b_1 = 0\\
k=2: &\qquad (2)(-2)b_2 = 3b_0 &&\implies b_2 = \frac{3}{2\cdot(-2)}b_0\\
k=3: &\qquad (3)(-1)b_3 = 3b_1 &&\implies b_3 = 0\\
k=4: &\qquad (4)(0)b_4 = 3b_2 = -\frac{3}{4}b_0 &&\implies \boxed{\;?\ ?\ ?\ ?\ ?\;}
\end{aligned}\tag{47}$$

We have arrived at a contradiction: the last equation implies that $b_0 = 0$, but we chose $b_0 \neq 0$. **There is no second solution of the form** $x^{-3}\sum_{n=0}^{\infty} b_n x^n$.

Let us try to understand what went wrong in the example by returning to the general equation (33) with a regular singular point at the origin (see (34)). If we look for a solution in the form (35), $y(x) = x^r \sum_{n=0}^{\infty} a_n x^n$, we will find a recursion

$$\gamma(n+r)a_n = (\text{some linear combination of } a_0, a_1, \ldots, a_{n-1}), \tag{48}$$

where again $\gamma(r) = r(r-1) + p_0 r + q_0$. The $n = 0$ case of this recursion is $\gamma(r)a_0 = 0$, and since $a_0 \neq 0$ we must again have $\gamma(r) = 0$. Let r_1 and r_2 denote the two roots of the quadratic equation $\gamma(r) = 0$; we will assume that the roots are real and number them so that $r_1 \geq r_2$. (Complex roots are not difficult to deal with in terms of what we are doing here; rather, the difficulty comes in interpreting x^r when x is complex. This is not hard but requires more knowledge of complex numbers than we want to assume.) Again we consider separately the cases $r = r_1$ and $r = r_2$.

Case 1: $r = r_1$**.** We set $r = r_1$ in (48) and solve successively for a_n when $n = 1, 2, \ldots$. Since $n \geq 1$, $n + r_1 > r_1 \geq r_2$, and then since $\gamma(r)$ can vanish only for $r = r_1$ or $r = r_2$ we have that $\gamma(n + r_1) \neq 0$, so that we can divide by $\gamma(n + r_1)$ in (48) and thus solve for a_n, for every n. We conclude that (33) *does* have a solution $y_1(x) = x^{r_1}\sum_{n=0}^{\infty} a_n x^n$; we will again make the specific choice $a_0 = 1$ so that y_1 denotes unambiguously a specific solution.

Case 2: $r = r_2$**.** Now we would like to set $r = r_2$ in (48) (but with the a_n replaced by b_n) and try to solve for the b_n as in the previous case (this is where we encountered trouble in Example 13). If $r_1 = r_2$, however, it is clear that this will simply produce the solution y_1 that we found in Case 1; we will have to look elsewhere for our second solution. But that is not the only difficulty. Suppose that $r_2 < r_1$ but that $m = r_1 - r_2$ is an integer; this was the case in Example 13, where $m = 1 - (-3) = 4$. Then when ,in solving for the b_n, we reach $n = m$, we will find that $\gamma(n + r_2) = \gamma(m + r_2) = \gamma(r_1) = 0$, and the left hand side of the n^{th} recursion will be $0 \cdot b_m$. **If the right hand side of this recursion is not zero we will have a contradiction.** This is precisely what happened in Example 13 and, as there, we will have to find a second solution by some other means. On the other hand, if $r_1 - r_2$ is not an integer we will have no difficulty solving the recursions and finding $y_2(x) = x^{-r_2}\sum_{n=0}^{\infty} b_n x^n$. In summary:

- We always have one solution $y_1(x) = x^{r_1} \sum_{n=0}^{\infty} a_n x^n$, with $a_0 = 1$.
- if $m = r_1 - r_2$ is not an integer then there is a second solution $y_2(x) = x^{r_2} \sum_{n=0}^{\infty} b_n x^n$, with $b_0 = 1$.
- if $m = r_1 - r_2$ is an integer (necessarily nonnegative) then we will have to find a second solution of a different form.

There is a systematic procedure, called *reduction of order*, which, when one knows one solution of a second order linear homogeneous ODE, enables one to find an independent second solution. Since for a regular singular point we may always find one solution, $y_1(x)$, corresponding to r_1, we could use reduction of order to find the form of a second solution $y_2(x)$ when $m = r_1 - r_2$ is an integer. We omit this step, however, and simply summarize the results:

Form of solutions near a regular singular point

Suppose that the ODE $y'' + p(x)y' + q(x) = 0$ has a regular singular point at the origin, that r_1 and r_2 are the two roots of the indicial equation $\gamma(r) = 0$, and that r_1 and r_2 are real and that $r_1 > r_2$. Then:

(a) if r_1 and r_2 are distinct and do not differ by an integer, then there are two linearly independent solutions:

$$y_1(x) = x^{r_1} \sum_{n=0}^{\infty} a_n x^n, \qquad y_2(x) = x^{r_2} \sum_{n=0}^{\infty} b_n x^n, \qquad a_0 = b_0 = 1. \tag{49}$$

(b) If $r_1 = r_2$, then there is one solution $y_1(x)$ of the form given in (49), and a second solution with the form

$$y_2(x) = y_1(x)(\ln x) + x^{r_1} \sum_{n=1}^{\infty} c_n x^n. \tag{50}$$

(c) If $r_1 - r_2 = m$, with m a positive integer, then there is one solution $y_1(x)$ as in (49), and a second solution with the form

$$y_2(x) = C y_1(x)(\ln x) + x^{r_2} \sum_{n=0}^{\infty} c_n x^n, \qquad c_0 = 1, c_m = 0. \tag{51}$$

The constant C may or may not be zero.

Each of the series in (49)– (50) has radius of convergence at least as large as the smaller of the radii of convergence (34) for $xp(x)$ and $x^2 q(x)$.

In these formulas we have "normalized" the solutions by choosing a_0 and b_0 in (49), the coefficient of $y_1(x)(\ln x)$ in (50), and c_0 in (51), all to have value 1. We could just as well have said only that they were nonzero, but it is convenient to have the solutions $y_1(x)$

and $y_2(x)$ unambiguously defined. We have also chosen to omit any constant term in (50), that is, to set c_0 to be 0 there, and similarly to set $c_m = 0$ in (51). If one did not do this, one would find that the values of c_0 in (50) and c_m in (51) would not be determined during the solution procedure: they could be chosen freely. Choosing a nonzero value, however, would amount to adding a multiple of $y_1(x)$ to the solution $y_2(x)$ as given in (50) or (51), and this is not of interest.

We now turn to the question of determining these solutions explicitly, that is, of finding the various unknown coefficients a_n, b_n, c_n, and/or C. We give a step by step summary.

Step I: As discussed above and demonstrated in Example 13, we begin by substituting the form $y(x) = x^r \sum_{n=0}^{\infty} a_n x^n$ into $y'' + p(x)y' + q(x) = 0$, obtaining

$$\sum_{n=0}^{\infty} \bigl[\gamma(n+r)a_n - F_n(r,\mathbf{a})\bigr] x^{n+r-2} = 0 \quad \Longrightarrow \quad \gamma(n+r)a_n = F_n(r,\mathbf{a}), \quad n = 0, 1, \ldots, \tag{52}$$

where $\mathbf{a} = (a_0, \ldots, a_{n-1})$ and $F_n(r,\mathbf{a})$ is some linear combination of a_0, a_1,…, and a_{n-1} (this is just (48) rewritten). The $n = 0$ instance of the recursion in (52) gives $\gamma(r) = 0$ and thus determines r_1 and r_2.

Step II: We find the coefficients a_n of the solution $y_1(x)$ by replacing r in (52) by r_1 and then solving successive recursions for a_1, a_2, …. This was also demonstrated in Example 13.

Step III(a): When $m = r_1 - r_2$ is not an integer we find the coefficients b_n of the solution $y_2(x)$ in (49) by replacing r in (52) by r_2 and a_n by b_n, and then solving successive recursions for b_1, b_2, …. This is just the same procedure as used in Step II above.

Step III(b): To find $y_2(x)$ when $m = r_1 - r_2$ is an integer one must first find $y_1(x)$ as in Step I. Then (50) or (51) tells us that $y_2(x)$ can be written as

$$y_2(x) = C y_1(x)(\ln x) + u(x), \tag{53}$$

where $C = 1$ in (50) and C is to be determined in (51), and

$$u(x) = \sum_{n=0}^{\infty} c_n x^{n+r_2}, \qquad \text{with } c_0 = 0 \text{ in (50) and } c_0 = 1,\ c_m = 0 \text{ in (51).} \tag{54}$$

Substituting (53) into

$$y'' + p(x)y' + q(x) = 0, \tag{55}$$

one finds that $u(x)$ must satisfy the equation

$$u'' + p(x)u' + q(x)u = \frac{C}{x^2}\bigl[y_1(x) - xp(x)y_1(x) - 2xy_1'(x)\bigr]. \tag{56}$$

One then substitutes the form (54) of the series for $u(x)$ into (56) and solves for $c_1, c_2, \ldots$ and, in (51), C. Because the left hand side of (56) is just the original ODE, with y replaced by u, the structure of the resulting recursion will be obtained from (52) by replacing r by r_2, replacing a_n by c_n, and adding the right hand side from (56):

$$\sum_{n=0}^{\infty}\bigl[\gamma(n+r_2)c_n - F_n(r,\mathbf{c})\bigr]x^{n+r_2-2} = \frac{C}{x^2}\sum_{k=0}^{\infty} d_k x^{k+r_1} = C\sum_{n=m}^{\infty} d_{n-m}x^{n+r_2-2}. \tag{57}$$

Here $\sum_{k=0}^{\infty} d_k x^{k+r_1}$ is the (known) power series for $y_1(x) - xp(x)y_1(x) - 2xy_1'(x)$ and to obtain the second equality we have made the change of summation index $n = k + m$ in (57). Thus we have the recursion

$$\gamma(n+r_2)c_n - F_n(r,\mathbf{c}) = \begin{cases} 0, & \text{if } n < m, \\ Cd_{n-m}, & \text{if } n \geq m. \end{cases} \tag{58}$$

When $m = 0$, that is, when solving (50) (so that $C = 1$) one will have $d_0 = 0$, so that the $n = 0$ case of (58) will be just $0 = 0$; we will then solve the $n = 1, 2, \ldots$ cases for $c_1, c_2, \ldots$. When $m > 0$ the $n = m$ case will be $-F_n(r,\mathbf{c}) = Cd_0$, d_0 will be nonzero, and we will solve this for C, then solve $n = 1, 2, \ldots$ cases for $c_1, c_2, \ldots$.

One must be careful about one point in using the procedure of Step III(b). As stated, it assumes that one begins with the form (55) of the ODE in which the coefficient of y'' is 1. In practice, however, one frequently begins rather with $M(x)y'' + P(x)y' + Q(x)y = 0$ (see (14)); for example, in Example 13 we substituted the form (35) of the solution into $x^2y'' + 3xy' - 3(1+x^2)y = 0$, not into $y'' + (3/x)y' - (3(1+x^2)/x^2)y = 0$. In this case it is better to multiply (56) by $M(x)$ to obtain

$$M(x)u'' + P(x)u' + Q(x)u = \frac{M(x)}{x^2}\bigl[y_1(x) - xp(x)y_1(x) - 2xy_1'(x)\bigr] \tag{59}$$

and start from there. If $M(x) = x^2$, however, this does not change much; in particular, (57) is multiplied by x^2 and the recursion (58) is the same.

Example 13 concluded: We show how this works by finding a second solution $y_2(x)$ for the equation (36), thus completing the solution of Example 13. We have $r_1 = 1$, $r_2 = -3$, and so $m = 4$: we are looking for a solution in the form (51). We will use (58) and first discuss separately left and right sides of this equation. The left hand side comes directly from (41); since $\gamma(n + r + 2) = n(n-4)$ it is $n(n-4)c_n - 3c_{n-2}$ (that is, $F_n(r,\mathbf{c}) = -3c_{n-2}$. For the right hand side we find the d_k from the solution $y_1(x)$ as given in (45); using $x(x) = 3$:

$$\begin{aligned} y_1(x) - xp(x)y_1(x) - 2xy_1'(x) &= -2\bigl(y_1(x) + xy_1'(x)\bigr) \\ &= -2\left(\sum_{k=0}^{\infty}\frac{3^k}{2^{2k-1}\,(k+2)!\,k!}\,x^{2k+1} + \sum_{k=0}^{\infty}\frac{(2k+1)3^k}{2^{2k-1}\,(k+2)!\,k!}\,x^{2k+1}\right) \\ &= -4\sum_{k=0}^{\infty}\frac{(k+1)3^k}{2^{2k-1}\,(k+2)!\,k!}\,x^{2k+1} \end{aligned} \tag{60}$$

Comparing (60) with (57) and using $r_1 = 1$ we see that

$$d_{2k} = -\frac{(k+1)3^k}{2^{2k-3}\,(k+2)!\,k!}, \qquad d_{2k+1} = 0, \qquad k = 0, 1, \ldots.$$

One finds easily then that the odd coefficients c_{2k+1} are all zero; we omit details. For the even coefficients we have

$$\begin{aligned}
n = 0: &\qquad 0\cdot c_0 - 3c_{-2} = 0; \quad \text{take } c_0 = 1.\\
n = 2: &\qquad (-4)c_2 - 3c_0 = 0; \quad c_2 = -\frac{3}{4}.\\
n = 4: &\qquad 0\cdot c_4 - 3c_2 = Cd_0 = -4C; \quad C = \frac{3}{4}c_2 = -\frac{9}{16}; \quad \text{take } c_4 = 0.\\
n = 6: &\qquad 12c_6 - 3c_4 = Cd_2 = \left(-\frac{9}{16}\right)\left(-\frac{6}{3}\right); \quad c_6 = \frac{3}{32}.\\
n = 8: &\qquad 32c_8 - 3c_6 = Cd_4 = \left(-\frac{9}{16}\right)\left(-\frac{27}{96}\right); \quad c_8 = \frac{1}{32}\left(\frac{81}{512} + \frac{9}{32}\right) = \frac{225}{16384}.\\
&\qquad \vdots
\end{aligned}$$

Thus

$$y_2(x) = -\frac{9}{16}y_1(x)(\ln x) + x^{-3} - \frac{9}{16}x^{-1} + \frac{3}{32}x^3 + \frac{225}{16384}x^5 + \cdots.$$

This result may also be obtained, as shown in class, by writing out a few terms on both sides of (57) explicitly.

Remark 14: An ordinary point of a differential equation may be considered, in some sense, as a special case of a regular singular point. If $x = 0$ is an ordinary point of (55) then the above analysis applies; one finds that $\gamma(r) = r(r-1)$ and hence that $r_1 = 1$, $r_0 = 0$ and $m = 1$. Thus we expect solutions $y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+1}$ with $a_0 = 1$ and, from (51), $y_2(x) = Cy_1(x)(\ln x) + \sum_{n=0}^{\infty} b_n x^n$ with $b_0 = 1$, $b_1 = 0$. However, we already know that in this case there are two linearly independent solutions, as power series in x, which do not contain $\ln x$; this means that necessarily $C = 0$. One can also find this directly from (58). Then $y_1(x)$ and $y_2(x)$ are respectively the two solutions $y_1(x)$ and $y_0(x)$ found in (28).