The Nonlinear Pendulum

The nonlinear pendulum equation is

$$\theta'' = -\frac{g}{l}\sin\theta - \gamma\theta',$$

where

- θ is the angle that the pendulum makes from a downward vertical axis, measured counterclockwise;
- g is the gravitational constant,
- l the length of the pendulum, and
- γ is a damping constant, here measured (MKS units) in sec^{-1}.

We take $g/l = 1$ for simplicity and set $x = \theta$, $y = \theta'$, so we are studying the nonlinear system

$$x' = y \qquad y' = \sin x - \gamma y.$$

Since $x = \theta$ is an angle, two points in the phase plane of the form (x, y) and $(x + 2n\pi, y)$ represent the same physical point.

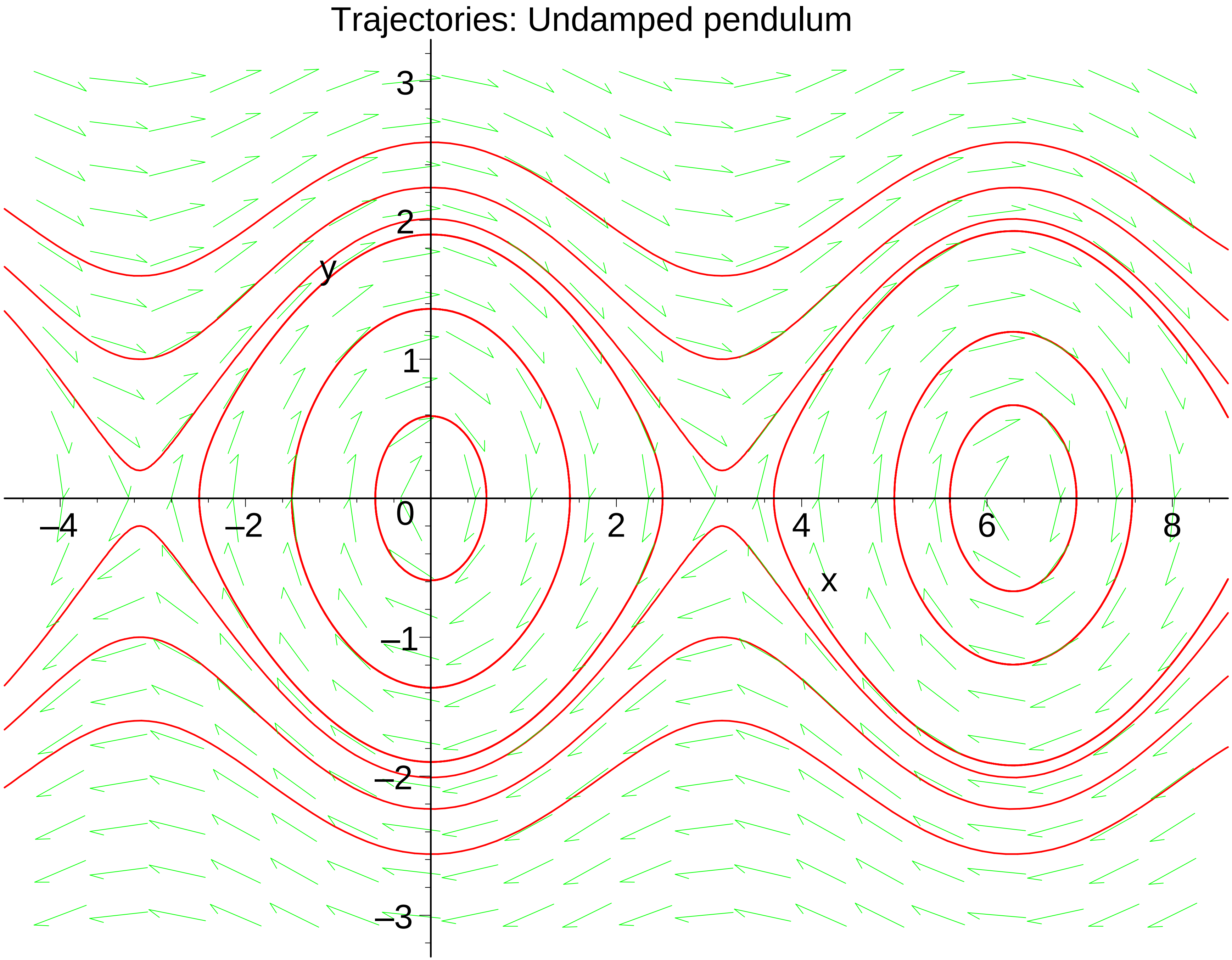
Here are the equations again:

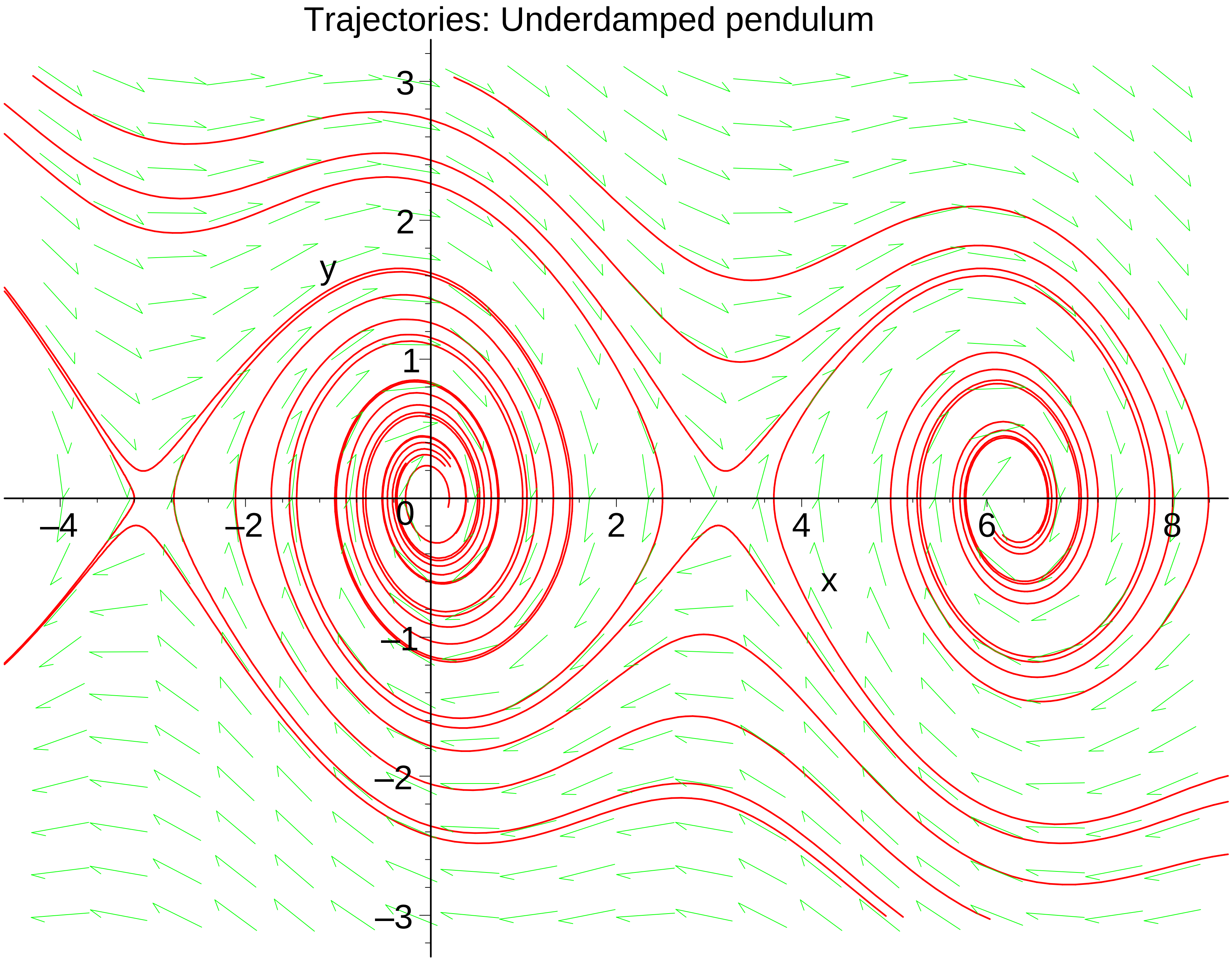
$$x' = y \qquad y' = \sin x - \gamma y.$$

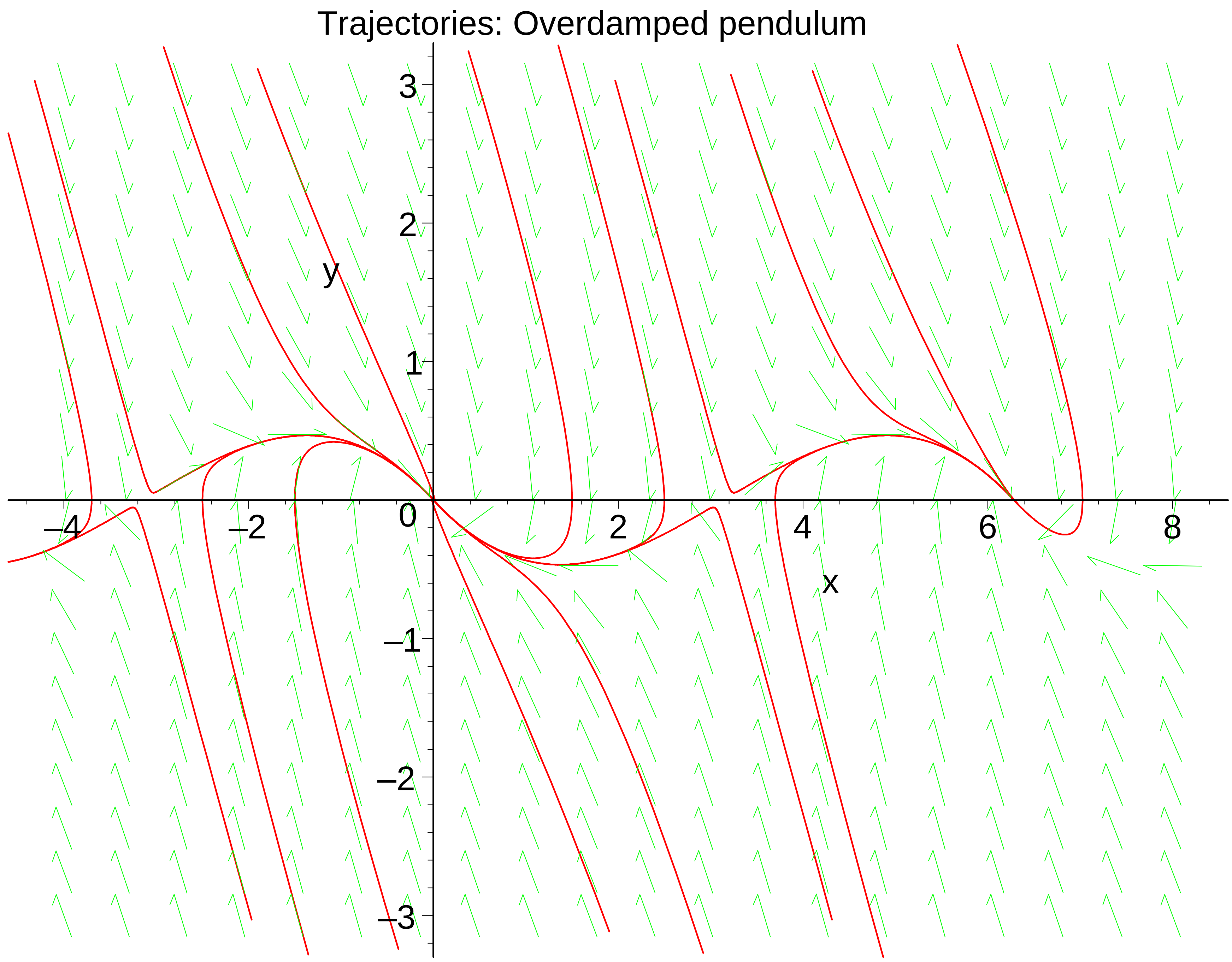
The system has critical points at $x = n\pi$, $y = 0$, where

- If n is even then the pendulum is motionless, hanging down;
- If n is odd then the pendulum is motionless, balanced straight up;

The critical points for odd n are always saddle points. The critical points for even n can be centers (undamped case, $\gamma = 0$), stable spirals (underdamped case) or stable nodes (overdamped case). We draw the phase plane in these three cases, taking $\gamma = 0$, $\gamma = 0.2$, and $\gamma = 2.1$, repectively.


Trajectories: Undamped pendulum
y
x
–4
–2
0
2
4
6
8
3
2
1
–1
–2
–3


Trajectories: Underdamped pendulum
y
x
3
2
1
0
–1
–2
–3
–4
–2
2
4
6
8



Trajectories: Overdamped pendulum
y
x
3
2
1
0
–1
–2
–3
–4
–2
2
4
6
8