Turn in starred problems Tuesday 4/4/2017.

8.A* Exercise 2(a) from the posted notes (by Professor Liu) on the calculus of variations. Take E, I, and q as constants.

8.B* An elastic membrane is stretched over a wire loop; the parametric equations in cylindrical coordinates of the loop are $r = r(\theta) = R$, $z = z(\theta) = A\sin 2\theta$, with θ the polar angle. The membrane surface is described by a function $z = w(x, y)$, $x^2 + y^2 \le R^2$, with the boundary condition (from the loop) $w(R\cos\theta, R\sin\theta) = A\sin 2\theta$. w will minimize the energy

$$E(w) = \int_{x^2+y^2\le R^2} \left[\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right] dx\,dy.$$

(a) Use the calculus of variations to obtain the partial differential equation satisfied by w.

(b) Use the chain rule to express $\partial w/\partial x$ and $\partial w/\partial y$ in terms of $\partial w/\partial r$ and$\partial w/\partial\theta$, and thus express $E(w)$ as an integral in polar coordinates involving the r and θ derivatives of w (remember that $dx\,dy = r\,dr\,d\theta$). Find directly from this the differential equation for $w(r, \theta)$ in polar coordinates. (Note: this is one way to obtain the form of the Laplacian in polar coordinates. See (18) on page 785 of Greenberg.)

(c) Find the solution $w(r, \theta)$. Hint: separation of variables. w should be bounded.

8.C Consider the situation of problem 8.B, but suppose now that $w(x, y)$ is to minimize the area $A(w)$ of the membrane, where

$$A(w) = \int_{x^2+y^2\le R^2} \sqrt{1 + \left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2}\, dx\,dy.$$

Find the partial differential equation, in Cartesian coordinates, which w will satisfy.

8.D* Consider vibrations of a rectangular membrane which covers the region Ω in which $0 \le x \le a$, $0 \le y \le b$; let $w(t, x, y)$ denote the (vertical) displacement of the membrane at the point (x, y) and time t. The kinetic and potential energies of the membrane are respectively

$$T = \frac{\rho}{2}\int_\Omega \left(\frac{\partial w}{\partial t}\right)^2 dx\,dy \quad \text{and} \quad V = \frac{\tau}{2}\int_\Omega \left[\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right] dx\,dy,$$

with ρ the density of the membrane and τ the surface tension. The motion of the membrane between times t_1 and t_2, with fixed initial and final configurations $w(t_1, x, y) = u(x, y)$ and $w(t_2, x, y) = v(x, y)$, will be one for which the action $\int_{t_1}^{t_2} L(t)\,dt$, where $L = T - V$, is stationary (that is, at which the first variation of the action is zero).

(a) Suppose that w satisfies homogeneous Dirichlet boundary conditions: for $t_1 \le t \le t_2$,

$$w(t, 0, y) = w(t, a, y) = 0, \quad 0 \le y \le b, \qquad w(t, x, 0) = w(t, x, b) = 0, \quad 0 \le x \le a.$$

Use the calculus of variations to obtain the partial differential equation satisfied by w.

(b) Suppose now that the edge of the membrane on which $y = 0$ is free to move transversely to the x-y plane. Show that w satisfies the PDE of (a) but with a homogeneous Neumann boundary condition $\partial w/\partial y = 0$ on this edge.

8.E* A string, whose rest position is the portion $0 \le x \le L$ of the x axis, vibrates in the x-y plane with amplitude $y = w(x, t)$. The left end of the string is fixed at its rest position (so that $w(0, t) = 0$) while the right end is free to move transversally to the x axis but is subject with a linear restoring force $-kw(L, t)$, corresponding to a potential energy $V_0(t) = kw(L, t)^2/2$ which must be added to the usual potential energy of the string. Find the total kinetic and potential energies of the system and the action $\int_{t_1}^{t_2}(T(t) - V(t))\,dt$, and thus show that $w(x, t)$ satisfies the wave equation with a Robin boundary condition at the right end.