These problems cannot cover everything from the course; I suggest that you also review the midterm exams, their review sheets, and homework problems, particularly those on assignments 12, 13, and 14, which cover material that has not yet appeared on an exam..

1. Let $y(t)$, $t \geq 0$, be the solution of

$$y''(t) - 6y'(t) + 9y(t) = f(t), \quad y(0) = 0, \quad y'(0) = 1.$$

Let $Y(s)$ be the Laplace transform of y and $F(s)$ be the Laplace transform of f.

(a) Find an expression for $Y(s)$ in terms of $F(s)$.

(b) Find $y(t)$ if $f(t) = \delta(t-1)$.

2. Consider the Sturm-Liouville problem on the interval $0 < x < 2$:

$$y'' + \lambda y = 0, \qquad y(0) + 2y'(0) = 0, \quad y(2) = 0.$$

You may use without proof the fact that all eigenvalues are nonnegative.

(a) Is there an eigenvalue $\lambda = \lambda_0 = 0$? If yes, give an eigenfunction. If not, show why not.

(b) Find a condition on $\lambda > 0$ that it must satisfy to be an eigenvalue, and for λ satisfying this condition write down an associated eigenfunction.

(c) There are an infinite number of positive eigenvalues $0 < \lambda_1 < \lambda_2 < \cdots$. Show graphically that $(\pi/2)^2 < \lambda_1 < (3\pi/4)^2$.

(d) An arbitrary function $f(x)$ defined for $0 < x < 2$ will have an expansion in terms of the eigenfunctions found above. Write down this expansion, and give integral formulas for all the coefficients which appear. Be as explicit as possible.

3. (a) Find the general solution to

$$\begin{aligned} 4u_{xx}(x,t) &= u_t(x,t), \qquad 0 < x < \pi, \qquad t > 0; \\ u_x(0,t) &= 0, \quad u(\pi,t) = 0, \qquad t > 0. \end{aligned}$$

Show all details necessary to obtain the solution.

(b) Find the solution $u(x,t)$ to

$$\begin{aligned} 4u_{xx}(x,t) &= u_t(x,t) + 1, \qquad 0 < x < \pi, \qquad t > 0; \\ u_x(0,t) &= 1, \quad u(\pi,t) = 1, \qquad t > 0; \\ u(x,0) &= 0. \end{aligned}$$

Give an explicit integral formula for the coefficients in your solution; you do not need to evaluate the integrals explicitly.

4. In this problem we study the PDE

$$u_t = -4u_x, \qquad -\infty < x < \infty, \quad t > 0.$$

(a) By taking the Fourier transform of both sides of the equation, obtain the differential equation satisfied by the Fourier transform $\hat{u}(\omega,t)$ and then solve it to find $\hat{u}(\omega,t)$. Your answer should involve an unknown function of ω.

(b) Suppose now that $u(x,t)$ is required to satisfy $u(x,0) = f(x)$ for some given function $f(x)$. Using your answer in (a), find $\hat{u}(\omega,t)$ in terms of $\hat{f}(\omega)$.

(c) By taking the inverse Fourier transform of $\hat{u}(\omega,t)$, find $u(x,t)$. Your answer, which should be expressed in terms of f, will be somewhat similar to d'Alembert's formula.

(d) Suppose that $f(x) = H(x) - H(x-1)$. Sketch $u(x,0)$ and $u(x,1)$.

5. (a) Find $f(x)$ if the Fourier transform of f is $\hat{f}(\omega) = e^{-(\omega+1)^2}$.

(b) Find the Fourier transform of $e^{-|x-3|} + e^{-2|x|}$.

6. Consider the following differential equation for $y(x)$, $x > 0$:

$$2x^2y'' + xy' - (1 + x^2)y = 0. \tag{1}$$

(a) Explain why $x = 0$ is a regular singular point of this equation.

(b) We want to solve (1) by the Frobenius method. Find the indicial equation and determine its roots r_1 and r_2.

(c) Let $y_1(x)$ be the Frobenius method solution corresponding to the larger root and with $a_0 = 1$. Find y_1 explicitly out to its first three non-zero terms.

(d) Write down the general recursion relation (relating a_n to previous a_k) for determining the coefficients of the Frobenius method solutions.

7. Consider the wave equation

$$\begin{aligned} u_{xx} &= u_{tt} - u, \qquad 0 < x < 1, \quad t > 0; && (2)\\ u(0,t) = 0, \quad u(1,t) &= 0, \qquad t > 0 && (3)\\ u(x,0) = 0, \quad u_t(x,0) &= x, \qquad 0 < x < 1. && (4) \end{aligned}$$

(a) Write down equations that must be satisfied by $X(x)$ and $T(t)$ if $u(x,t) = X(x)T(t)$ is to be a product solution to (2).

(b) Find all possible separable solutions to (2) and (3), and write down the general solution to these two equations.

(c) Explain how to choose the coefficients of the general solution to solve (2) and (4) together with the initial conditions given in (4). You should give explicit formulas, but these may have integrals in them.

8. This problem is about the nonlinear system in the plane:

$$x' = y, \qquad y' = -x + x^3 + y.$$

(a) Find the singular points and give the linearized system of differential equatios near each. Classify the type and stability of each singular point, in so far as possible using linearization. If the linearization method does not fully classify the singular point, indicate what possibilities it does allow.

(b) Sketch the phase portrait for the linearized system near each singular point. Include arrows of flow directions and eigenvectors, where appropriate.

(c) Sketch the nullclines (curves on which the flow is vertical or horizontal) and indicate by arrows on your sketch the direction of flow on the nullclines and in the regions of the plane that they define.

9. Solve Laplace's equation in the rectangle $0 \le x \le 4,$, $0 \le y \le 3$, with the boundary conditions:

(a) $u(0,y) = 0$, $u(4,y) = 1$, $u(x,0) = 0$, $u(x,3) = 0$;

(b) $u(0,y) = 0$, $u(4,y) = 1$, $u(x,0) = 0$, $u(x,3) = 2\sin 3\pi x$;

(c) $u_x(0,y) = 0$, $u(4,y) = 0$, $u(x,0) = \cos 7\pi x/8$, $u(x,3) = 0$.