FORMULAS

You will be given a copy of Appendix F of Greenberg, as well as these formulas:

$$\cos(x+iy) = \cos x \cosh y - i \sin x \sinh y, \quad \sin(x+iy) = \sin x \cosh y + i \cos x \sinh y$$

$$\sin w = \sum_{n=0}^{\infty} \frac{(-1)^n w^{2n+1}}{(2n+1)!} \qquad \cos w = \sum_{n=0}^{\infty} \frac{(-1)^n w^{2n}}{(2n)!}$$

$$e^w = \sum_{n=0}^{\infty} \frac{w^n}{n!} \qquad \frac{1}{1-w} = \sum_{n=0}^{\infty} w^n, \ |w| < 1$$

$$\operatorname{Res}_{z=a} f(z) = \frac{1}{(n-1)!} \lim_{z \to a} \frac{d^{n-1}}{dz^{n-1}} [(z-a)^n f(z)]$$

$$f^{(n)}(z) = \frac{n!}{2\pi i} \oint \frac{f(\zeta)}{(\zeta - z)^{n+1}} \, d\zeta$$

REVIEW PROBLEMS

1. Find all values of $(5i)^i$ and give a plot showing the locations of these numbers in the complex plane. You may use the fact that $\ln 5 = 1.609\ldots$ is a little larger than $\pi/2$.

2. (a) Find a function $u(x,y)$ which is harmonic in the upper half plane $\{z = x + iy \mid y > 0\}$ and takes boundary values $u(x,0) = 1$ for $x > 2$, $u(x,0) = 3$ for $-2 < x < 2$, and $u(x,0) = 0$ for $x < -2$. Express your answer in terms of x and y and give a full, unambiguous definition of the function u.
(b) Find a harmonic conjugate $v(x,y)$ of $u(x,y)$.
Note: this problem requires no computation!

3. Let D be the strip $0 < x < \pi$ in the z plane (where $z = x + iy$), and consider the mapping $w = \cos z = \cos x \cosh y - i \sin x \sinh y$. NOTE: The questions below ask you about images of lines and of D under this mapping. A sketch will no doubt be helpful in answering these questions, but you should also describe the images in words and explain your reasoning.
(a) Describe carefully what happens to each of the two straight lines bounding D, and to the line $x = \pi/2$, under this mapping. Find the image of the point on each line at which $y = 0$ and describe the image of points on the line at which y is near $+\infty$ and near $-\infty$.
(b) Using (a), describe the image of the domain D.

4. Let D be the portion of the first quadrant in the complex plane which lies inside the circle $|z| = 2$:
$$D = \{z = x + iy \mid x > 0,\ y > 0, \text{ and } x^2 + y^2 < 4\}.$$
(a) Let $f(z) = z^2$. Describe the image of D (in the w-plane) under the mapping $w = f(z)$.
(b) Find a conformal mapping which takes D onto the *entire* first quadrant in the ζ-plane: $\{\zeta = \xi + i\eta \mid \xi > 0,\ \eta > 0\}$. **Hint:** part (a) should help.
(c) Solve Laplace's equation $u_{xx} + u_{yy} = 0$ in D, with boundary conditions $u = 1$ on the x axis, $u = -1$ on the y axis, and $u = 0$ on the circular arc.

5. The formula $f(z) = \log\bigl(z + i\sqrt{z^2 - 2}\bigr)$ defines a multiple-valued function of z. Find all possible values of $f(i)$.

6. Let D denote the portion of the upper half plane lying *outside* the circle $|z| = 3$. Find a harmonic function $u(x, y)$ in D which takes boundary value $u = 5$, on the semicircle $|z| = 3, \operatorname{Im} z > 0$, and $u = 0$, on the real axis $\operatorname{Im} z = 0$, $|\operatorname{Re} z| > 3$. Hint: Appendix F.

7. In this problem we let C be the straight line from 0 to $2 + i$.

(a) Evaluate $\displaystyle\int_C (z^2 + \bar{z})\, dz$.

(b) Find an "ML" bound for $\displaystyle\left|\int_C \frac{e^{2iz}}{(z+1)(z+4)}\, dz\right|$.

8. Let $f(z) = \dfrac{1}{3 - z}$. Find the Taylor series of $f(z)$ with center 1. In what region does this series represent $f(z)$?

9. Use the Residue Theorem to evaluate $\displaystyle\int_0^\pi \frac{d\theta}{5 - 4\cos\theta}$.

10. (a) We know that $\sin n\pi = 0$ for $n = 0, \pm 1, \pm 2, \ldots$. Show that each of these zeros of $\sin z$ is of order 1.

In the remainder of this problem we define $f(z) = \dfrac{z \sin^2 z}{e^z \sin 4z}$.

(b) Find all singular points of $f(z)$ and for each, if it is isolated, classify it as removable, essential, or a pole of specified order. Explain your reasoning.

(c) Find all zeros of this function, after any removable singularites have been removed, and the order of each. Explain your reasoning.

(d) If $f(z)$ is expanded in a Taylor series about the center $z = i$, what will the radius of convergence of the series be? **Do not attempt to calculate the series.**

11. Use the Residue Theorem to evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{x \sin 2x\, dx}{(x^2+1)(x^2+9)}$. Justify all steps of your calculation.

12. Suppose that the function $f(z)$ is analytic in a domain D which contains the closed disk $|z| \le R$. Show formally from Cauchy's integral formula that, in the interior of this disk, $f(z)$ is equal to the sum of its Taylor series with center $z = 0$. Here "formally" means that you are not asked to give a proof that your manipulations are justified.