Turn in starred problems Tuesday 3/7/2017.

1*. Consider the problem of finding extrema of $I(y) = \int_0^1 (1 + y'(x)^2)\,dx$, subject to the conditions $y(0) = y_1$, $y(1) = y_2$.

(a) Determine the Euler-Lagrange equation for the problem and show that it has a unique solution $y_0(x)$ satisfying the endpoint conditions.

(b) By computing $I(y_0 + \eta)$, where $\eta(x)$ is a differentiable function with $\eta(0) = \eta(1) = 0$, show that $I(y_0)$ is an absolute minimum of $I(y)$.

2. Find the Euler-Lagrange equation for extremals of $I(y) = \int_{x_1}^{x_2} f(x, y(x), y'(x))\,dx$ in each case below. Simplify to the extent practical. The equation in (b) should be familiar.

(a) $f(x, y, y') = xy'^2 - yy' + y$; (b) $f(x, y, y') = p(x)y'^2 - q(x)y^2 - \lambda w(x)y^2$.

3*. Consider the problem of finding extrema of

$$I(y) = \int_0^a (y'(x)^2 - y(x)^2)\,dx, \qquad a > 0,$$

subject to the conditions $y(0) = y_1$, $y(a) = y_2$. Determine the Euler-Lagrange equation for the problem and show that if $a$ is not of the form $n\pi$ for some integer $n$ then there is a unique solution satisfying the end-point conditions. Show further that if $a = n\pi$ then either there is no solution or there are many solutions, and identfy the conditions under which each possibilitiy occurs.

4*. In this problem we find the shortest path (geodesic) between two points on the sphere. We use the notation of Greenberg, Section 14.6.3 for spherical coordinates $\theta$ and $\phi$. (This problem is discussed in Section 3-5(c) of Weinstock.)

(a) Show that if a path on the surface of a sphere of radius $R$ is described by a function $\theta(\phi)$, defined for $\phi_1 \le \phi \le \phi_2$ and satisfying $\theta(\phi_1) = \theta_1$, $\theta(\phi_2) = \theta_2$, then the path length is

$$I(\theta) = \int_{\text{path}} \sqrt{dx^2 + dy^2 + dz^2} = R\int_{\theta_1}^{\theta_2} \sqrt{\sin^2\phi\,\theta'(\phi)^2 + 1}\,d\phi.$$

(b) Determine the Euler-Lagrange equation satisfied by a minimizing path $\theta(\phi)$ and, from the fact that the integrand $f(\phi, \theta')$ is independent of $\theta$, show that $\theta(\phi)$ satisfies the first order equation

$$\frac{d\theta}{d\phi} = \pm\frac{\csc^2\phi}{\sqrt{C^2 - \csc^2\phi}}$$

for some constant $C$. Integrate this equation (a preliminary substitution $u = \cot\phi$ is helpful) and show that the resulting solutions lies on a plane through the origin: for some $a, b, c$

$$ax + by + cz = aR\sin\phi\cos\theta + bR\sin\phi\sin\theta + cR\cos\phi = 0,$$

on the curve, and hence the curve is a portion of a great circle.

5*. (Weinstock, page 46, 7 (a)). Derive the differential equation satisfied by the four-times differentiable function $y(x)$ which extremizes the integral

$$I(y) = \int_{x_1}^{x_2} f(x, y, y', y'')\,dx,$$

under the condition that both $y$ and $y'$ are prescribed at $x_1$ and $x_2$.