TEXAS TECH UNIVERSITY

5 Perturbation Theory

Generally finding the exact solution of most interesting problems is impossible or at least so difficult that it is not practical to obtain it. Sometimes it is possible with a bit of work to obtain a so-called asymptotic series approximation of the solution that gives a good approximation to the solution. In what follows I hope to provide, mostly by way of numerous examples, some insight into this important branch of mathematics which I refer to as perturbation theory. Our goal is to examine several examples including the asymptotic analysis of solutions of algebraic, transcendental, differential equations and the evaluation of integrals.

5.1 Preliminary material

The binomial Theorem states

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 \; + \cdots . \tag{5.1.1}$$

If n is a positive integer this formula terminates and we have the more familiar formula

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 \; + \cdots + b^n.$$

More generally for $n \in \mathbb{R}$ the series is infinite and converges for $\left|\frac{b}{a}\right| < 1$ and diverges otherwise.

As examples consider

$$(a+b)^{1/2} = a^{1/2} + \frac{1}{2}a^{-1/2}b + \frac{\left(\frac{1}{2}\right)\left(\frac{-1}{2}\right)}{2!}a^{-3/2}b^2 + \cdots \tag{5.1.2}$$

$$(a+b)^{-1} = a^{-1} - a^{-2}b + a^{-3}b^2 - a^{-4}b^3 + \cdots . \tag{5.1.3}$$

We will also use the classical Taylor series (and Taylor Polynomial) expansion for a smooth function which expanded about $x = x_0$ is given by

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n. \tag{5.1.4}$$

Examples with $x_0 = 0$ include

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \tag{5.1.5}$$

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{(2n+1)}}{(2n+1)!} \tag{5.1.6}$$

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{(2n)}}{(2n)!}. \tag{5.1.7}$$

We are interested in

$$\lim_{\epsilon \to 0} f(\epsilon), \quad \epsilon > 0.$$

We will usually assume that $\epsilon > 0$ in order to minimize confusion that might arise otherwise, e.g.,

$$\lim_{\epsilon \downarrow 0} e^{-1/\epsilon} = 0, \quad \lim_{\epsilon \uparrow 0} e^{-1/\epsilon} = \infty.$$

We are also interested in the case when the limit exists (i.e., it does not have an essential singularity like $\sin(1/\epsilon)$). Thus we consider the cases

$$\left\{ \begin{array}{l} f(\epsilon) \to 0, \\ \\ f(\epsilon) \to A, \\ \\ f(\epsilon) \to \infty, \end{array} \right\} \text{ as } \epsilon \to 0, \quad 0 < A < \infty.$$

If $f(\epsilon) \to 0, \infty$ as $\epsilon \to 0$ then we are also interested in the rate at which the limit is approached, e.g.,

$$\lim_{\epsilon \to 0} \sin(\epsilon) = 0, \quad \lim_{\epsilon \to 0} (1 - \cos(\epsilon)) = 0, \quad \lim_{\epsilon \to 0} (\epsilon - \sin(\epsilon)) = 0, \quad \lim_{\epsilon \to 0} [\ln(1+\epsilon)]^4 = 0, \quad \lim_{\epsilon \to 0} e^{-1/\epsilon} = 0.$$

In order to determine the rate at which such a limit is achieved we introduce the concept of *gauge functions*. The simplest such functions are the powers of ϵ which for $\epsilon < 1$ satisfy

$$1 > \epsilon > \epsilon^2 > \cdots, \quad \epsilon^{-1} < \epsilon^{-2} < \epsilon^{-3} < \cdots.$$

Thus for example we can compute, using Taylor series or using L'Hospital's rule

$$\lim_{\epsilon \to 0} \frac{\sin(\epsilon)}{\epsilon} = \lim_{\epsilon \to 0} \left(1 - \frac{\epsilon^2}{3!} + \frac{\epsilon^4}{5!} - \cdots \right) = 1$$

$$\lim_{\epsilon \to 0} \frac{(1 - \cos(\epsilon))}{\epsilon^2} = \frac{1}{2!}$$

$$\lim_{\epsilon \to 0} \frac{(\epsilon - \sin(\epsilon))}{\epsilon^3} = \frac{1}{3!}$$

$$\lim_{\epsilon \to 0} \frac{[\ln(1+\epsilon)]^4}{\epsilon^4} = \lim_{\epsilon \to 0} \left(1 - \frac{\epsilon}{2} + \frac{\epsilon^2}{3} + \cdots \right) = 1.$$

Note also that functions like $f(\epsilon) = e^{-1/\epsilon}$ go to zero faster than any power of ϵ, i.e., by L'Hospital's rule

$$\lim_{\epsilon\to 0} \frac{e^{-1/\epsilon}}{\epsilon^n} = \lim_{x\to\infty} \frac{x^n}{e^x} = \lim_{x\to\infty} \frac{n!}{e^x} = 0.$$

Furthermore, $f^{(n)}(0) = 0$ for all n. For example, $f'(\epsilon) = e^{-1/\epsilon}/\epsilon^2$, and once again using L'Hospital's rule

$$f'(0) = \lim_{\epsilon\to 0} \frac{e^{-1/\epsilon}}{\epsilon^2} = \lim_{x\to\infty} \frac{x^2}{e^x} = \lim_{x\to\infty} \frac{2}{e^x} = 0.$$

Thus f cannot be expanded in a taylor series about $\epsilon = 0$.

On the other hand the function $f(\epsilon) = e^{1/\epsilon} \to \infty$ as $\epsilon \to 0$ and this limit is faster than any power of $1/\epsilon$, i.e.,

$$\lim_{\epsilon\to 0} \frac{e^{1/\epsilon}}{1/\epsilon^n} = \lim_{x\to\infty} \frac{e^x}{x^n} = \lim_{x\to\infty} \frac{e^x}{n!} = \infty.$$

We also have slowly converging functions like $f(\epsilon) = \ln(1/\epsilon)$ which goes to zero slower than any power of $1/\epsilon$. Namely,

$$\lim_{\epsilon\to 0} \frac{\ln(1/\epsilon)}{\epsilon^{-\alpha}} = \lim_{x\to\infty} \frac{\ln(x)}{x^\alpha} = \lim_{x\to\infty} \frac{1}{\alpha x^\alpha} = 0.$$

So, in addition to all powers of ϵ (i.e., ϵ^j, for $j = \cdots, -2, -1, 0, 1, 2, \cdots$ we also might need functions like $e^{\pm 1/\epsilon}$ and $\ln(1/\epsilon)$, $\ln(\epsilon)$, etc.

A very useful notation that we will use often is the "Big Oh" and "Little Oh" notation.

Definition 5.1. *We say that $f(\epsilon) = \mathcal{O}(g(\epsilon))$ as $\epsilon \to 0$ if*

$$\lim_{\epsilon\to 0} \frac{f(\epsilon)}{g(\epsilon)} = A, \;\; 0 < |A| < \infty. \tag{5.1.8}$$

We say that $f(\epsilon) = o(g(\epsilon))$ as $\epsilon \to 0$ if

$$\lim_{\epsilon\to 0} \frac{f(\epsilon)}{g(\epsilon)} = 0. \tag{5.1.9}$$

Examples of "Big-Oh" include

$$\cos(\epsilon) = \mathcal{O}(1), \;\; 1 - \cos(\epsilon) = \mathcal{O}(\epsilon^2), \;\; \tanh(\epsilon) = \mathcal{O}(\epsilon), \;\; \sec(\epsilon) = \mathcal{O}(1), \;\; \frac{\epsilon^{3/2}}{\sin(\epsilon)} = \mathcal{O}(\epsilon^{1/2}).$$

One of the main uses of the "little-oh" notation comes from the fact that sometimes it may be hard to determine the exact rate of convergence but it is sufficient to determine whether the rate is faster or slower than a given gauge. Examples of "little-h" include

$$\sin(\epsilon) = o(1), \;\; \sin(\epsilon) = o(\epsilon^{1/2}), \;\; \cos(\epsilon) = o(\epsilon^{-1}), \;\; e^{-1/\epsilon} = o(\epsilon^{-10^{-8}}), \;\; \ln(1/\epsilon) = o(\epsilon^{-.000001}).$$

Since we may not be able to use only powers of ϵ to identify the rate of convergence of a function, we are lead to supposing that we have a general set of *gauge functions*.

Definition 5.2. *We say that*

$$\sum_{n=0}^{N} c_n f_n(\epsilon)$$

*is an **asymptotic expansion** of $f(\epsilon)$ at $\epsilon = 0$ if the following hold:*

1. *The sequence $\{f_n\}$, $n = 0, \cdots, (N+1)$, is a **gauge sequence**; i.e., $f_n(\epsilon) = o(f_{n-1}(\epsilon))$ as $\epsilon \to 0$ for $n = 1, \cdots, (N+1)$, and*

2. *we have*

$$f(\epsilon) - \sum_{n=0}^{N} c_n f_n(\epsilon) = \mathcal{O}(f_{N+1}(\epsilon)) \quad \textit{as} \;\; \epsilon \to 0.$$

As an example, if $f(\epsilon)$ has $(N+1)$ continuous derivatives at $\epsilon = 0$ then it can be approximated (near $\epsilon = 0$) by a Taylor polynomial of degree N

$$f(\epsilon) = f(0) + f^{(1)}(0)\epsilon + \cdots + f^{(N)}(0)\epsilon^N/N! + \mathcal{O}(\epsilon^{N+1}).$$

The error term is given by

$$\frac{f^{(N+1)}(\eta)}{(N+1)!}\epsilon^{N+1}, \text{ for some } \; 0 < \eta < \epsilon.$$

Here the gauge functions are $f_n = \epsilon^n$.

Theorem 5.1 (Fundamental Theorem of Perturbation theory). *If an asymptotic expansion satisfies*

$$A_0 + A_1\epsilon + \cdots A_N\epsilon^N + \mathcal{O}(\epsilon^{N+1}) \equiv 0,$$

for all sufficiently small ϵ and the coefficients $\{A_j\}$ are independent of ϵ, then

$$A_0 = A_1 = \cdots = A_N = 0.$$

5.2 Algebraic Equations

Example 5.1. We would expect that the roots of the quadratic equation

$$x^2 - 2x + .001 = 0$$

would be close to the roots $x = 0$ and $x = 2$ of $x^2 - 2x = 0$ since $.001$ is small. The question is can we say approximately how close they are. To answer this we consider obtaining an asymptotic expansion for the roots of the more general problem

$$x^2 - 2x + \epsilon = 0 \text{ for } \epsilon \ll 1$$

where $\epsilon \ll 1$ means that ϵ is much less than 1.

For this example let us let $x_1 = 0$, $x_2 = 2$ and note that the exact solutions to the perturbed problem are

$$x_1(\epsilon) = 1 - \sqrt{1-\epsilon}, \quad x_2(\epsilon) = 1 + \sqrt{1-\epsilon}.$$

Then based on the binomial theorem given in (5.1.1) with $a = 1$, $b = -\epsilon$ and $n = 1/2$ which implies

$$\sqrt{1-\epsilon} = 1 - \frac{1}{2}\epsilon - \frac{1}{8}\epsilon^2 - \cdots .$$

Thus we can write

$$x_1(\epsilon) = \frac{1}{2}\epsilon + \frac{1}{8}\epsilon^2 + \frac{1}{16}\epsilon^3 + \cdots ,$$

and

$$x_2(\epsilon) = 2 - \frac{1}{2}\epsilon - \frac{1}{8}\epsilon^2 - \frac{1}{16}\epsilon^3 + \cdots .$$

So we can easily the rate at which the perturbed roots converge to the exact roots at $\epsilon = 0$.

An important question is whether or not this is actually an asymptotic expansion. If we can show that the series represents a convergent power series, then it is an asymptotic series.

One of the best methods to check whether a power series is convergent is the ratio test which states

$$\sum_{n-1}^{\infty} u_k \quad \text{converges if} \quad \lim_{k\to\infty} \frac{u_{k+1}}{u_k} < 1.$$

For our example we have the kth term given by

$$\frac{(1/2)(1/2-1)\cdots(1/2-k)(-\epsilon)^k}{k!}$$

so

$$\lim_{k\to\infty} \frac{(k+1)\text{st term}}{k\text{th term}} = \lim_{k\to\infty} \frac{(k+1/2)}{(k+1)}\,\epsilon = \epsilon$$

so the series converges for $0 < \epsilon < 1$.

Thus would not be a very useful method if we had to use the quadratic formula as we did above. Let us consider a more direct method – the method of regular perturbation theory. We suspect that there is an asymptotic series in the form

$$x(\epsilon) = a_0 + a_1\epsilon + a_2\epsilon^2 + \cdots .$$

We substitute this formal series into the perturbed equation and appeal to (5.1) by successively setting the terms corresponding to powers of ϵ equal to zero.

For this example we would have

$$(a_0 + a_1\epsilon + a_2\epsilon^2 + \cdots)^2 - 2(a_0 + a_1\epsilon + a_2\epsilon^2 + \cdots) + \epsilon = 0$$

or, collecting powers of ϵ,

$$[a_0^2 + 2a_0a_1\epsilon + (a_1 + 2a_0a_2)\epsilon^2 + \cdots] - 2[a_0 + a_1\epsilon + a_2\epsilon^2 + \cdots] + \epsilon = 0$$

or

$$(a_0^2 - 2a_0) + (2a_0a_1 - 2a_1 + 1)\epsilon + (a_1^2 + 2a_0a_2 - 2a_2)\epsilon^2 + \cdots$$

which gives

$$\begin{aligned}
\epsilon^0 &: \ a_0^2 - 2a_0 = 0, \ \Rightarrow a_0 = 0, 2, \\
\epsilon^1 &: \ 2a_0a_1 - 2a_1 + 1 = 0, \ \ \Rightarrow a_1 = \frac{-1}{2(a_0-1)}, \\
\epsilon^2 &: \ a_1^2 + 2a_0a_2 - 2a_2 = 0, \ \ \Rightarrow a_2 = \frac{-a_1^2}{2(a_0-1)}.
\end{aligned}$$

If we take $a_0 = 0$ then this gives

$$a_1 = \frac{-1}{2(0-1)} = \frac{1}{2}, \ \ a_2 = \frac{-(1/2)^2}{2(0-1)} = \frac{1}{8},$$

and for $a_0 = 2$ then this gives

$$a_1 = \frac{-1}{2(2-1)} = -\frac{1}{2}, \ \ a_2 = \frac{-(1/2)^2}{2(2-1)} = -\frac{1}{8},$$

in agreement with our calculations above.

Sometimes we might want to use the knowledge that the result can be given in a power series expansion, i.e., that

$$a_j = \frac{x^{(j)}(0)}{j!}, \qquad \text{where } \ x^{(j)}(0) = \frac{d^j}{d\epsilon^j} x(\epsilon)\big|_{\epsilon=0}.$$

So in this case we compute

$$\frac{d}{d\epsilon} f(x(\epsilon)) = \frac{d}{d\epsilon}(x(\epsilon)^2 - 2x(\epsilon) + \epsilon) = 2x(\epsilon)\frac{dx}{d\epsilon}(\epsilon) - 2\frac{dx}{d\epsilon}(\epsilon) + 1 = 0$$

which, for $x(0) = 0$ gives

$$\frac{dx}{d\epsilon}(0) = \frac{1}{2}$$

and, for $x(0) = 2$ gives

$$\frac{dx}{d\epsilon}(0) = -\frac{1}{2}.$$

We could continue but I think you get the idea.

Unfortunately it is very common that a regular asymptotic expansion does not suffice. Consider the following example.

Example 5.2. Consider the quadratic equation

$$\epsilon x^2 + x + 1 = 0.$$

For every nonzero ϵ this equation has two roots but for $\epsilon = 0$ the equation becomes $x+1 = 0$ which has only one root $x = -1$. Thus we say that there is a *singularity* in the roots of the perturbed equation at $\epsilon = 0$ (hence the name “singular perturbation problem”).

If we proceed as we did in Example 5.1 by assuming an expansion

$$x(\epsilon) = a_0 + a_1\epsilon + a_2\epsilon^2 + \cdots ,$$

then we get

$$\epsilon(a_0 + a_1\epsilon + a_2\epsilon^2 + \cdots)^2 + (a_0 + a_1\epsilon^+ a_2\epsilon^2 + \cdots) + 1 = 0,$$

or

$$(a_0 + 1) + \epsilon(a_1 + a_0^2) + \cdots$$

which implies

$$a_0 = -1, \quad a_1 = -a_0^2, \ \cdots \ \Rightarrow \ x(\epsilon) = -1 - \epsilon + \cdots .$$

But this only gives information about one perturbed root.

The exact roots are

$$x(\epsilon) = \frac{1}{2\epsilon}\left(-1 \pm \sqrt{1 - 4\epsilon}\right). \tag{5.2.1}$$

If we once again apply the binomial theorem with $a = 1$, $b = -4\epsilon$ and $n = 1/2$ then we have

$$\begin{aligned}(1 - 4\epsilon)^{1/2} &= 1 - 2\epsilon + \frac{(1/2)(-1/2)}{2!}16\epsilon^2 + \cdots \\ &= 1 - 2\epsilon - 2\epsilon^2 + \cdots\end{aligned}$$

Thus using the exact solutions (5.2.1) with the plus sign we arrive at

$$x(\epsilon) = \frac{-1 + 1 - 2\epsilon - 2\epsilon^2 + \cdots}{2\epsilon} = -1 - \epsilon + \cdots$$

just as above. But when we use (5.2.1) with the minus sign we get

$$x(\epsilon) = \frac{-1 - 1 + 2\epsilon + 2\epsilon^2 + \cdots}{2\epsilon} = -\frac{1}{\epsilon} + 1 + \epsilon + \cdots .$$

Thus the two roots go into powers of ϵ but one starts out with ϵ^{-1}.

Thus we see that we cannot expect to have asymptotic expansions only in the form

$$x(\epsilon) = a_0 + a_1\epsilon + a_2\epsilon^2 + \cdots ,$$

and we need additional information to determine the form of the expansions.

This is one of the main ideas in the area of singular perturbation theory.

We could have argued that since the number of roots of the unperturbed problem is less than the number of roots of the perturbed problem we should expect that some perturbed roots must go to infinity as $\epsilon \to 0$. With this in mind we might seek an expansion in the form

$$x(\epsilon) = \frac{y}{\epsilon^\nu}, \ \text{ for } \ \nu \geq 0.$$

Substituting into our equation

$$0 = f(x(\epsilon)) = \epsilon x(\epsilon)^2 + x(\epsilon) + 1 = \epsilon\left(\frac{y}{\epsilon^\nu}\right)^2 + \left(\frac{y}{\epsilon^\nu}\right) + 1$$

we end up with

$$\epsilon^{1-2\nu}y^2 + \epsilon^{-\nu}y + 1 = 0.$$

In order to see that one root must go to infinity (like $1/\epsilon$) we give a heuristic argument that suggests this should be the case. Let $x_1(\epsilon)$, $x_2(\epsilon)$ be the two roots. Then

$$\epsilon[x - x_1(\epsilon)][x - x_2(\epsilon)] \equiv 0.$$

If we multiply out the terms and recall that for every ϵ this must be $\epsilon x^2 + x + 1 = 0$ we get

$$\epsilon x^2 - [x_1(\epsilon) + x_2(\epsilon)]x + x_1(\epsilon)x_2(\epsilon) = 0$$

or

$$-\epsilon[x_1(\epsilon) + x_2(\epsilon)] = 1$$

and

$$\epsilon x_1(\epsilon)x_2(\epsilon) = 1.$$

Definition 5.3. *We say that $f \sim g$ if and only if $\dfrac{f(\epsilon)}{g(\epsilon)} \to 1$ as $\epsilon \to 0$.*

Now since we expect $x_1(\epsilon) \sim 1$ which implies that $x_2(\epsilon) \sim b/\epsilon$ where b is some constant.

We have just learned that we do not obtain a regular perturbation problem when the number of roots is less for the unperturbed problem than for the perturbed problem. This is not the only way a singular problem can arise from these types of perturbation problems. The next example shows that if the original (unperturbed) problem has multiple roots there can also be a problem.

Example 5.3. Consider the quadratic equation

$$P(\epsilon) = x^2 - 2\epsilon x - \epsilon = 0.$$

For this problem no roots are lost when $\epsilon = 0$, namely, $x = 0$ is a double root. If we try to proceed to obtain an asymptotic expansion for the roots as a regular perturbation expansion

$$x = a_0 + a_1 x + a_2 x^2 + \cdots , \tag{5.2.2}$$

then when we truncate at $N = 2$ we get

$$a_0^2 + (2a_0a_1 - 2a_0 - 1)\epsilon + (a_1^2 + 2a_0a_2 - 2a_1)\epsilon^2 + \mathcal{O}(\epsilon^3) = 0.$$

Which implies

$$a_0^2 = 0, \;\; 2a_0a_1 - 2a_0 - 1 = 0, \;\; a_1^2 + 2a_0a_2 - 2a_1 = 0.$$

This implies that

$$a_0 = 0,$$

and then that $-1 = 0$ which is a contradiction.

Thus no roots of $P(\epsilon)$ have the form (5.2.2). Just as in the previous examples we could apply the quadratic formula to find out what the form of the roots is but this really defeats the purpose of our investigation. In particular for polynomials of degree greater than or equal five there is no formula like the quadratic formula to use.

Let us consider the following line of reasoning. We know that there must be two roots which we denote by $x_1(\epsilon)$ and $x_2(\epsilon)$ which must approach zero as $\epsilon \to 0$. Let us assume that the zeros satisfy

$$x_j(\epsilon) \sim \epsilon^p b_0, \quad p > 0, \;\; \beta_0 \neq 0.$$

Just as in the last example, let us make the change of variables

$$x(\epsilon) = \epsilon^p w(\epsilon), \;\; w(0) \neq 0.$$

Then we get

$$Q(w, \epsilon) \equiv P(\epsilon) = \epsilon^{2p} w^2(\epsilon) - 2\epsilon^{p+1} w(\epsilon) - \epsilon = 0.$$

As ϵ goes to zero the largest of $\{\epsilon^{2p}, \epsilon^{p+1}, \epsilon^1\}$ is the one with the smallest exponent (since a fraction raised to a higher power is smaller). Thus to find the dominant term we must determine the smallest of these exponents.

Note that if $1/2 < p < 1$ then

$$\min\{2p, p + 1, 1\} = 1$$

which gives

$$\epsilon^2 w^2(\epsilon) - 2\epsilon^2 w(\epsilon) - \epsilon = 0.$$

But if we divide by ϵ and let ϵ tend to zero we are left with $-1 = 0$ which is a contradiction.

If, on the other hand, $0 < p < 1/2$ then

$$\min\{2p, p + 1, 1\} = 2p$$

which, on multiplying by ϵ^{-2p}, gives

$$0 \equiv \epsilon^{-2p} Q(w, \epsilon) = w^2(\epsilon) - 2\epsilon^{-p+1} w(\epsilon) - \epsilon^{1-2p} \sim w^2(\epsilon) \sim w^2(0) \;\; \Rightarrow \;\; w(0) = 0$$

which, once again, is a contradiction.

The only other possibility is that $p = 1/2$. In this case we have

$$0 \equiv \epsilon^{-1} Q(w, \epsilon) = w^2(\epsilon) - 2\epsilon^{1/2} w(\epsilon) - 1 \sim w^(0) - 1, \;\; \Rightarrow \;\; w(0) = \pm 1.$$

This could work.

Note that with this substitution the resulting polynomial in w is

$$w^2(\epsilon) - 2\epsilon^{1/2} w(\epsilon) - 1 = 0.$$

Now working with a fractional power of ϵ is not so convenient so let us make one final adjustment and set

$$\beta = \epsilon^{1/2}$$

to obtain the regular perturbation problem

$$w^2 - 2\beta w - 1 = 0,$$

for which we seek an expansion for $\beta \sim 0$ in the form

$$w(\beta) = b_0 + b_1 \beta + b_2 \beta^2 + \cdots + b_N \beta^N + \mathcal{O}(\beta^{N+1}), \quad b_0 \neq 0.$$

Substitution in the polynomial gives

$$\begin{aligned} &b_0^2 - 1 = 0, \\ &2b_0b_1 - 2b_0 = 0, \\ &b_1^2 + 2b_0b_2 - 2b_1 = 0, \\ &\qquad\vdots \end{aligned}$$

which implies

$$\begin{aligned} &b_0 = \pm 1, \\ &b_1 = 1, \\ &b_2 = \pm 1/2, \\ &\qquad\vdots \end{aligned}$$

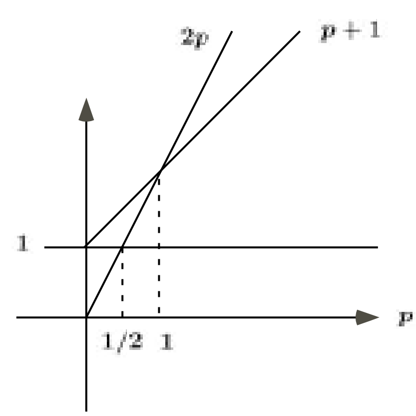
Thus we obtain the expansions

$$x_1(\epsilon) = \epsilon^{1/2} + \epsilon + \frac{1}{2}\epsilon^{3/2} + \mathcal{O}(\epsilon^2),$$

$$x_2(\epsilon) = -\epsilon^{1/2} + \epsilon - \frac{1}{2}\epsilon^{3/2} + \mathcal{O}(\epsilon^2).$$

Immediately following this example we turn to the problem of determining the asymptotic development of a wide class of algebraic equations. This discussion will include a method for determining the value of p as in this example. Since we have a simple example here let us take this opportunity to show graphically how the general method works.

To determine the minimal value for the set $\{\epsilon^{2p}, \epsilon^{p+1}, \epsilon^1\}$ we proceed as follows: In the (p, q) plane, plot the lines $q = 2p$, $q = p + 1$, $q = 1$. These lines will intersect in several places. Namely, two of the lines intersect at $(2/1, 1)$ and two others intersect at $(0, 1)$. Each of these points on intersection determine the asymptotic behavior of one branch of the roots of our equation.



The three lines form two intersections

At this point we present a quite general result for a class of singular perturbation problems. We consider polynomial equations of the form

$$\begin{aligned} P(x,\epsilon) =& (1+b_0\epsilon+c_0\epsilon^2+\cdots)+A_1\epsilon^{\alpha_1}(1+b_1\epsilon+c_1\epsilon^2+\cdots)x \\ &+\cdots+A_n\epsilon^{\alpha_n}(1+b_n\epsilon+c_n\epsilon^2+\cdots)x^n=0, \end{aligned} \tag{5.2.3}$$

where α_i are rational, b_i, c_i, etc are constants and $(1+b_i\epsilon+\cdots)$ are regular asymptotic series, i.e., they have the form

$$a_0+a_1\epsilon+\cdots+a_N\epsilon^N+R_{N+1}(\epsilon), \quad \text{with} \quad R_{N+1}(\epsilon)=\mathcal{O}(\epsilon^{N+1}).$$

We note that with this definition $P(x,\epsilon)$ can have roots that approach zero, a finite number or infinity (see the homework problems).

Much of the material in this section is taken from the Dover book [13] including the following Theorem.

Theorem 5.2. *Each zero $x(\epsilon)$ of (5.2.3) is of the form*

$$x(\epsilon)=\epsilon^p w(\epsilon), \quad w(0)\neq 0 \tag{5.2.4}$$

where $w(\epsilon)$ is a continuous function of ϵ for $\epsilon\sim 0$.

Sketch of Proof. If $x(\epsilon)=\epsilon^p w(\epsilon), \quad w(0)\neq 0$ then $P(\epsilon^p w,\epsilon)$ can be written as

$$P(\epsilon^p w,\epsilon)=Q(w,\epsilon)+\epsilon(b_0+\epsilon^{\alpha_1+p}b_1A_1w+\cdots+\epsilon^{\alpha_n+np}b_nA_nw^n)+\cdots,$$

where

$$Q(w,\epsilon)=1+\epsilon^{\alpha_1+p}A_1w+\cdots\epsilon^{\alpha_n+np}A_nw^n.$$

The main point is that the exponents

$$E=\{0,\alpha_1+p,\cdots,\alpha_n+np\} \tag{5.2.5}$$

determines a set of, so-called, proper values $\{p_1,p_2,\cdots,p_m\}$. These numbers are determined as follows: Draw the graphs of the lines $q=\alpha_j+jp$ in the (p,q) plane. Starting on the right, we note that for p sufficiently large the smallest exponent will be 0. As p decreases, we imagine a vertical line moving with us through the graphs of the various lines, there will be a first point at which (at least) two lines intersect at a point $(p_1,0)$ (here $e_1=0$). At this point one and only one line will have the largest slope n_1. Now continue to the left along the intersection of your vertical line and the line with maximum slope until you encounter the next point of intersection with one of the set of lines. This point is denoted (p_2,e_2). The slopes, at this point, range from a minimum of n_1 to a maximum of n_2. Continue in the same fashion until the last and smallest proper value p_m is reached (i.e., there are no more intersection lines to the left). At least one of the lines that intersect at this last point must have the maximum slope of all the lines which is n. In this way we have generated a set of pairs $\{(p_j,e_j)\}_{j=1}^m$

Now for each j define the polynomials

$$T^{(j)}(w,\epsilon)=\epsilon^{-e_j}P(\epsilon^{p_j}w,\epsilon).$$

Each $T^{(j)}(w,\epsilon)$ can be written as

$$T^{(j)}(w,\epsilon) = T^{(j)}(w) + E^{(j)}(w,\epsilon)$$

where

$$T^{(j)}(w) = A_{n_j}\left(w^{n_j} + \cdots + B_j w^{n_{j-k}}\right), \quad E^{(j)}(w,0) = 0.$$

We note that if the nonzero roots of $T^{(j)}(w)$ are given by x_k for $k = 1, \cdots, n_j$, then $T^{(j)}(w,\epsilon)$ has n_j roots denoted by $x_k(\epsilon)$ satisfying

$$\lim_{\epsilon \to \infty} x_k(\epsilon) = x_k, \quad k = 1, \cdots, n_j.$$

That is, the roots of $T^{(j)}(w,\epsilon)$ approach the roots of $T^{(j)}(w)$ as $\epsilon \to \infty$.

Unfortunately, the non-zero roots of $T^{(j)}(w,\epsilon)$ need not be regular: the α's and the associated proper values and exponents, (p_j, e_j), may be non-integer rational or $T^{(j)}(w)$ may have repeated roots. Thus to obtain regular expansions, new parameters must be introduced. Namely, we introduce a new parameter β by

$$\epsilon = \beta^{q_j} \tag{5.2.6}$$

where

$$q_j = \text{lcd}\{0, \alpha_1 + p_j, \cdots, \alpha_n + np_j\}, \quad \text{lcd means least common denominator}.$$

then let

$$R^{(j)}(w,\beta) = T^{(j)}(w,\beta^{q_j}) = \beta^{-q_j e_j} P(\beta^{q_j p_j} w, \beta^{q_j}).$$

The roots of $T^{(j)}(w,\epsilon)$ are identical to those of $R^{(j)}(w,\beta)$ but the nonzero roots of $R^{(j)}(w,\beta)$ will have a regular expansion in w of the form

$$w(\beta) = b_0 + b_1\beta + \cdots + b_N\beta^N + \mathcal{O}(\beta^{N+1}).$$

SUMMARY:

Every root of (5.2.3) can be expressed in the form (5.2.4). The set of exponents (5.2.5) determines a set of proper values $\{p_1, \cdots, p_m\}$. For each proper value we introduce a new parameter β through (5.2.6) and an associated polynomial $R^{(j)}(w,\beta)$. The simple non-zero roots $R^{(j)}(w,\beta)$ have regular perturbation expansions in β. The total number of non-zero roots of all the $R^{(j)}(w,\beta)$ is n. These yield expansions for each of the roots of (5.2.3).

□

Example 5.4. At this point an example showing how to choose the proper values is probably the best way to see what this means. Consider

$$P(x,\epsilon) = 1 + x^3 + \epsilon^6 x^6 + 2\epsilon^9 x^7 + \epsilon^{12} x^8 + \epsilon^{18} x^9.$$

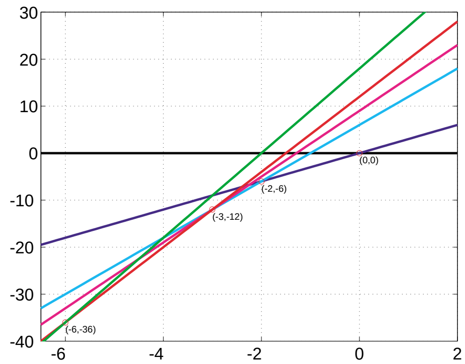
The substitution of $x = \epsilon^p w$ gives

$$P(\epsilon^p w, \epsilon) = 1 + \epsilon^{3p} w^3 + \epsilon^{6+6p} w^6 + 2\epsilon^{9+7p} w^7 + \epsilon^{12+8p} w^8 + \epsilon^{12+8p} w^9,$$

so the set of exponents is

$$E = \{0, 3p, 6 + 6p, 9 + 7p, 12 + 8p, 12 + 8p\}.$$

From the graph we can read off the proper values p_j and the minimal exponents e_j.



Determination of the Proper values

We obtain

1. $(0, 0):\ T^{(1)}(w, \epsilon) = \epsilon^0 P(\epsilon^0 w, \epsilon) = 1 + w^3 + \epsilon^6 w^6 + 2\epsilon^9 w^7 + \epsilon^{12} w^8 + \epsilon^{18} w^9$
2. $(-2, -6):\ T^{(2)}(w, \epsilon) = \epsilon^6 P(\epsilon^{-2} w, \epsilon) = w^3 + w^6 + 2\epsilon w^7 + \epsilon^2 w^8 + \epsilon^6 (1 + w^9)$
3. $(-3, -12):\ T^{(3)}(w, \epsilon) = \epsilon^{12} P(\epsilon^{-3} w, \epsilon) = w^6 + 2w^7 + w^8 + \epsilon^3 (w^3 + w^9) + \epsilon^{12}$
4. $(-6, -36):\ T^{(4)}(w, \epsilon) = \epsilon^{36} P(\epsilon^{-6} w, \epsilon) = w^8 + w^9 + 2\epsilon^3 w^7 + \epsilon^6 w^6 + \epsilon^{18} w + \epsilon^{36}$

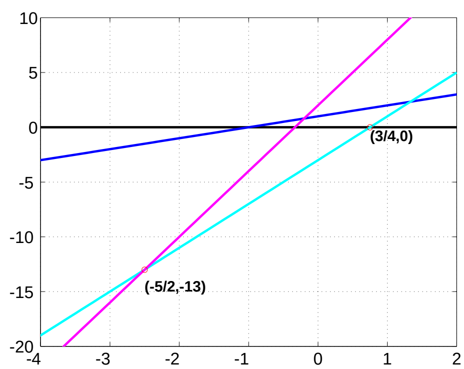
Example 5.5. Let us consider the following problem in some detail

$$P(x, \epsilon) = 1 - \epsilon + \epsilon(2 + 3\epsilon^2)x - \epsilon^{-3}(16 - \epsilon)x^4 + \epsilon^2(4 - \epsilon + \epsilon^3)x^6.$$

1. Set $x = \epsilon^p w$ and determine the exponents $E = \{0, 1 + p, -3 + 4p, 2 + 6p\}$ gathered from

$$P(\epsilon^p, \epsilon) = 1 - \epsilon + \epsilon^{1+p}(2 + 3\epsilon^2)w - \epsilon^{-3+4p}(16 - \epsilon)w^4 + \epsilon^{2+6p}(4 - \epsilon + \epsilon^3)w^6.$$

2. Determine the proper values and the polynomials $T^{(j)}(w, \epsilon)$.



(a) $(3/4, 0)$: $T^{(1)}(w,\epsilon) = 1 - \epsilon + \epsilon^{7/4}(2+3\epsilon^2)w - (16-\epsilon)w^4 + \epsilon^{13/2}(4-\epsilon+\epsilon^3)w^6$.

(b) $(-5/2, -13)$: $T^{(2)}(w,\epsilon) = \epsilon^{13}(1-\epsilon) + \epsilon^{23/2}(2+3\epsilon^2)w - (16-\epsilon)w^4 + (4-\epsilon+\epsilon^3)w^6$.

3. For each j determine q_j, set $\epsilon = \beta^{q_j}$ and compute the polynomials $R^{(j)}(w,\beta)$:

(a) $\epsilon = \beta^4$: $$R^{(1)}(w,\beta) = \beta^{-q_1 e_1} P(\beta^{q_1 p_1} w, \beta^{q_1}) = 1 + \beta^4 + \beta^7(2+3\beta^8)w - (16-\beta^4)w^4 + \beta^{26}(4-\beta^4+\beta^{12})w^6.$$

(b) $\epsilon = \beta^2$: $$R^{(2)}(w,\beta) = \beta^{-q_2 e_2} P(\beta^{q_2 p_2} w, \beta^{q_2}) = \beta^{26}(1-\beta^2) + \beta^{23}(2+3\beta^4)w - (16-\beta^2)w^4 + \beta^{26}(4-\beta^2+\beta^6)w^6.$$

4. Each $R^{(j)}(w,\beta)$ has non-zero roots of the form

$$w(\beta) = b_0 + b_1 w + \cdots + b_N \beta^N + \mathcal{O}(\beta^{N+1}). \tag{5.2.7}$$

Substitute (5.2.7) into $R^{(j)}(w,\beta) \equiv 0$, collect and equate to zero coefficients of like powers of β. Solve, one-by-one, for the unknowns $b_0, b_1, \cdots$.

(a) For $j = 1$ the root will have the form $w(\beta) = b_0 + b_4\beta^4 + \mathcal{O}(\beta^7)$ from part 3 .
We have

$$\begin{aligned} 0 = R^{(1)}(w,\beta) =& 1 - \beta^4 + \mathcal{O}(\beta^7) - (16-\beta^4)(b_0 + b_4\beta^4 + \mathcal{O}(\beta^7))^4 + \mathcal{O}(\beta^7) \\ =& 1 - \beta^4 - (16-\beta^4)(b_0^4 + 4b_0^3 b_4 \beta^4) + \mathcal{O}(\beta^7) \\ =& 1 - \beta^4 - 16b_0^4 - 64b_0^3 b_4\beta^4 + b_0^4\beta^4 + \mathcal{O}(\beta^7). \end{aligned}$$

Setting like powers of β to zero we arrive at

$$1 - 16b_0^4 = 0 \;\; \Rightarrow \;\; b_0 = \left(\frac{1}{16}\right)^{1/4} = \frac{1}{2}e^{(k-1)i\pi/2}, \quad k = 1,2,3,4.$$

and

$$-1 - 64b_0^3 b_4 + b_0^4 = 0, \;\; \Rightarrow \;\; b_4 = \frac{(1-b_0^4)}{-64b_0^3} = \frac{(1-1/16)b_0}{-4} = -\frac{15}{64}b_0$$

where we have multiplied the top and bottom by b_0 and used the fact that $b_0^4 = 1/16$.

(b) For $j = 2$ the root will have the form $w(\beta) = b_0 + b_2\beta^2 + \mathcal{O}(\beta^4)$.
Once again we have We have

$$0 = R^{(2)}(w,\beta) = (4-\beta^2+\beta^6)(b_0+b_4\beta^2+\mathcal{O}(\beta^4))^6 - (16-\beta^2)(b_0+b_4w^2+\mathcal{O}(\beta^4))^4 + \mathcal{O}(\beta^4)$$

$$= (4-\beta^2)(b_0^6+6b_0^5b_2\beta^2+\mathcal{O}(\beta^4)) - (16-\beta^2)(b_0^4+4b_0^3b_4\beta^2)+\mathcal{O}(\beta^4)$$

$$= (4b_0^6-16b_0^4)+(24b_0^5b_2-b_0^6-64b_0^3b_2+b_0^4)\beta^2+\mathcal{O}(\beta^4).$$

From this we conclude that

$$4b_0^6 - 16b_0^4 = 0 \text{ (the nonzero values satisfy) } b_0^2 = 4, \ \Rightarrow \ b_0 = 2(-1)^k, \ k = 5, 6,$$

and

$$0 = 24b_0^5b_2 - b_0^6 - 64b_0^3b_2 + b_0^4 \ \Rightarrow \ (16)(8)b_2b_0 = 48 \ \Rightarrow \ b_2 = \frac{3b_0}{32}.$$

5. Finally we write down the roots $x_j(\epsilon)$ for $j = 1, 2, 3, 4, 5, 6$.

(a) $x_k(\epsilon) = \dfrac{1}{2}e^{(k-1)i\pi/2}\epsilon^{3/4}\left(1 - \left(\dfrac{15}{64}\right)\epsilon + \mathcal{O}(\epsilon^{7/4})\right)$ for $k = 1, 2, 3, 4$.

(b) $x_k(\epsilon) = 2(-1)^k\epsilon^{-5/2}\left(1 + \left(\dfrac{3}{32}\right)\epsilon + \mathcal{O}(\epsilon^2)\right)$ for $k = 5, 6$.

Example 5.6. Consider the equation

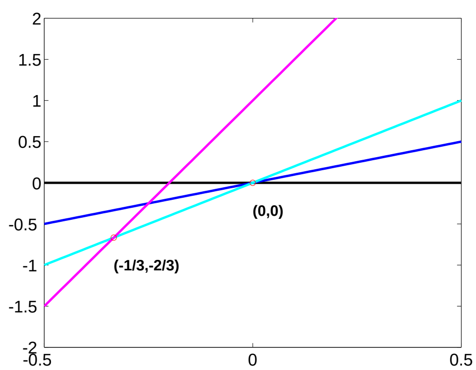
$$P(x,\epsilon) = 1 - 2x + x^2 + \epsilon x^5 = 0.$$

1. For this problem we seek $x = \epsilon^p w$ so we get

$$P(\epsilon^p w,\epsilon) = 1 - 2\epsilon^p w + \epsilon^{2p}w^2 + \epsilon^{1-5p}w^5$$

and

$$E = \{0, p, 2p, 1+5p\}$$



Determination of the Proper values

2. Now for $j = 1, 2$ we try to carry out the procedure described above.

   (a) For $j = 1$ we have $(0, 0)$ and

$$T^{(1)}(w, \epsilon) = \epsilon^0 P(\epsilon^0 w, \epsilon) = 1 - 2w + w^2 + \epsilon w^5.$$

   (b) For $j = 2$ we have $(-1/3, -2/3)$ and

$$T^{(2)}(w, \epsilon) = \epsilon^{2/3} P(\epsilon^{-1/3} w, \epsilon) = \epsilon^{2/3} - 2\epsilon^{1/3} w + w^2 + w^5.$$

3. Determine q_j and set $\epsilon = \beta^{q_j}$.

   (a) For $j = 1$ we run into trouble. $q_1 = \text{lcd}\{0, 1\} = 1$ so $\epsilon = \beta$ and

$$R^{(1)}(w, \beta) = 1 - 2w + w^2 + \beta w^5.$$

   If we try a regular perturbation series $w = b_0 + b_1 w + \cdots$ we arrive at

$$1 - 2\,b_0 - 2\,b_1\,\beta - 2\,b_2\,\beta^2 + \left(b_0 + b_1\,\beta + b_2\,\beta^2\right)^2 + \beta\,\left(b_0 + b_1\,\beta + b_2\,\beta^2\right)^5$$

   and the coefficients of β^0 is $1 - 2\,b_0 + {b_0}^2$ which has a double root 1 so we see that this is not a regular perturbation and we must proceed differently.

   (b) For $j = 2$ we have $q_2 = \text{lcd}\{0, -1/3, -2/3\} = 3$ so $\epsilon = \beta^3$ and

$$R^{(2)}(w, \beta) = \beta^2 - 2\beta w + w^2 + w^5.$$

   At this point we stop and begin again.

Because of the double root must seek

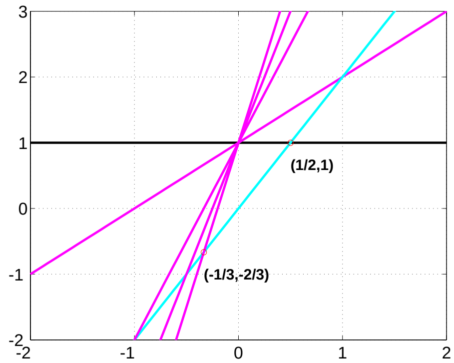
$$w = 1 + \beta^\nu u, \quad u = u_0 + u_1\beta + u_2\beta^2 + \cdots, \; u_0 \neq 0.$$

Thus we obtain

$$\begin{aligned} T^{(1)}(1 + \beta^\nu u, \beta) =& 1 - 2(1 + \beta^\nu u) + (1 + \beta^\nu u)^2 + \beta(1 + \beta^\nu u)^5 \\ =& 1 - 2(1 + \beta^\nu u) + (1 + 2\beta^\nu u + \beta^{2\nu} u^2) + \beta(1 + 5\beta^\nu u + 10\beta^{2\nu} u^2 + \cdots) \\ =& \beta + 5\beta^{\nu+1} u + \beta^{2\nu}(1 + 10\beta) u^2 + 10\beta^{1+3\nu} u^3 + 5\beta^{1+4\nu} u^4 + \beta^{1+5\nu} u^5. \end{aligned}$$

The set of exponents is

$$E = \{1, 2\nu, \nu + 1, 3\nu + 1, 4\nu + 1, 5\nu + 1\}.$$



Determination of the Proper values

There are two proper values $(p_1, e_1) = (1/2, 1)$ and $(p_2, e_2) = (-1/3, -2/3)$

1. For $j = 1$ multiply by β^{-1} and we have
$$1 + 5\beta^{1/2}u + (1 + 10\beta)u^2 + 10\beta^3u^3 + 5\beta^2u^4 + \beta^{5/2}u^5 = 0.$$
At this point we set $\beta = \gamma^2$ to obtain
$$1 + 5\gamma u + (1 + 10\gamma^2)u^2 + 10\gamma^3u^3 + 5\gamma^4u^4 + \gamma^5u^5 = 0.$$
We now substitute $u = u_0 + u_1\gamma + \cdots$ and after some calculations we arrive at
$$1 + u_0^2 = 0, \quad \Rightarrow \quad u_0 = \pm i,$$
$$2u_0u_1 + 5u_0 = 0, \quad \Rightarrow \quad u_1 = -\frac{5}{2},$$
$$2u_0u_2 + 5u_1 + 10u_0^2 + u_1^2 = 0, \quad \Rightarrow \quad u_2 = -\frac{1}{2}\frac{5u_1 + 10u_0^2 + u_1^2}{u_0} = \mp\frac{65}{8}i.$$
We arrive at
   (a) $y_1 = 1 + \epsilon^{1/2}\left(i - \frac{5}{2}\epsilon^{1/2} - \frac{65}{8}i\epsilon^1 + \cdots\right)$,
   (b) $y_1 = 1 + \epsilon^{1/2}\left(-i - \frac{5}{2}\epsilon^{1/2} + \frac{65}{8}i\epsilon^1 + \cdots\right)$,
2. For $j = 2$ we have $\gamma = \beta^{1/3}$ and
$$\gamma^5 + 5\gamma^4u + (1 + 10\gamma^3)u^2 + 10\gamma^2u^3 + 5\gamma u^4 + u^5 = 0.$$
We now substitute $u = u_0 + u_1\gamma + \cdots$ and after some calculations we arrive at
$$u_0^2 + u_0^5 = 0, \text{ (only nonzero roots)} \quad \Rightarrow \quad u_0 = (-1)^{1/3} = \{-1, \frac{1}{2} \pm \frac{1}{2}\sqrt{3}i\},$$
$$2u_0u_1 + 5u_0^4 + 5u_0^4(1 + u_1) = 0, \quad \Rightarrow \quad u_1 = -5\frac{u_0^3}{2 + 5u_0^3} = -\frac{5}{3},$$
$$2u_0u_2 + 5u_1 + 10u_0^2 + u_1^2 = 0, \; \Rightarrow \; u_2 = -\frac{(u_1^2 + 10u_0^3(1 + u_1^2) + 20u_0^3u_1)}{u_0(2 + 5u_0^3)} = -\frac{5}{9u_0}.$$
We arrive at

(a) for $u_0 = -1$: $y_3 = 1 + \epsilon^{-1/3}\left(-1 - \dfrac{5}{3}\epsilon^{1/3} + \dfrac{5}{9}\epsilon^{2/3} + \cdots\right)$,

(b) for $u_0 = \dfrac{1}{2} + \dfrac{1}{2}\sqrt{3}i$: $y_4 = 1 + \epsilon^{-1/3}\left(u_0 - \dfrac{5}{3}\epsilon^{1/3} - \dfrac{5}{9u_0}\epsilon^{2/3} + \cdots\right)$,

(c) for $u_0 = \dfrac{1}{2} + \dfrac{1}{2}\sqrt{3}i$: $y_5 = 1 + \epsilon^{-1/3}\left(u_0 - \dfrac{5}{3}\epsilon^{1/3} - \dfrac{5}{9u_0}\epsilon^{2/3} + \cdots\right)$.

For this example we have set $\epsilon = .001$ and computed the approximate solutions using Maple and the following commands:

```
r :=fsolve(subs(epsilon=.001,1-2*x+x^2+epsilon*x^5),x,complex);
```

This resulted in the answers:

$$r(1) = -10.61837874,$$
$$r(2) = .9975295090 - (.03136955287)i,\ r(3) = .9975295090 + (.03136955287)i,$$
$$r(4) = 4.311659861 - (8.715473330)i, r(5) = 4.311659861 + (8.715473330)i$$

Next we compared these answers with our results and obtained the following errors

1. $y_1 = .9975000000 + (.03136584154)i \;\Rightarrow\; |r(3) - y_1| = .00002974147023,$
2. $y_2 = .9975000000 - (.03136584154)i \;\Rightarrow\; |r(2) - y_2| = .00002974147023,$
3. $y_3 = -10.61111111 \;\Rightarrow\; |r(1) - y_3| = .00726763,$
4. $y_4 = 4.305555556 - (8.708366563)i \;\Rightarrow\; |r(4) - y_4| = .009368493834,$
5. $y_5 = 4.305555556 + (8.708366563)i \;\Rightarrow\; |r(5) - y_5| = .009368493834.$

5.3 Transcendental Equations

This is a much more difficult situation and because of that we cannot expect such a complete answer. One of the main tools that we present will be the *Lagrange Inversion Formula*.

More generally, one situation we often encounter is that we are given

$$f(x, t) = 0 \tag{5.3.1}$$

and we are interested in describing the roots $x = \varphi(t)$ for $t \to \infty$.

For the Lagrange Inversion Formula let us assume that $f(z)$ is analytic in a neighborhood of $z = 0$ in $\mathbb{C}$ and $f(0) \neq 0$. Consider

$$w = \frac{z}{f(z)}. \tag{5.3.2}$$

There exists $a, b > 0$ such that for $|w| < a$, the equation (5.3.2) has one solution in $|z| < b$ and this solution is an analytic function of w:

$$z = \sum_{k=1}^{\infty} c_k w^k, \quad c_k = \frac{1}{k!}\left\{\left(\frac{d}{dz}\right)^{k-1} (f(z))^k\right\}\Big|_{z=0}. \tag{5.3.3}$$

The Lagrange Inversion Formula (5.3.3) is a special case of a very general result on *Implicit Functions* that states:

If $f(z,w)$ is an analytic function of z and w in $|w| < a_1$, $|z| < b_1$ and $f(0,0) = 0$ and $\dfrac{\partial f}{\partial z}(0,0) \neq 0$, then there exists $a \leq a_1$ and $b \leq b_1$ such that for all $|w| < a$ the equation $f(z,w) = 0$ has exactly one solution $z(w)$ in $|z| < b$ and

$$z(w) = \sum_{k=1}^{\infty} c_k w^k.$$

Example 5.7. Consider the equation

$$xe^x = t^{-1}, \quad t \to \infty.$$

Note that as $t \to \infty$ we have $t^{-1} \to 0$

To use the Lagrange Inversion Formula we observe that our equation can be written as

$$ze^z = w \text{ with } x = z, \ w = t^{-1},$$

so we have

$$\frac{z}{f(z)} = w \text{ where } f(z) = e^{-z}.$$

We apply the Lagrange Inversion Formula which implies that there exists $a, b > 0$ such that for $|w| < a$ so that the one and only one solution z with $|z| < b$ is given by

$$z = \sum_{k=1}^{\infty} (-1)^{k-1} k^{k-1} \frac{w^k}{k!}.$$

This series converges for $|w| < e^{-1}$ so for $t \gg e$ (i.e., for t sufficiently large.) Thus we finally arrive at

$$x = \sum_{k=1}^{\infty} (-1)^{k-1} k^{k-1} \frac{t^{-k}}{k!}.$$

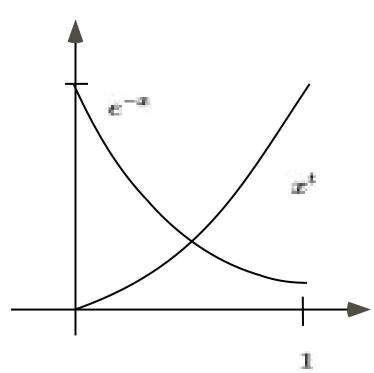
Note that we have used $f(z)^k = e^{-kz}$ so

$$c_k = \frac{1}{k!} \left\{ \left(\frac{d}{dz} \right)^{k-1} (e^{-kz} \right\} \Bigg|_{z=0} = \frac{(-1)^{k-1} k^{k-1} e^{-kz}}{k!} \Bigg|_{z=0} = \frac{(-1)^{k-1} k^{k-1}}{k!}.$$

Example 5.8. Consider the equation

$$x^t = e^{-x}, \quad t \to \infty.$$

For all $|x| < 1$, $x^t \to 0$ as $t \to \infty$ so, as we can see from the figure there is a root in $0 < x < 1$.



Single Intersection

For $x = 1$ we have $1^t = 1$ for all $t > 0$. To use the Lagrange Inversion Formula we set

$$x = 1 + z, \;\; t^{-1} = w$$

so that $x^t = e^{-x}$ becomes

$$\frac{z}{f(z)} = w \;\text{ where }\; f(z) = -\frac{z(1+z)}{\log(1+z)}.$$

This follows from

$$(z+1)^{w^{-1}} = e^{-(z-1)}$$

which implies

$$w^{-1}\log(z+1) = -(z+1),$$

which implies

$$w = -\frac{\log(z+1)}{(z+1)} = \frac{z}{\left(-\dfrac{z(z+1)}{\log(z+1)}\right)}.$$

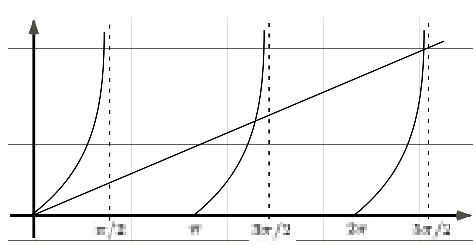
We obtain

$$x = 1 - \frac{1}{t} + \mathcal{O}(t^{-2}).$$

Example 5.9. In determining the eigenvalues of certain Sturm-Liouville problems we often have need of determining the zeros of transcendental equations of the form

$$\tan(x) = x.$$

It is not possible to solve such equations in closed form but by graphing the functions $y = x$ and $y = \tan(x)$ we can see that there are infinitely many solutions and we can use perturbation theory to get a good idea of the asymptotic form of these zeros for $x \gg 0$.



Infinitely many intersections

We note that there are roots in each interval

$$\left(n\pi, \frac{(2n+1)\pi}{2}\right), \quad n = 1, 2, \cdots .$$

Let x_n denote the root in the nth interval. We want to determine the behavior of x_n as $n \to \infty$. Since $\tan(x)$ has a an asymptote at each $x = (2n+1)\pi/2$ and approaches $+\infty$ as $x \to (2n+1)\pi/2$

we see that x_n getting closer and closer to $(2n+1)\pi/2$ for increasing n. So to apply the Lagrange Inversion Formula we set $x = (2n+1)\pi/2 - z$ where z will be a small parameter. Also a little shows that

$$\sin(x) = \sin\left((2n+1)\pi/2 - z\right) = \sin((2n+1)\pi/2)\cos(z),$$
$$\cos(x) = \cos\left((2n+1)\pi/2 - z\right) = \sin((2n+1)\pi/2)\sin(z),$$

so with $w^{-1} = (2n+1)\pi/2$ we have

$$\cos(z) = (w^{-1} - z)\sin(z)$$

which implies

$$\frac{(\cos(z) + z\sin(z))}{\sin(z)} = w^{-1}$$

or

$$w = \frac{z}{f(z)} \quad \text{with} \quad f(z) = \frac{z(\cos(z) + z\sin(z))}{\sin(z)}.$$

We note that since $f(0) = 1 \neq 0$ we can apply the Lagrange Inversion Formula. Here z is given as a series

$$z = w + c_2 w^2 + \cdots$$

where $c_1 = 1$ since $f(0) = 1$. We also note that since f is even so that $(f(z))^k$ is even. Also if k is even then the $(k-1)$st derivative of an even function is odd so that

$$c_k = \frac{1}{k!}\left\{\left(\frac{d}{dz}\right)^{k-1}(f(z)^k)\right\}\bigg|_{z=0} = 0 \quad (\text{ for } k \text{ even. })$$

So we get

$$x_n = \frac{(2n+1)\pi}{2} - c_1\frac{2}{(2n+1)\pi} - c_3\left(\frac{2}{(2n+1)\pi}\right)^3 + \cdots .$$

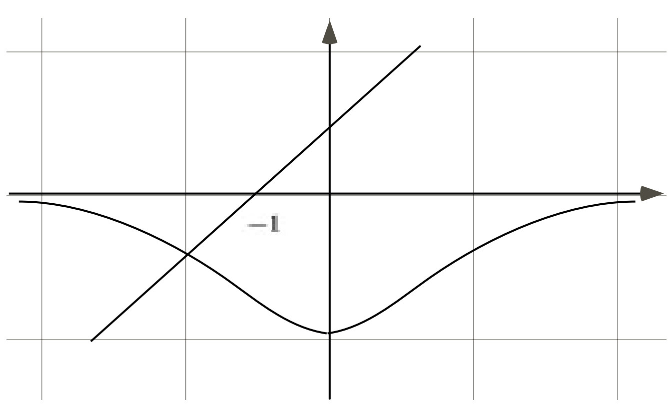
Example 5.10. As was mentioned in the introduction, sometimes it is not possible to obtain an asymptotic expansion in the simple form

$$x = a_0 + a_1\epsilon + a_2\epsilon^2 + \cdots .$$

As an example consider

$$x + 1 + \epsilon\,\text{sech}\left(\frac{x}{\epsilon}\right) = 0, \quad \epsilon \sim 0.$$

By considering the intersections of the graphs of $y = x+1$ and $y = -\epsilon\,\text{sech}\left(\frac{x}{\epsilon}\right)$



If we seek an expansion in the form $x = a_0 + a_1\epsilon + a_2\epsilon^\alpha + \cdots$ then it will follow that $a_0 = -1$ but nothing will allow us to determine α or a_1. The reason is that the behavior of the hyperbolic sech function will not allow it.

In this case we seek an asymptotic form of the solution in the form $x = -1 + \mu(\epsilon)$ under the assumption that $\mu \ll 1$ where part of our job is to determine the behavior in ϵ. We obtain

$$\mu + \epsilon \,\mathrm{sech}(-1/\epsilon + \mu/\epsilon) = 0.$$

Now we use the fact that

$$\mathrm{sech}(-1/\epsilon + \mu/\epsilon) \sim \mathrm{sech}(-1/\epsilon) \sim 2e^{-1/\epsilon}$$

so we have

$$\mu \sim -2\epsilon e^{-1/\epsilon}$$

and

$$x \sim -1 - 2\epsilon e^{-1/\epsilon}, \quad \epsilon \to 0.$$

A problem related to te material in this section is that of finding an asymptotic expansion for the eigenvalues and eigenfunctions of a Sturm-Liouville problem. At this point we give a heuristic treatment of such a problem and note that while these calculations are only formal they can be made more rigorous and give quite good approximations. We will return to this idea when we study the WKB method and perturbation methods for solutions of differential equations.

Let us consider the Sturm-Liouville problem

$$w'' - q(x)w = \lambda w \tag{5.3.4}$$

$$w(0) = 0, \;\; w(1) = 0 \tag{5.3.5}$$

where we assume that $q(x)$ is a smooth real valued function on $[0, 1]$. Let $\lambda = -\rho^2$ and rewrite the problem as

$$w'' + \rho^2 w = q(x)w \tag{5.3.6}$$

$$w(0) = 0, \;\; w(1) = 0 \tag{5.3.7}$$

Applying the method of variation of parameters we can write the solution as an integral equation for w as

$$w(x) = A\cos(\rho x) + B\sin(\rho x) + \frac{1}{\rho}\int_0^x \sin(\rho(x-y))q(y)w(y)\,dy. \tag{5.3.8}$$

Now we note that $w(0) = 0$ implies that $A = 0$ and we can compute that $w'(0) = \rho B$. So we seek

$$w(x) = \sin(\rho x) + \frac{A(x,\rho)}{\rho}.$$

We substitute this expression into the right hand side of (5.3.8) to obtain

$$\begin{aligned}
w(x) =& \sin(\rho x) + \frac{\sin(\rho x)}{\rho}\int_0^x \cos(\rho y)q(y)\left(\sin(\rho y) + \frac{A(y,\rho)}{\rho}\right)\,dy \\
&- \frac{\cos(\rho x)}{\rho}\int_0^x \sin(\rho y)q(y)\left(\sin(\rho y) + \frac{A(y,\rho)}{\rho}\right)\,dy. \\
=& \sin(\rho x) + \frac{\sin(\rho x)}{\rho}\int_0^x \sin(\rho y)\cos(\rho y)q(y)\,dy \\
&- \frac{\cos(\rho x)}{\rho}\int_0^x \sin^2(\rho y)q(y)\,dy + \mathcal{O}(1/\rho^2) \\
=& \sin(\rho x) - \frac{\cos(\rho x)}{2\rho}\int_0^x q(y)\,dy + \frac{\cos(\rho x)}{2\rho}\int_0^x \cos(2\rho y)q(y)\,dy \\
&+ \frac{\sin(\rho x)}{2\rho}\int_0^x \sin(2\rho y)q(y)\,dy + \mathcal{O}(1/\rho^2) \\
=& \sin(\rho x) - \frac{\cos(\rho x)}{2\rho}\int_0^x q(y)\,dy + \frac{\cos(\rho x)}{2\rho}\int_0^x \left(\frac{\sin(2\rho y)}{2\rho}\right)' q(y)\,dy \\
&+ \frac{\sin(\rho x)}{2\rho}\int_0^x \left(\frac{-\cos(2\rho y)}{2\rho}\right)' q(y)\,dy + \mathcal{O}(1/\rho^2) \\
=& \sin(\rho x) - \frac{\cos(\rho x)}{2\rho}\int_0^x q(y)\,dy + \mathcal{O}(1/\rho^2).
\end{aligned}$$

Let

$$G(x) = \frac{1}{2}\int_0^x q(y)\,dy,$$

and we have

$$w(x) = \sin(\rho x) - \frac{\cos(\rho x)}{\rho}G(x) + \mathcal{O}(1/\rho^2).$$

Now from the boundary condition at $x = 1$ we have

$$0 = w(1) = \sin(\rho) - \frac{\cos(\rho)}{\rho}G(1) + \mathcal{O}(1/\rho^2).$$

So for $|\rho| \gg 1$ we need $\sin(\rho) \sim 0$ so $\rho \sim n\pi$.

Let

$$\rho_n = n\pi + a/n + \mathcal{O}(1/n^2)$$

so, with

$$h_1 \equiv G(1) = \frac{1}{2}\int_0^1 q(y)\,dy$$

we must have

$$
\begin{aligned}
0 \sim& \sin(n\pi + a/n + \mathcal{O}(1/n^2)) - \frac{\cos(n\pi + a/n + \mathcal{O}(1/n^2))}{n\pi + a/n + \mathcal{O}(1/n^2)} h_1 + \mathcal{O}(1/n^2) \\
=& \sin(a/n + \mathcal{O}(1/n^2)) - \frac{\cos(n\pi + a/n + \mathcal{O}(1/n^2))}{n\pi + a/n + \mathcal{O}(1/n^2)} h_1 + \mathcal{O}(1/n^2) \\
=& \frac{a}{n} - \frac{h_1}{n\pi + a/n + \mathcal{O}(1/n^2)} + \mathcal{O}(1/n^2).
\end{aligned}
$$

This implies that

$$
a = \frac{h_1}{\pi},
$$

so we have

$$
\rho_n = n\pi + \frac{h_1}{n\pi} + \mathcal{O}(1/n^2).
$$

Now we can substitute this into our approximation for w to obtain

$$
\begin{aligned}
w_n(x) =& \sin\left([n\pi + \frac{h_1}{n\pi} + \mathcal{O}(1/n^2)]x\right) - \frac{\cos([n\pi + \frac{h_1}{n\pi} + \mathcal{O}(1/n^2)]x)}{[n\pi + \frac{h_1}{n\pi} + \mathcal{O}(1/n^2)]} G(x) + \mathcal{O}(1/n^2) \\
=& \sin(n\pi x)\cos\left(\left[\frac{h_1}{n\pi} + \mathcal{O}(1/n^2)\right]x\right) + \cos(n\pi x)\sin\left(\left[\frac{h_1}{n\pi} + \mathcal{O}(1/n^2)\right]x\right) \\
&- \frac{G(x)}{n\pi}\left\{\cos(n\pi x)\cos\left(\left[\frac{h_1}{n\pi} + \mathcal{O}(1/n^2)\right]x\right)\right. \\
&\left. - \sin(n\pi x)\sin\left(\left[\frac{h_1}{n\pi} + \mathcal{O}(1/n^2)\right]x\right)\right\} + \mathcal{O}(1/n^2) \\
=& \sin(n\pi x) + \cos(n\pi x)\left(\frac{h_1 x}{n\pi} - \frac{G(x)}{n\pi}\right) + \mathcal{O}(1/n^2)
\end{aligned}
$$

where we have used

$$
\sin\left(\left[\frac{h_1}{n\pi} + \mathcal{O}(1/n^2)\right]x\right) = \frac{h_1 x}{n\pi} + \mathcal{O}(1/n^2).
$$

So, finally, we have

$$
w_n(x) = \sin(n\pi x) + \frac{\cos(n\pi x)}{n\pi}(h_1 x - G(x)) + \mathcal{O}(1/n^2). \tag{5.3.9}
$$

With a bit more work we can also analyze the more challenging problem Let us consider the Sturm-Liouville problem

$$
w'' - q(x)w = \lambda w \tag{5.3.10}
$$

$$
w'(0) - hw(0) = 0, \quad w'(1) + Hw(1) = 0 \tag{5.3.11}
$$

where we assume that $q(x)$ is a smooth real valued function on $[0,1]$. Let $\lambda = -\rho^2$ and rewrite the problem as

$$w'' + \rho^2 w = q(x)w \tag{5.3.12}$$
$$w'(0) - hw(0) = 0, \tag{5.3.13}$$
$$w'(1) + Hw(1) = 0 \tag{5.3.14}$$

Applying the method of variation of parameters we can write the solution as an integral equation for w as

$$w(x) = A\cos(\rho x) + B\sin(\rho x) + \frac{1}{\rho}\int_0^x \sin(\rho(x-y))q(y)w(y)\,dy. \tag{5.3.15}$$

Once again we recall that a particular solution of $u'' + \rho^2 u = v$ is given by

$$u = \frac{1}{\rho}\int_0^x \sin(x-y)v(y)\,dy.$$

Therefore we can write the general solution of (5.3.12)-(5.3.14) as (5.3.15).

Now from (5.3.13) and the seeking $w(0) = 1$, $w'(0) = h$ we have

$$w(x) = \cos(\rho x) + \frac{h}{\rho}\sin(\rho x) + \frac{1}{\rho}\int_0^x \sin(x-y)w(y)q(y)\,dy.$$

Applying the boundary condition (5.3.14) we obtain

$$\tan(\rho) = \frac{C}{\rho - D} \tag{5.3.16}$$

where

$$C = h + H + \int_0^1 \left[\cos(\rho y) - \frac{H}{\rho}\sin(\rho y)\right] q(y)w(y)\,dy, \tag{5.3.17}$$
$$D = \frac{Hh}{\rho} + \int_0^1 \left[\sin(\rho y) + \frac{H}{\rho}\cos(\rho y)\right] q(y)w(y)\,dy. \tag{5.3.18}$$

At this point we make a couple of assumptions and draw some conclusions.

Remark 5.1. *1. Assume that q is $C^2[0,1]$.*

2. *We assume that $\sup_x |w(x)| \leq M < \infty$ independent of ρ. This is always true but it would require an extra digression to prove it so we simply assume that it is true.*

3. *We note the previous assumption implies that C and D are bounded independent of ρ.*

4. *For large ρ we can assume that $\rho \sim n\pi$*

5. *We see that*

$$w(x) = \cos(\rho x) + \frac{a(x,\rho)}{\rho}$$

with $a(x,\rho)$ bounded in ρ and x.

Under these assumptions, applying integration by parts and simplifying the result for w we arrive at

$$w(x) = (1 + \mathcal{O}(1/\rho^2))\cos(\rho x) + \left(\frac{G(x)}{\rho} + \mathcal{O}(1/\rho^2)\right)\sin(\rho x), \tag{5.3.19}$$

$$G(x) = h + \frac{1}{2}\int_0^x q(y)\,dy. \tag{5.3.20}$$

Now we repeat this same procedure for the expressions for C and D to obtain

$$C = h + H + h_1 + \mathcal{O}(1/p), \quad h_1 = \frac{1}{2}\int_0^1 q(y)\,dy, \tag{5.3.21}$$

$$D = \mathcal{O}(1/\rho). \tag{5.3.22}$$

From this we obtain

$$\tan(\rho) = \frac{h + H + h_1 + \mathcal{O}(1/p)}{\rho + \mathcal{O}(1/\rho)} \equiv \frac{b}{\rho} + \mathcal{O}(1/\rho), \tag{5.3.23}$$

$$b = h + H + h_1. \tag{5.3.24}$$

Writing

$$\rho_n = n\pi + \frac{a}{n} + \mathcal{O}(1/n^2)$$

in the above we find that

$$a = \frac{b}{\pi}$$

so that

$$\rho_n = n\pi + \frac{b}{n\pi} + \mathcal{O}(1/n^2). \tag{5.3.25}$$

We now substitute (5.3.25) into (5.3.19) to obtain

$$w_n(x) = \cos(n\pi x) + \left[\frac{G(x) - bx}{n\pi}\right]\sin(n\pi x) + \mathcal{O}(1/n^2), \tag{5.3.26}$$

where ρ_n is given in (5.3.25) with $b = h + H + h_1$

$$h_1 = \frac{1}{2}\int_0^1 q(y)\,dy, \quad G(x) = h + \frac{1}{2}\int_0^x q(y)\,dy.$$

As a final note, we usually are interested in normalized eigenfunctions s we would like to compute the norm of our asymptotic eigenfunction so we could take the square root of this and divide to obtain a asymptotic formula for normalized eigenfunction.

Let us set

$$g(x) = h + \frac{1}{2}\int_0^x q(y)\,dy - bx$$

Then we have

$$\int_0^1 w_n^2(x)\,dx = \int_0^1 \left[\cos(n\pi x) + \frac{g(x)}{n\pi}\sin(n\pi x) + \mathcal{O}(1/n^2)\right]^2\,dx$$

$$= \int_0^1 \left[\cos^2(n\pi x) + 2\frac{g(x)}{n\pi}\sin(n\pi x)\cos(n\pi x) + \frac{g(x)^2}{n^2\pi^2}\sin^2(n\pi x)\right]\,dx + \mathcal{O}(1/n^2)$$

$$= \int_0^1 \left[\cos^2(n\pi x) + \frac{g(x)\sin(2n\pi x)}{n\pi}\right]\,dx + \mathcal{O}(1/n^2)$$

$$= \frac{1}{2} + \int_0^1 g(x)\left(\frac{-\cos(2n\pi x)}{n\pi}\right)'\,dx + \mathcal{O}(1/n^2)$$

$$= \frac{1}{2} + \frac{1}{n\pi}\left\{\frac{-g(x)\cos((2n\pi x)}{2n\pi} + \int_0^1 \frac{g'(x)\cos((2n\pi x)}{2n\pi}\,dx\right\} + \mathcal{O}(1/n^2)$$

$$= \frac{1}{2} + \frac{1}{n\pi}\left\{\frac{\left(-h - 1/2\int_0^1 q(y)\,dy\right)}{2n\pi}\right\} + \mathcal{O}(1/n^2)$$

$$= \frac{1}{2} + \frac{1}{(n\pi)^2}\left(h + 1/2\int_0^1 q(y)\,dy\right) + \mathcal{O}(1/n^2)$$

$$= \frac{1}{2} + \mathcal{O}(1/n^2)$$

which implies

$$\|w_n\| = \left(\int_0^1 w_n^2(x)\,dx\right)^{1/2} = \frac{1}{\sqrt{2}} + \mathcal{O}(1/n). \tag{5.3.27}$$

Example 5.11. As an example consider the Sturm-Liouville problem

$$w'' - xw = \lambda w \tag{5.3.28}$$

$$w(0) = 0, \;\; w(1) = 0 \tag{5.3.29}$$

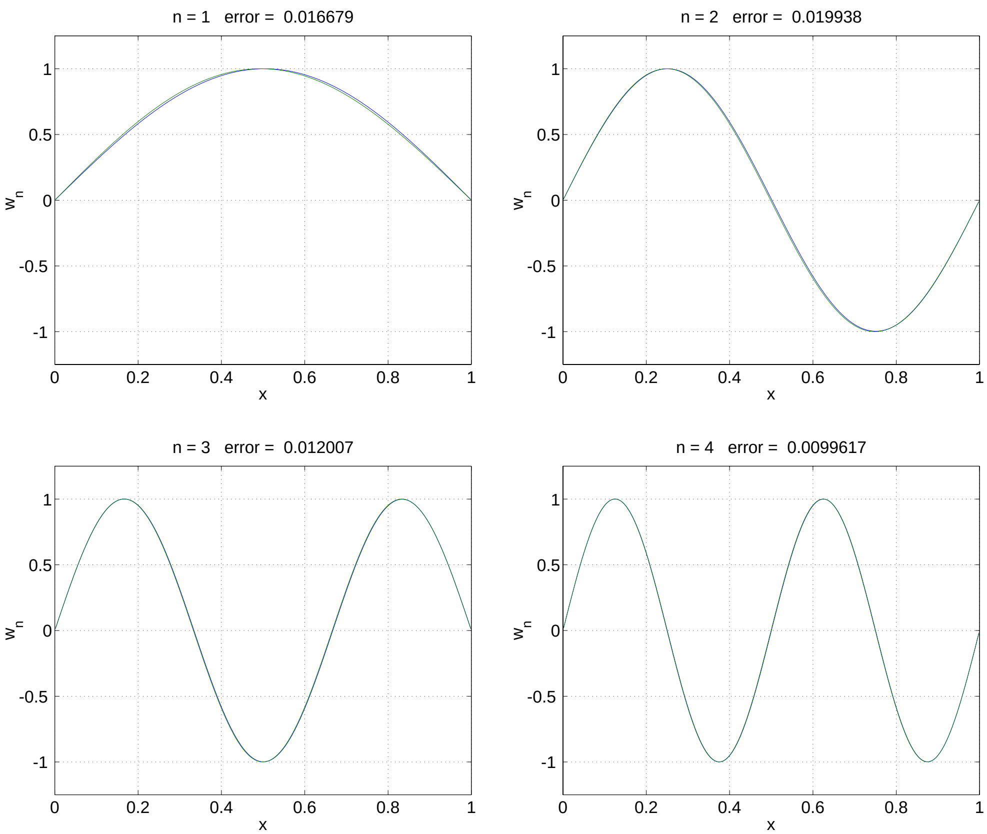
In this case we have $q(x) = x$ and

$$G(x) = \frac{1}{2}\int_0^x t\,dt = \frac{x^2}{4}, \quad h_1 = G(1) = \frac{1}{4},$$

and

$$\rho_n = n\pi + \frac{1}{4n\pi}\mathcal{O}(\left(\frac{1}{n^2}\right),$$

$$w_n(x) = \sin(n\pi x) + \frac{\cos(n\pi x)}{n\pi}\left(\frac{x}{4} - \frac{x^2}{4}\right) + \mathcal{O}\left(\frac{1}{n^2}\right).$$



Eigenfunctions

1. Find the first three terms in the asymptotic expansion for the roots of $x^3 - 5x^2 + 4x + \epsilon = 0$. Note that $x^3 - 5x^2 + 4x = 0$ has roots $x = 0$, $x = 1$ and $x = 4$.

2. Use the quadratic formula to obtain the roots of $x^2 - 2\epsilon x - \epsilon = 0$ (see Example 5.3) and use it to verify our findings in the notes. Namely that the two perturbed roots have expansions

$$x_1(\epsilon) = \epsilon^{1/2} + \epsilon + \frac{1}{2}\epsilon^{3/2} + \mathcal{O}(\epsilon^2),$$

$$x_2(\epsilon) = -\epsilon^{1/2} + \epsilon - \frac{1}{2}\epsilon^{3/2} + \mathcal{O}(\epsilon^2).$$

3. Find the first two terms in the expansion of the roots of $x^3 - \epsilon x^2 - \epsilon^2 = 0$.

4. Show that the equation (which is in the form (5.2.3) from the notes)

$$P(x, \epsilon) = 1 + \epsilon^{-1}x + \epsilon^{-1}x^2 + x^3 = 0$$

has roots that approach zero, a finite number and infinity.

5. For small ϵ, find the first two terms in the expansion of each of the roots and compare with the approximate answers obtained from Maple with $\epsilon = .01$

$$P(x, \epsilon) = x^3 - (3 + \epsilon)x - 2 + \epsilon = 0$$

6. For small ϵ, find the first two terms in the expansion of each of the roots and compare with the approximate answers obtained from Maple with $\epsilon = .01$

$$P(x,\epsilon) = \epsilon x^4 - x^3 + 3x - 2 = 0.$$

7. Sketch the graph of $f_\epsilon(x) = x^2 + e^{\epsilon x}$ for $x \in \mathbb{R}$ and a small positive ϵ. Use a little calculus to convince yourself that the graph is correct. From the graph note that the equation

$$x^2 + e^{\epsilon x} = 5$$

has two roots for small postive ϵ and find the first two terms in an asymptotic expansion of these roots.

8. Determine a two term expansion for the large roots of $x\tan(x) = 1$

5.4 Evaluation of Integrals

Consider the differential equation

$$y' + y = \frac{1}{x}.$$

The function e^x is an integrating factor which reduces the above equation to

$$\frac{d}{dx}(ye^x) = \frac{e^x}{x}$$

which, after integration, gives

$$y = e^{-x}\int_{x_0}^{x} \frac{e^\tau}{\tau}\,d\tau + ce^{-x}$$

where x_0 is arbitrary and c is a constant. If, for example, we impose the initial condition $y(1) = a$ then

$$a = \int_{x_0}^{1} \frac{e^\tau}{\tau}\,d\tau + c$$

which implies

$$c = ae - \int_{x_0}^{1} \frac{e^\tau}{\tau}\,d\tau$$

which , in turn implies,

$$y = e^{-x}\left(ae + \int_{1}^{x} \frac{e^\tau}{\tau}\,d\tau\right)$$

or

$$y(x) = ae^{1-x} + e^{-x}\int_{1}^{x} \frac{e^\tau}{\tau}\,d\tau.$$

This seems great, we have solved the initial value problem. But wait! We cannot evaluate the integral since we cannot find an anti-derivative for

$$\frac{e^\tau}{\tau}.$$

This is but one example of a situation in which we would like to obtain approximate values for integrals. In most of our examples the integrals will depend on a parameter ϵ. The main techniques in this subject are

1. *Expansion of Integrands*
2. *Integration by Parts*
3. *Laplace's Method*
4. *Method of Stationary Phase*

By way of several examples I hope to introduce you to some of the most elementary concepts in this area.

5.4.1 Expansion of Integrands

First let us consider the method of *Expansion of Integrands*.

Example 5.12. Consider the integral

$$I(\epsilon) = \int_0^1 \sin(\epsilon x^2)\, dx. \tag{5.4.1}$$

If we *expansion the Integrand* in a power series

$$\sin(\epsilon x^2) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(\epsilon x^2)^{2n-1}}{(2n-1)!} = \epsilon x^2 - \frac{1}{6}\epsilon^3 x^6 + \frac{1}{120}\epsilon^5 x^{10} + \mathcal{O}(\epsilon^7).$$

Note, by the way, that this series coverges uniformly and absolutely for all ϵx^2 since, using the ratio test, we have

$$\lim_{n\to\infty} \frac{n\text{th term}}{(n-1)\text{st term}} = \lim_{n\to\infty} \frac{(-1)^{(n+1)}(\epsilon x^2)^{(2n-1)}(2n-3)!}{(2n-1)!(-1)^n(\epsilon x^2)^{(2n-3)}} = \lim_{n\to\infty} \frac{-(\epsilon x^2)^2}{(2n-1)(2n-2)} = 0.$$

Thus we can substitute the series into the integral, interchange the sum and integral and integrate each term to get

$$\begin{aligned} I(\epsilon) &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\epsilon^{2n-1}}{(2n-1)!} \int_0^1 x^{4n-2}\, dx \\ &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\epsilon^{2n-1}}{(2n-1)!(4n-1)} \\ &= \frac{1}{3}\epsilon - \frac{1}{42}\epsilon^3 + \frac{1}{1320}\epsilon^5 + \mathcal{O}(\epsilon^7). \end{aligned} \tag{5.4.2}$$

Example 5.13. Consider the integral

$$I(x) = \int_0^x t^{-3/4} e^{-t}\, dt \quad \text{for small} \quad x. \tag{5.4.3}$$

Once again we expand (part of) the integrand as

$$e^{-t} = \sum_{n=0}^{\infty} \frac{(-1)^n t^n}{n!} = 1 - t + \frac{1}{2}t^2 - \frac{1}{6}t^3 + \frac{1}{24}t^4 + \mathcal{O}(t^5).$$

By the ratio test we see that the series converges for all t since we have

$$\lim_{n\to\infty} \frac{n\text{th term}}{(n-1)\text{st term}} = \lim_{n\to\infty} \frac{(-1)^n t^n (n-1)!}{n!(-1)^{n-1} t^{n-1}} = \lim_{n\to\infty} \frac{-t}{n} = 0.$$

Thus once again we substitute the series into the integral to get

$$\begin{aligned} I(x) =& \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \int_0^x t^{n-3/4}\, dt \\ =& \sum_{n=1}^{\infty} \frac{(-1)^n x^{n+1/4}}{n!(n+1/4)} \\ =& 4x^{1/4} - \frac{4}{5}x^{5/4} + \frac{2}{9}x^{9/4} - \frac{2}{39}x^{13/4} + \mathcal{O}(x^{17/4}). \end{aligned} \tag{5.4.4}$$

5.4.2 Integration by Parts

Sometimes expanding the integrand in a power series is not appropriate. Sometimes a useful alternate is the method of *Integration by Parts*.

Example 5.14. Consider the integral

$$I(x) = \int_x^{\infty} \frac{e^{-t}}{t^2}\, dt \quad \text{for large} \quad x. \tag{5.4.5}$$

Rather than try to expand in a series we use integration by parts

$$\int_a^b f'(x)g(x)\, dx = fg\Big|_a^b - \int_a^b f(x)g'(x)\, dx.$$

$$\begin{aligned}
I(x) &= \int_x^\infty \left(-e^{-t}\right)' t^{-2}\, dt \\
&= \left(-e^{-t}\right) t^{-2}\Big|_x^\infty - (-2)\int_x^\infty \left(-e^{-t}\right)\left(t^{-3}\right)\, dt \\
&= \frac{e^{-x}}{x^2} - 2!\int_x^\infty \frac{e^{-t}}{t^3}\, dt \\
&\quad\vdots \\
&= \frac{e^{-x}}{x^2} - \frac{2!e^{-x}}{x^3} + \frac{3!e^{-x}}{x^4} + \cdots + \frac{(-1)^{n-1}n!e^{-x}}{x^{n+1}} \\
&\quad + (-1)^n(n+1)!\int_x^\infty \frac{e^{-t}}{t^{n+2}}\, dt.
\end{aligned}$$

For $x \le t < \infty$ we have $(t/x) \ge 1$ which implies $(t/x)^{n+2} \ge 1$ or $t^{n+2} \ge x^{n+2}$. Thus we have

$$\frac{1}{t^{n+2}} \le \frac{1}{x^{n+2}}$$

and we can write

$$\int_x^\infty \frac{e^{-t}}{t^{n+2}}\, dt \le \frac{1}{x^{n+2}}\int_x^\infty e^{-t}\, dt = \frac{e^{-x}}{x^{n+2}}.$$

Thus we have an asymptotic expansion

$$I(x) = e^{-x}\sum_{n=1}^{N} \frac{(-1)^{n-1}n!}{x^{n+1}} + e^{-x}\mathcal{O}\left(\frac{1}{x^{N+2}}\right).$$

But we note that the infinite series diverges:

$$\lim_{n\to\infty} \frac{n\text{th term}}{(n-1)\text{st term}} = \lim_{n\to\infty} \frac{(-1)^{n-1}n!x^n}{(-1)^{n-2}(n-1)!x^{n+1}} = \lim_{n\to\infty} \frac{-n}{x} = -\infty.$$

On the other hand, for any fixed N we can make the error small by taking x large.

Example 5.15. Consider the Laplace transform integral

$$I(x) = \int_0^\infty e^{-xt} f(t)\, dt \quad \text{for large} \quad x. \tag{5.4.6}$$

Here we assume that f is an analytic function and that the integral exists.

$$
\begin{aligned}
I(x) &= \int_0^\infty \left(\frac{e^{-xt}}{-x}\right)' f(t)\,dt \\
&= \left(\frac{e^{-xt}}{-x}\right) f(t)\Big|_0^\infty - \int_0^\infty \left(\frac{e^{-xt}}{-x}\right) f'(t)\,dt \\
&= \frac{f(0)}{x} + \frac{1}{x}\int_0^\infty e^{-xt} f'(t)\,dt \\
&\quad \vdots \\
&= \frac{f(0)}{x} + \frac{f'(0)}{x^2} + \cdots + \frac{f^{(n)}(0)}{x^{n+1}} + \frac{1}{x^{n+1}}\int_0^\infty e^{-xt} f^{(n+1)}(t)\,dt.
\end{aligned}
$$

If we assume that

$$\sup_{0\le t<\infty} |f^{(n+1)}(t)| \le M$$

then

$$
\begin{aligned}
&\left|\frac{1}{x^{n+1}}\int_0^\infty e^{-xt} f^{(n+1)}(t)\,dt\right| \\
\le&\frac{M}{x^{n+1}}\int_0^\infty e^{-xt}\,dt \\
=&\frac{M}{x^{n+1}}\left(\frac{e^{-xt}}{-x}\right)\Big|_{t=0}^{t=\infty} \\
=&\frac{M}{x^{n+2}}.
\end{aligned}
$$

So we have

$$I(x) = \sum_{n=0}^{N} \frac{f^{(n)}(0)}{x^{n+1}} + \mathcal{O}\left(\frac{1}{x^{(N+2)}}\right).$$

Example 5.16. Consider the Fourier transform of a function $f \in C^\infty(\mathbb{R}^+)$ and $f^{(k)} \in L^1(\mathbb{R}^+)$ for all k

$$I(\alpha) = \int_0^\infty e^{i\alpha t} f(t)\,dt \quad \text{for large} \quad \alpha > 0. \tag{5.4.7}$$

Here $f \in C^\infty(\mathbb{R}^+)$ means that f is infinitely differentialbe and $f^{(k)} \in L^1(\mathbb{R}^+)$ means that the integral

$$\int_0^\infty |f^{(k)}(t)|\,dt < \infty.$$

Thus we can apply integration by parts as often as we like and, just as in the last example, we obtain

$$I(\alpha) = \sum_{n=0}^{N} \frac{f^{(n)}(0)}{(-i\alpha)^{n+1}} + \mathcal{O}\left(\frac{1}{\alpha^{(N+2)}}\right).$$

In this case the estimate of the error term

$$E(\alpha) = \left| \frac{1}{(-i\alpha)^{N+1}} \int_0^\infty e^{i\alpha t} f^{(n+1)}(t)\, dt \right|$$

proceeds as follows

$$\begin{aligned} E(\alpha) =& \left| \frac{1}{(-i\alpha)^{N+1}} \left[\frac{e^{i\alpha t} f^{(N+1)}(t)}{(i\alpha)} \bigg|_0^\infty - \int_0^\infty \frac{e^{i\alpha t} f^{(N+2)}(t)}{(i\alpha)} \right] \right| \\ \leq& \frac{1}{\alpha^{(N+2)}} \left[\left| f^{(N+1)}(0) \right| + \int_0^\infty |f^{(N+2)}(t)|\, dt \right] \\ \equiv& \frac{M}{\alpha^{(N+2)}}. \end{aligned}$$

The method of integration by parts, while quite appealing, is not very flexible. it can only produce asymptotic series in a very special form, e.g. for

$$I(x) = \int_a^b f(t) e^{x\varphi(t)}\, dt$$

only an expressing of the form

$$I(x) = e^{x\varphi(b)} \sum_{n=1}^{\infty} A_n x^{-n}, \quad x \to \infty$$

(i.e., powers of $1/x$). Attempts to use integration by parts will break down when you arrive at a step with a term that doesn't exist, e.g., integration by parts is not going to work when $\varphi'(t)$ has a zero on $[a, b]$. This would be true for example for

$$\int_0^\infty e^{-xt^2}\, dt.$$

In this case $\varphi(t) = t^2$, $\varphi'(t) = 2t$, and $\varphi'(0) = 0$. If we try to use integration by parts then

$$\begin{aligned} I(x) = \int_0^\infty e^{-xt^t}\, dt &= \int_0^\infty \frac{1}{-2xt} \left(-2xte^{-xt^2}\right)\, dt \\ &= \int_0^\infty \frac{1}{-2xt} \left(e^{-xt^2}\right)'\, dt \\ &= \left(\frac{e^{-xt^2}}{-2xt}\right) \bigg|_0^\infty - \int_0^\infty \left(\frac{1}{-2xt^2}\right) e^{-xt^2}\, dt. \end{aligned}$$

Note that these terms don't exist.

5.4.3 Laplace's Method

Consider integrals of the form

$$I(x) = \int_a^b e^{xh(t)} f(t)\, dt \tag{5.4.8}$$

where $f(t)$ is real, the integral exists and x is large positive. According to Laplace's method only the immediate neighborhood of the maximum of $h(t)$ on $[a, b]$ contribute to the integral for x large. If there are several places where the maximum occurs thenthe expansion will have contributions from each of these.

Assumption 5.1. *We assume that $h(\cdot)$ is a continuous function with a maximum on $[a, b]$ at $t = c$ and $f(c) \neq 0$ and f is continuous and real valued.*

The result is that the integral (5.4.8) only depends (asymptotically) only on t near c, i.e., If $I(x, \epsilon)$ denotes the integral

$$I(x, \epsilon) = \int_a^{a+\epsilon} f(t) e^{xh(t)}\, dt, \quad c = a.$$

$$I(x, \epsilon) = \int_{c-\epsilon}^{c+\epsilon} f(t) e^{xh(t)}\, dt, \quad a < c < b,$$

$$I(x, \epsilon) = \int_{b-\epsilon}^{b} f(t) e^{xh(t)}\, dt, \quad c = b.$$

In order for this to work the following are critical:

1. The expansion of $I(x, \epsilon)$ does not depend on ϵ;
2. The expansion of $I(x, \epsilon)$ is identical to the full expansion of $I(x)$.

It turns out that 1 and 2 are both true. If, for example, $a < c < b$ then

$$\left| \int_a^{c-\epsilon} f(t) e^{xh(t)}\, dt \right| + \left| \int_{c+\epsilon}^{b} f(t) e^{xh(t)}\, dt \right|$$

is exponentially small with respect to $I(x)$ as $x \to \infty$. This follows because for all $t \in [a, c - \epsilon]$ and $t \in [c + \epsilon, b]$, the expression $e^{xh(t)}$ is exponentially smaller than $e^{xh(c)}$ as $x \to \infty$. To show that $I(x) - I(x, \epsilon)$ is exponentially small as $x \to \infty$ you can use integration by parts. It is helpful to replace $I(x)$ by $I(x, \epsilon)$ because for $\epsilon > 0$ may be chosen so small that it is valid to replace $f(t)$ and $h(t)$ by their Taylor Series expansions at $t = c$.

Rather than attempt a general proof we will consider a series of examples.

Example 5.17. Consider the integral

$$I(x) = \int_0^{10} \frac{e^{-xt}}{(t+1)}\, dt. \tag{5.4.9}$$

In this case $h(t) = -t$ which on $[0, 10]$ has a max at $t = 0$ and $h(0) = 0$. We will use Laplaces Method to find the first term in the asymptotic expansion of

$$I(x, \epsilon) = \int_0^{\epsilon} (1+t)^{-1} e^{-xt}\, dt.$$

If we exapnd $(1+t)^{-1}$ is a Taylor series about $t = 0$ we can write

$$(1+t)^{-1} = 1 + \mathcal{O}(t).$$

Then we have

$$I(x, \epsilon) = \int_0^{\epsilon} (1+t)^{-1} e^{-xt}\, dt = \int_0^{\epsilon} (1 + \mathcal{O}(t))\, e^{-xt}\, dt \sim \frac{(1 - e^{-\epsilon x})}{x}, \quad x \to \infty$$

and

$$e^{-x\epsilon} \ll 1 \quad x \to \infty \quad \text{for any} \quad \epsilon > 0$$

so

$$I(x) = \int_0^{10} (1+t)^{-1} e^{-xt}\, dt \sim \frac{1}{x}, \quad x \to \infty.$$

Laplace Method Steps

1. Replace $I(X)$ by $I(x, \epsilon)$.
2. Expand the functions $f(t)$ and $h(t)$ in series which are valid near the location of the max of $h(t)$. Thus we obtain $I(x, \epsilon)$ as a sum of integrals.
3. Extend each of these integrals to $\displaystyle\int_0^{\infty}$, i.e., replace the upper limit ϵ by ∞.

Let us reexamine Example 5.17.

Example 5.18. Once again we consider the integral (5.4.9)

$$I(x) = \int_0^{10} \frac{e^{-xt}}{(t+1)}\, dt.$$

If we try to expand $(1+t)^{-1}$ in powers of t using the binomial formula we get

$$(1+t)^{-1} = 1 - t + t^2 - t^3 + \cdots .$$

But by the ratio test

$$\lim_{n\to\infty} \frac{n\text{th}}{(n-1)\text{st}} = \lim_{n\to\infty} \frac{(-1)^n t^n}{(-1)^{n-1} t^{n-1}} = \lim_{n\to\infty} (-t) = -t$$

so the series converges only for $|t| < 1$ and we cannot integrate from 0 to 10. So let us break up the integral to get

$$I(x) = \int_0^\delta \frac{e^{-xt}}{(t+1)}\,dt + \int_\delta^{10} \frac{e^{-xt}}{(t+1)}\,dt = I_1(x,\delta) + I_2(x,\delta).$$

We shall show that $I_2(x,\delta)$ is exponentially small for large x and so gives no significant contribution to the value of $I(x)$. Not that for all

$$t > 0, \quad \frac{1}{(1+t)} < 1$$

so

$$I_2(x,\delta) = \int_\delta^{10} \frac{e^{-xt}}{(t+1)}\,dt < \int_\delta^{10} e^{-xt}\,dt = -\frac{1}{x}\left(e^{-10x} - e^{-\delta x}\right).$$

The last term goes to zero exponentially as $x \to \infty$. So we have

$$I(x) = \int_0^\delta \frac{e^{-xt}}{(t+1)}\,dt + (\text{exponentially small term for large } x).$$

This is true for all $\delta > 0$ so we conclue that the value of $I(x)$ only depends on the immediate neighborhood of $t = 0$ for large x.

Note that

$$\sup_{t\geq 0}\left(\frac{1}{(1+t)}\right) = 1$$

and this sup occurs at $t = 0$.

Now for $\delta < 1$ we can expand $(1+t)^{-1}$ in a convergent power series in t and $I_1(x,\delta)$ can be written as

$$\begin{aligned} I_1(x,\delta) &= \int_0^\delta e^{-xt}\left(\sum_{n=0}^\infty (-1)^n t^n\right)\,dt \\ &= \sum_{n=0}^\infty (-1)^n \left(\int_0^\delta e^{-xt} t^n\,dt\right). \end{aligned} \tag{5.4.10}$$

Let $\tau = xt$ so that $d\tau = xdt$ and we get

$$\int_0^\delta e^{-xt} t^n\,dt = \frac{1}{x^{n+1}} \int_0^{\delta x} \tau^n e^{-\tau}\,d\tau.$$

Now we can us repeated integration by parts to get further information.

Note that

$$\begin{aligned}\int_0^{\delta x} \tau^n e^{-\tau}\, d\tau =& \int_0^{\delta x} \tau^n \left(\frac{e^{-\tau}}{-1}\right)' d\tau \\ =& \left(\frac{e^{-\tau}\,\tau^n}{-1}\right)\Bigg|_0^{\delta x} + n\int_0^{\delta x} \tau^{(n-1)} e^{-\tau}\, d\tau \\ =& - e^{-\delta x}(\delta x)^n - ne^{-\delta x}(\delta x)^{n-1} + n(n-1)\int_0^{\delta x} e^{\tau}\tau^{(n-2)}\, d\tau \\ &\vdots \\ =& - e^{-\delta x}(\delta x)^n - ne^{-\delta x}(\delta x)^{n-1} - \cdots - n(n-1)\cdots 3\int_0^{\delta x} \tau^2 e^{-\tau}\, d\tau \\ =& - e^{-\delta x}\left[(\delta x)^n + n(\delta x)^{n-1} + \cdots + n(n-1)\cdots 4(\delta x)^3\right] \\ &+ \frac{n!}{2}\int_0^{\delta x} e^{-\tau}\tau^2\, d\tau.\end{aligned} \tag{5.4.11}$$

Now we notice that by integration by parts once again

$$\begin{aligned}\frac{n!}{2}\int_0^{\delta x} e^{-\tau}\tau^2\, d\tau =& \frac{n!}{2}\left[-e^{-\tau}\tau^2\Big|_0^{\delta x} + 2\int_0^{\delta x} e^{-\tau}\tau\, d\tau\right] \\ =& -\frac{n!}{2}e^{-\delta x}(\delta x)^2 + n!\int_0^{\delta x} e^{-\tau}\tau\, d\tau \\ =& -\frac{n!}{2}e^{-\delta x}(\delta x)^2 - n!e^{-\delta x}(\delta x) + n!\int_0^{\delta x} e^{-\tau}\, d\tau \\ =& -\frac{n!}{2}e^{-\delta x}(\delta x)^2 - n!e^{-\delta x}(\delta x) - n!e^{-\delta x} + n!\end{aligned} \tag{5.4.12}$$

Combining (5.4.12) with (5.4.11) we arrive at

$$\begin{aligned}\int_0^{\delta} t^n e^{-xt}\, dt =& \frac{1}{x^{n+1}}\int_0^{\delta x} \tau^n e^{-\tau}\, d\tau \\ =& \frac{n!}{x^{(n+1)}} - \frac{e^{-\delta x}}{x^{(n+1)}}\left[(\delta x)^n + n(\delta x)^{n-1} + \cdots + \frac{n!}{2}(\delta x)^2 + n!(\delta x) + n!\right] \\ =& \frac{n!}{x^{(n+1)}} - e^{-\delta x}\left[\frac{\delta^n}{x} + \frac{n\delta^{n-1}}{x^2}\right. \\ &\left. + \frac{n(n-1)\delta^{n-2}}{x^3} + \cdots + \frac{n!\delta^2}{2x^{(n-1)}} + \frac{n!\delta}{x^n} + \frac{n!}{x^{(n+1)}}\right].\end{aligned} \tag{5.4.13}$$

Now as $x \to \infty$, the term $e^{-\delta x} \to 0$ faster than any power of $\left(\frac{1}{x}\right)$, so we have

$$\int_0^{\delta} t^n e^{-xt}\, dt = \frac{n!}{x^{(n+1)}} \ + \ \text{exponentially small terms}. \tag{5.4.14}$$

Remark 5.2. 1. So all this work has taught us that, up to the order of exponentially small terms, the integral $I(x, \delta)$ satisfies (5.4.14) which is independent of $\delta \in \mathbb{R}$.

2. With this in mind we follow our next step in the Laplace method. That is, instead of the messy calculations above let us replace the upper limit of integration δ by ∞ and from the theory of Laplace transforms we have

$$\int_0^\infty t^n e^{-xt}\, dt = \frac{n!}{x^{(n+1)}}. \tag{5.4.15}$$

So our above work justifies Laplaces claim.

Next we substitute (5.4.15) into (5.4.10) to obtain (up to exponentially small terms which we neglect)

$$I(x) = \sum_{n=0}^{\infty} \frac{(-1)^n n!}{x^{n+1}}, \qquad x \to \infty. \tag{5.4.16}$$

Concerning the infinite sum in (5.4.16) we note that

$$\lim_{n\to\infty} \frac{n\text{th}}{(n-1)\text{st}} = \lim_{n\to\infty} \frac{(-1)^n n! x^n}{(-1)^{n-1}(n-1)! x^{n+1}} = \lim_{n\to\infty} \frac{-n}{x} = -\infty.$$

The series diverges! So actually we cannot use "$=$" in (5.4.16). Thus more precisely we write,

$$I(x) \sim \sum_{n=0}^{\infty} \frac{(-1)^n n!}{x^{n+1}}, \qquad x \to \infty. \tag{5.4.17}$$

This is a famous formula known as *Watson' Lemma*.

The method of this last example is applicable to integrals of the form

$$I(x) = \int_0^b f(t) e^{-xt}\, dt. \tag{5.4.18}$$

Assumption 5.2. *The function f is assumed to be continuous on $[0, b]$ and possess an asymptotic expansion*

$$f(t) \sim t^\alpha \sum_{n=0}^{\infty} a_n t^{n\beta}, \qquad \textit{as} \quad t \to 0^+, \ \ \alpha > -1, \ \ \beta > 0. \tag{5.4.19}$$

The conditions on α and β gaurantee that the integral exists near $t = 0$. Also if we have $b = \infty$ then we must also assume that

$$f(t) \le M e^{ct}, \quad \textit{for some} \quad M, c > 0,$$

so that the integral exists at $t = \infty$.

Lemma 5.1 (Watson's Lemma). *If f satisfies the conditions in Assumption 5.2 then*

$$I(x) \sim \sum_{n=0}^{\infty} \frac{a_n \Gamma(\alpha+\beta n+1)}{x^{\alpha+\beta n+1}}, \quad x \to \infty. \tag{5.4.20}$$

Remark 5.3. 1. If the expansion for f satisfies, instead of (5.2), the condition

$$f(t) \sim \sum_{m=1}^{\infty} a_m t^{m\beta-1}, \qquad \text{as} \quad t \to 0^+, \;\; \beta > 0,$$

(note the sum starts at $m = 1$ but $\alpha = -1$) then Watson's Lemma gives

$$I(x) \sim \sum_{m=1}^{\infty} a_m \int_0^{\infty} t^{m\beta-1} e^{-xt}\, dt = \sum_{m=1}^{\infty} a_m \frac{\Gamma(m\beta)}{x^{m\beta}}.$$

2. Recall that our original interest in Laplace's Method was in integrals of the form (5.4.8). The integrals we consider for Watson's Lemma are a special case with $h(t) = -t$ and on the interval $(0, \infty)$ the maximum of this function is $c = 0$ at $t = 0$.

Outline of Proof

1. First we define

$$I(x, \epsilon) = \int_0^{\epsilon} f(t) e^{-xt}\, dt.$$

2. Next we choose ϵ so small that the first N terms in the asymptotic series for f are a good approximation to $f(t)$, i.e.,

$$\left| f(t) - t^{\alpha} \sum_{m=0}^{N} a_m t^{n\beta} \right| \le K t^{\alpha+\beta(N+1)}, \quad 0 \le t \le \epsilon, \;\; K > 0.$$

3. We substitute the above series into the integral to get

$$\begin{aligned} \left| I(x,\epsilon) - \sum_{n=0}^{N} a_n \int_0^{\epsilon} t^{\alpha+\beta n} e^{-xt}\, dt \right| & \\ \le K \int_0^{\epsilon} t^{\alpha+\beta(N+1)} e^{-xt}\, dt & \\ \le K \int_0^{\infty} t^{\alpha+\beta(N+1)} e^{-xt}\, dt & \\ = \frac{K\Gamma(\alpha+\beta(N+1)+1)}{x^{\alpha+\beta(N+1)+1}}. & \end{aligned}$$

4. Finally, since the right-hand-side of 3. is independent of ϵ we can replace the upper limit of integration on the left-hand-side by ∞. The resulting integrals can be evaluated (using formulas from Laplace transform theory) to obtain

$$\int_0^\infty t^{\alpha+\beta n} e^{-xt}\, dt = \frac{\Gamma(\alpha+\beta n+1)}{x^{\alpha+\beta n+1}}.$$

So we obtain

$$\left| I(x) - \sum_{m=0}^{N} a_n \frac{\Gamma(\alpha+\beta n+1)}{x^{\alpha+\beta n+1}} \right| \le \frac{K\Gamma(\alpha+\beta(N+1)+1)}{x^{\alpha+\beta(N+1)+1}}.$$

5. For every N (recall that $\beta > 0$) the term

$$\frac{K\Gamma(\alpha+\beta(N+1)+1)}{x^{\beta}}$$

can be made as small as we like by taking x sufficiently large. Thus, for x sufficiently large, we can write

$$\left| I(x) - \sum_{m=0}^{N} a_n \frac{\Gamma(\alpha+\beta n+1)}{x^{\alpha+\beta n+1}} \right| \ll \frac{1}{x^{\alpha+\beta N+1}}.$$

Since this is valid for all N we have obtained a valid asymptotic expansion.

Example 5.19. Consider the integral

$$I(x) = \int_0^5 \frac{e^{-xt}}{(1+t^2)}\, dt, \quad \text{for large} \quad x.$$

For small t we have the convergent Taylor series

$$f(t) = \frac{1}{(1+t^2)} = 1 - t^2 + t^4 - t^6 + \cdots$$

so applying Watson's Lemma we get

$$I(x) \sim \frac{1}{x} - \frac{2!}{x^3} + \frac{4!}{x^7} - \frac{6!}{x^9} + \cdots, \quad x \to \infty.$$

Remark 5.4. For more general integrals (5.4.8) we cannot directly use Watson's Method; it only works for $h(t) = -t$. There are a couple of cases that we mention briefly. Thus we consider integrals of the form (5.4.8), i.e.,

$$I(x) = \int_a^b e^{xh(t)} f(t)\, dt$$

1. If h is very simple try setting $s = -h(t)$ so that $ds = -h'(t)dt$ and, provided $h'(t) \neq 0$ we can set

$$I(x) = \int_{-h(a)}^{-h(b)} F(s) e^{-xs}\, ds, \quad F(s) = -\frac{f(t)}{h'(t)}.$$

2. If h has a max at $t = c$ then we can replace the integral by an integral near c, i.e., we pick small ϵ and integrate over the region $|t - c| < \epsilon$. In the region $|t - c| < \epsilon$ we replace $h(t)$ by the first few terms of a Taylor series about $t = c$. Provided that $h'(c) \neq 0$ we have

$$h(t) \approx h(c) + (t - c)h'(c).$$

If $h'(c) = 0$ then (if $h''(c) \neq 0$) we would have

$$h(t) \approx h(c) + \frac{1}{2}(t - c)^2 h''(c).$$

More generally, if $h^{(p)}(c) \neq 0$ is the first nonvanishing derivative at $t = c$ then

$$h(t) \approx h(c) + \frac{1}{p!}(t - c)^p h^{(p)}(c).$$

In each case we expand $f(t)$ about $t = c$ and retain the leading term (assume for simplicity that $f(c) \neq 0$ so the leading term is $f(c)$) There are three possibilities: either 1) $c = a$, 2) $c = b$ or 3) $a < c < b$. We will consider the cases 1) and 3) and leave 2) as an exercise.

(1) We assume that a maximum of h occurs at $c = a$ and that $h'(a) \neq 0$. This implies that $h'(a) < 0$, $h(t) \approx h(a) + (t - c)h'(a)$ and we replace $f(t)$ by $f(a)$ to obtain

$$\begin{aligned} I(x, \epsilon) \approx &\int_a^{a+\epsilon} f(a) e^{x[h(a)+(t-a)h'(a)]}\, dt \\ \approx &f(a)e^{xh(a)} \int_a^{\infty} e^{x(t-a)h'(a)}\, dt \\ \approx &f(a)e^{xh(a)} \left(\left. \frac{e^{x(t-a)h'(a)}}{xh'(a)} \right|_{t=a}^{t=\infty} \right) \\ \approx &- \frac{f(a)e^{xh(a)}}{xh'(a)} \quad \text{as} \quad x \to \infty. \end{aligned}$$

Thus we have

$$I(x) \sim -\frac{f(a)e^{xh(a)}}{xh'(a)} \quad \text{as} \quad x \to \infty. \tag{5.4.21}$$

(2) We assume that a maximum of h occurs at $c = b$, $f(b) \neq 0$ and $h'(b) \neq 0$ and we obtain

$$I(x) \sim \frac{f(b)e^{xh(b)}}{xh'(b)} \quad \text{as} \quad x \to \infty. \tag{5.4.22}$$

The proof of this is left as an exercise.

(3) Finally consider the case that $a < c < b$ which means that (since it is an interior maximum) $h'(c) = 0$. Let us assume that $h''(c) \neq 0$, then we must have $h''(c) < 0$ (why?) and we have

$$h(t) \approx h(c) + \frac{1}{2}(t-c)^2 h''(c).$$

We also assume that $f(c) \neq 0$. Recall that

$$\int_{-\infty}^{\infty} e^{-s^2}\, ds = \int_0^{\infty} u^{-1/2} e^{-u}\, du = \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$$

With this we can write

$$\begin{aligned} I(x,\epsilon) \approx & \int_{c-\epsilon}^{c+\epsilon} f(c) e^{x[h(c)+1/2(t-c)^2 h''(c)]}\, dt \\ \approx & f(c) e^{xh(c)} \int_{-\infty}^{\infty} e^{x(t-c)^2 h''(c)/2}\, dt. \end{aligned}$$

At this point we make the substitution

$$s = \sqrt{\frac{-h''(c)x}{2}}(t-c), \quad \Rightarrow ds = \sqrt{\frac{-h''(c)x}{2}}\, dt,$$

and we have

$$\begin{aligned} I(x,\epsilon) \approx & f(c) e^{xh(c)} \sqrt{\frac{2}{-h''(c)x}} \int_{-\infty}^{\infty} e^{-s^2}\, ds \\ \approx & \frac{f(c) e^{xh(c)} \sqrt{2\pi}}{\sqrt{-h''(c)x}}, \quad x \to \infty. \end{aligned}$$

$$I(x) \sim \frac{f(c) e^{xh(c)} \sqrt{2\pi}}{(-h''(c)x)^{1/2}}, \quad x \to \infty. \tag{5.4.23}$$

Remark 5.5. (a) In the cases $c = a$ or $c = b$, if $h'(c) = 0$ but $h''(c) \neq 0$ then the answers given in (5.4.21) or (5.4.22) is simply are replaced by (5.4.23) multiplied by $\frac{1}{2}$.

(b) If $h^{(p)}(c) \neq 0$ is the first nonvanishing derivative ($h^{(j)}(c) = 0$ for $j = 1, \cdots, (p-1)$) then (cf. [1])

$$I(x) \sim \frac{2\Gamma(1/p)(p!)^{1/p}}{p\left[-xh^{(p)}(c)\right]^{1/p}} f(c) e^{xh(c)}, \quad x \to \infty. \tag{5.4.24}$$

Verification of this result requires the formula

$$\int_{-\infty}^{\infty} e^{-s^p}\, ds = \frac{2}{p}\Gamma\left(\frac{1}{p}\right).$$

1. Show that as $\epsilon \sim 0$
$$\int_0^1 \frac{\sin(\epsilon t)}{t}\, dt \sim \epsilon - \frac{1}{18}\epsilon^3 + \frac{1}{600}\epsilon^5.$$

2. Show that as $x \to \infty$
$$\int_x^\infty \frac{e^{-t}}{t}\, dt \sim e^{-x}\left(\frac{1}{x} - \frac{1!}{x^2} + \frac{2!}{x^3} - \frac{3!}{x^4}\right).$$

3. Find an expansion in the form $I(x) = c_0 + c_1 x + c + c_2 x^2 + c_3 x^4 + \cdots c_8 x^8 + \mathcal{O}(x^9)$ for small x for
$$I(x) = \int_x^\infty e^{-t^2}\, dt.$$
$\left(\text{Hint: write } I(x) = \int_0^\infty \exp(-t^2)\, dt - \int_0^x \exp(-t^2)\, dt\right)$ Use (1) your answer and (2) Maple to find approximations to $I(x)$ for
$$x = .1, .25, .5, 1.$$
Compare your answers.

4. Find an asymptotic expansion, for large x, for $I(x) = \displaystyle\int_0^\infty \frac{e^{-t}}{(x+t)}\, dt$. Show that the infinite series you obtain diverges as $N \to \infty$ (hint: use ratio test). Use, (1) your answer, and (2) Maple to find approximations to $I(x)$ for $N = 2$ and $N = 4$ and $x = 10, 100, 1000, 10000$. Compare your answers.

5. Show that as $x \to \infty$, $\displaystyle\int_x^\infty e^{-t^2}\, dt = \frac{e^{-x^2}}{2x}\left[1 + \sum_{n=1}^\infty \frac{(1)(3)(5)\cdots(2n-1)}{(2x^2)^n}\right].$

6. Show that as $x \to \infty$, $\displaystyle\int_0^1 e^{-xt}\sin(t)\, dt \sim \frac{1}{x^2}.$

7. Show that as $x \to \infty$, $\displaystyle\int_1^2 e^{-x[t+(1/t)]}\, dt \sim \frac{\sqrt{\pi}}{2\sqrt{x}}\, e^{-2x}.$

8. Show that as $x \to \infty$, $\displaystyle\int_0^{\pi/2} e^{-x\tan(t)}\, dt \sim \frac{1}{x}.$

9. Show that as $x \to \infty$, $\displaystyle\int_0^1 \frac{e^{-xt^n}}{(1+t)}\, dt \sim \frac{\Gamma(1/n)}{nx^{1/n}}.$

5 Perturbation Theory

5.5 Perturbation Methods for ODEs

In this section we will consider the use of perturbation methods applied to finding approximate solutions to a variety of initial and boundary value problems for ordinary differential equations. This is only a very basic introduction to a very large subject. You can learn more about this topic by going to some of the references in the bibliography. The main point is that for most differential equations it is not possible to obtain an exact, explicit answer so we do what we can to obtain useful information about the solution. Sometimes a problem already contains a parameter which is known to be small (or large) and in some cases we can introduce such a parameter. This parameter is used as a perturbation parameter to obtain an asymptotic series expansion of the solution. Just as we saw in earlier sections sometimes a straightforward expansion is possible but other times we must be much more careful and use some alternative devise to arrive at a uniform asymptotic expansion.

Problems are classified as regular or singular. A regular problem is one for which a simple asymptotic expansion can be found with the property that the expansion is uniform in the independent variable. Usually, but not always, this is applicable for problems on a finite interval, e.g., for all t satisfying $0 \le t \le T < \infty$. The problem

$$\dot{x} = -x^3 + \epsilon x, \;\; x(0,\epsilon) = \xi_0 + \epsilon\xi_1 + \mathcal{O}(\epsilon^2), \;\; \text{here} \;\; \dot{x} \equiv \frac{d}{dt}x,$$

is a regular problem on any fixed interval $0 \le t \le T < \infty$. It is not regular on $(0,\infty)$. For smooth enough data and some stability a regular expansion can hold on an infinite interval. For example the problem

$$\dot{x} = -x + \epsilon, \;\; x(0,\epsilon) = \xi_0 + \epsilon\xi_1 + \mathcal{O}(\epsilon^2),$$

is regular on $(0,\infty)$. Finally a problem like

$$\epsilon\dot{x} = -x, \quad x(0,\epsilon) = \xi_0 + \epsilon\xi_1 + \mathcal{O}(\epsilon^2)$$

is singular even on $0 \le t \le T < \infty$ since the solution depends on ϵ in a singular way (as you will see later).

5.5.1 Regular Perturbation

Let us begin with a simple example

Example 5.1. Consider the first order initial value problem

$$\dot{x} + 2x + \epsilon x^2 = 0, \quad x(0) = \cosh(\epsilon), \quad 0 < \epsilon \ll 1. \tag{5.5.1}$$

We seek an approximate solution to this problem as an asymptotic series in powers of ϵ. Namely, we seek

$$x(t) = x_0(t) + x_1(t)\epsilon + x_2(t)\epsilon^2 + \cdots . \tag{5.5.2}$$

Just as we have done in each previous example we substitiute (5.5.2) into (5.5.1) to get

$$(\dot{x}_0 + 2x_0) + \epsilon(\dot{x}_1 + 2x_1 + x_0^2) + \epsilon^2(\dot{x}_2 + 2x_2 + 2x_0x_1) + \cdots = 0. \tag{5.5.3}$$

Similarly for the initial condition we obtain

$$x_0(0) + \epsilon x_1(0) + \epsilon^2 x_2(0) + \cdots = \cosh(\epsilon) = 1 + \frac{\epsilon^2}{2} + \frac{\epsilon^4}{24} + \cdots . \tag{5.5.4}$$

For this to hold for all ϵ we equate corresponding powers of ϵ in both (5.5.3) and (5.5.4). In this way we obtain an infinite sequence of linear nonhomogenous equations to solve for the functions $x_j(t)$.

$$\epsilon^0 : \quad \dot{x}_0 + 2x_0 = 0, \;\; x_0(0) = 1, \tag{5.5.5}$$

$$\epsilon^1 : \quad \dot{x}_1 + 2x_1 = -x_0^2, \;\; x_1(0) = 0, \tag{5.5.6}$$

$$\epsilon^2 : \quad \dot{x}_2 + 2x_2 = -2x_0x_1, \;\; x_2(0) = \frac{1}{2}. \tag{5.5.7}$$

From the ϵ^0 terms we can easily solve the homogeneous first order linear equation to get

$$x_0(t) = e^{-2t}.$$

The remaining problems are first order linear nonhomogeneous problems. Recall that these problems are easily solved (up to quadrature) as follows: For

$$y' + py = q, \quad y(0) = y_0$$

we multiply by the exponential (here we mean the indefinite integral or antiderivative) $\exp\left(\int p(t)\,dt\right)$ to obtain

$$\left[\exp\left(\int p(t)\,dt\right) y(t)\right]' = q(t)\exp\left(\int p(t)\,dt\right).$$

Nett we integrate this to obtain

$$\exp\left(\int p(t)\,dt\right) y(t) = \int q(t)\exp\left(\int p(t)\,dt\right) ds + C_0$$

or

$$y(t) = \exp\left(-\int p(t)\,dt\right)\left[\int_0^t q(t)\exp\left(\int p(t)\,dt\right) ds + C_0\right].$$

Substituting $t = 0$ we have $C_0 = y_0$ so that finally

$$y(t) = \exp\left(-\int p(t)\,dt\right)\left[\int_0^t q(t)\exp\left(\int p(t)\,dt\right) ds + y_0\right] \tag{5.5.8}$$

Applying (5.5.8) to (5.5.6) we have

$$[e^{2t}x_1(t)]' = -x_0^2$$

or

$$x_1(t) = e^{-2t}\left[\int_0^t e^{2\tau} e^{-4\tau}\, d\tau + C_0\right].$$

Thus we have

$$x_1(t) = -\frac{e^{-2t}}{2}\left[1 - e^{-2t}\right].$$

Finally for the ϵ^2 term we have from (5.5.7)

$$\dot{x}_2 + 2x_2 = -2x_2x_1 = e^{-4t} - e^{-6t}.$$

Once again appealing to (5.5.8) we have

$$x_2(t) = e^{-2t}\left[-\frac{1}{2}e^{-2t} + \frac{1}{4}e^{-4t} + C\right]$$

and

$$\frac{1}{2} = x_2(0) = \left[-\frac{1}{2} + \frac{1}{4} + C\right],$$

so

$$C = \frac{3}{4},$$

and

$$x_2(t) = e^{-2t}\left[\frac{3}{4} - \frac{1}{2}e^{-2t} + \frac{1}{4}e^{-4t}\right].$$

Combining these terms we arrive at

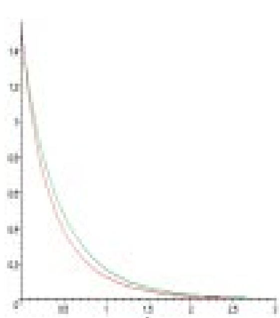
$$x(t) = e^{-2t}\left(1 + -\frac{1}{2}\left[1 - e^{-2t}\right]\epsilon + \frac{1}{2}\left[\frac{3}{2} - e^{-2t} + \frac{1}{2}e^{-4t}\right]\epsilon^2\right). \tag{5.5.9}$$

The exact solution to this problem is given by

$$x(t) = \frac{2\cosh(\epsilon)}{2e^{2t} + \epsilon\cosh(\epsilon)\left[e^{2t} - 1\right]}. \tag{5.5.10}$$

It can be shown, using Maple for example, that the first three terms in the Taylor series expansion of the exact solution in (5.5.10) is exactly (5.5.9).

We set $\epsilon = 1$ (which is not small) and plot both the aymptotic formula and the exact solution.



This example exhibits the basic feature of the regular perturbation method. What we have not done is address the question of whether this is actually an asymptotic expansion which is is uniform in t.

At this point we give a theorem from [7] for regular first order systems on a finite interval. Thus we consider

$$\dot{x} = f(t,x,\epsilon) \quad x(0) = \xi(\epsilon), \quad 0 \le t \le T < \infty. \tag{5.5.11}$$

We also impose the following assumptions

Assumption 5.1. *We assume that the unperturbed problem has a unique solution $x_0(t)$ on $0 \le t \le T$. That is, x_0 is the unique solution of*

$$\dot{x_0} = f(t,x_0,0) \quad x_0(0) = \xi(0).$$

Assumption 5.2. *We assume that f and ξ are smooth. More precisely for some desired n we assume that f and ξ are C^{n+1}.*

This last assumption implies, for example, that

$$\xi(\epsilon) = \xi_0 + \xi_1\epsilon + \cdots \xi_n\epsilon^n + \mathcal{O}(\epsilon^{n+1}).$$

Theorem 5.1 (Regular Perturbation Theorem Finite Interval). *Under the Assumptions 5.1, 5.2, for sufficiently small ϵ the problem (5.5.11) has a unique solution defined on $0 \le t \le T$, it is $(n+1)$ times continuously differentiable with respect to ϵ and has a Taylor series expansion*

$$x(t) = x_0(t) + x_1(t)\epsilon + \cdots + x_n(t)\epsilon^n + \mathcal{O}(\epsilon^{n+1}),$$

where the error estimate holds as $\epsilon \to 0$ uniformly for $0 \le t \le T$.

Example 5.2. Consider the first order initial value problem

$$\dot{x} = -\frac{x}{(1+\epsilon)}, \quad x(0) = \cos(\epsilon), \quad 0 < \epsilon \ll 1. \tag{5.5.12}$$

We seek an approximate solution to this problem as an asymptotic series in powers of ϵ. Namely, we seek

$$x(t) = x_0(t) + x_1(t)\epsilon + x_2(t)\epsilon^2 + \cdots . \tag{5.5.13}$$

For $\epsilon < 1$ we have

$$\frac{1}{(1+\epsilon)} = 1 - \epsilon + \epsilon^2 - \epsilon^3 + \cdots$$

and we can write

$$\begin{aligned}\frac{d}{dt}(x_0(t) + x_1(t)\epsilon + \cdots) &= -(x_0(t) + x_1(t)\epsilon + \cdots)(1 - \epsilon + \epsilon^2 + \cdots) \\ &= -\left[x_0 + (x_0 - x_1)\epsilon + (x_0 - x_1 + x_2)\epsilon^2 + \cdots\right].\end{aligned}$$

So equating powers of ϵ we obtain

$$\epsilon^0: \quad \dot{x}_0 = -x_0, \qquad x_0(0) = 1, \tag{5.5.14}$$
$$\epsilon^1: \quad \dot{x}_1 = -x_1 + x_0, \qquad x_1(0) = 0, \tag{5.5.15}$$
$$\epsilon^2: \quad \dot{x}_2 = -x_2 + x_1 - x_0, \qquad x_2(0) = -\frac{1}{2}. \tag{5.5.16}$$

Equations (5.5.14), (5.5.15) and (5.5.16) are easily solved and we get

$$x_0(t) = e^{-t}, \quad x_1(t) = te^{-t}, \quad x_2(t) = e^{-t}\left[\frac{t^2}{2} - t - \frac{1}{2}\right].$$

The exact solution to this problem is

$$x(t) = \cos(\epsilon)e^{-\frac{t}{1+\epsilon}}$$

which is bounded for all $0 \le t < \infty$. Furthermore on any bounded interval $0 \le t \le T < \infty$ our asymptotic expansion

$$x(t) \sim e^{-t} + te^{-t}\epsilon + e^{-t}\left[\frac{t^2}{2} - t - \frac{1}{2}\right]\epsilon^2 + \mathcal{O}(\epsilon^3)$$

can be compared with the exact answer. For $\epsilon = .1$ the maximum deviation of the exact and asymptotic solution with just three terms (second order in ϵ) is on the order of 10^{-5}.

On the other hand, as we have mentioned, there are problems which are regular on the any finite interval but not on the infinite interval. Consider the following two problems:

1. $\dot{x} = -x^3 + \epsilon x$ and $x(0) = a$.
2. $\dot{x} = \epsilon x$ and $x(0) = a$.

As for 1. we have the exact solution

$$x(t) = \frac{\pm\sqrt{\epsilon\,(1 - a^{-2}\exp(-2\epsilon t(a^2 - \epsilon))}}{(1 - a^{-2}\exp(-2\epsilon t(a^2 - \epsilon))},$$

which converges to $\pm\sqrt{\epsilon}$ as $t \to \infty$ which is not a smooth function. As for 2. the solution diverges to infinity and we see that $e^{\epsilon t}$ does not converge univormly for $0 \le t < \infty$ as $\epsilon \to 0$.

Nevertheless we do have a result concerning regular expansions on unbounded intervals.

Assumption 5.3. *We assume that the unperturbed problem has a unique solution $x_0(t)$ on $0 \le t < \infty$. That is, x_0 is the unique solution of*

$$\dot{x_0} = f(t, x_0, 0) \quad x_0(0) = \xi(0).$$

Assumption 5.4. *We assume that f and ξ are smooth functions on $0 \le t < \infty$. More precisely for some desired n we assume that f and ξ are C^{n+1}.*

Assumption 5.5. *We assume that the linearization about $x_0(t)$ is exponentially stable, i.e., If $Y(t)$ is the solution of*

$$\frac{dY}{dt} = f_x(t, x_0(t), 0)Y, \quad Y(0) = I.$$

Then

$$\|Y(t)\| \le Ke^{\alpha t}, \quad K > 0, \ \ \alpha > 0, \ \ 0 \le t < \infty.$$

here f_x is the jacobian matrix with (i, j) entry given by $\dfrac{\partial f_i}{\partial x_j}$ in the vector case.

Theorem 5.2 (Regular Perturbation Theorem Infinite Interval). *Under the Assumptions 5.3, 5.4, 5.5, for sufficiently small ϵ the problem (5.5.11) has a unique solution defined on $0 \le t < \infty$, it is $(n+1)$ times continuously differentiable with respect to ϵ and has a Taylor series expansion*

$$x(t) = x_0(t) + x_1(t)\epsilon + \cdots + x_n(t)\epsilon^n + \mathcal{O}(\epsilon^{n+1}),$$

where the error estimate holds as $\epsilon \to 0$ uniformly for $0 \le t < \infty$.

Example 5.3. Consider the two dimensional first order initial value problem

$$\dot{x} = Ax + \epsilon f(x), \ \ x(0) = \xi, \ 0 < \epsilon \ll 1. \tag{5.5.17}$$

where A is an $n \times n$ matrix and f is a nonlinear function from $\mathbb{R}^n$ to $\mathbb{R}^n$ with $f(0) = 0$. If in addition $\sigma(A)$ is contained in the left half complex plane, then from Theorem 5.2 we can conclude that

$$x(t, \epsilon) = e^{At}\xi + \mathcal{O}(\epsilon).$$

As an example consder

$$\dot{x}_1 = -2x_1 + x_2 + \epsilon x_2^2, \tag{5.5.18}$$
$$\dot{x}_2 = -2x_2 + x_1 + \epsilon x_1^2, \tag{5.5.19}$$
$$x_1(0) = 1, \quad x_2(0) = 1.$$

In this case we have

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \ \ A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}, \ \ f(x) = \begin{bmatrix} x_2^2 \\ x_1^2 \end{bmatrix}.$$

The eigenvalues of A are given by the zeros of the characteristic polynomial

$$0 = \det \begin{bmatrix} (\lambda + 2) & -1 \\ -1 & (\lambda + 2) \end{bmatrix} = (\lambda + 2)^2 - 1$$

which implies

$$\lambda_1 = -1, \lambda_2 = -3.$$

Since these eignevalues are in the left half plane and $f(0) = 0$ we can apply the theorem.

A maple code that finds the asymptotic expansion for x_1 and x_2 using basic principles can be found on my web page.

Example 5.4. Finally we consider a general class of problems that naturally led to the material in the next section. Consider the second order initial value problem

$$y'' - f(x)y = 0, \quad y(0) = 1, \quad y'(0) = 1. \tag{5.5.20}$$

There are very few functions f for which it is possible to obtain a closed form solution to this problem. Nevertheless this is a standard model for many problems that arise in practical applications.

Note the problem has no ϵ. To see how pereturbation methods are useful in practice let us introduce ϵ into (5.5.20) to obtain

$$y'' - \epsilon f(x)y = 0, \quad y(0) = 1, \quad y'(0) = 1. \tag{5.5.21}$$

Now we seek a regular perturbation expansion as

$$y(x) = \sum_{n=0}^{\infty} \epsilon^n \, y_n(x)$$

where

$$y_0(0) = 1, \;\; y_0'(0) = 1, \quad y_n(0) = 0, \quad y_n'(0) = 0, \quad n = 1, 2, \cdots .$$

For $\epsilon = 0$ we have

$$y'' = 0, \; y(0) = 1, \;\; y'(0) = 1$$

which implies

$$y_0(x) = 1 + x.$$

Furthermore, in general we have

$$\sum_{n=0}^{\infty} \epsilon^n y_n'' - f(x) \sum_{n=0}^{\infty} \epsilon^{n+1} y_n = 0,$$

or

$$y_0'' + \sum_{n=1}^{\infty} \epsilon^n \left(y_n'' - f(x) y_{n-1} \right) = 0.$$

Thus we obtain the recurcise formulas

$$y_n'' = f(x) y_{n-1}, \quad y_n(0) = 0, \;\; y_n'(0) = 0.$$

Since we know that $y_0(x) = 1 + x$ these equations can be solved (in principle) by quadrature, i.e.,

$$y_n'(x) = \int_0^x f(s) y_{n-1}(s) \, ds,$$

and

$$y_n(x) = \int_0^x \left(\int_0^t f(s) y_{n-1}(x) \, ds \right) \, dt.$$

From this we have

$$y(x) = (1+x) + \epsilon \int_0^x \left(\int_0^t f(s)(1+s)\, ds \right) dt +$$

$$\epsilon^2 \int_0^x \left(\int_0^t f(s) \left[\int_0^s \left\{ \int_0^v (1+u) f(u)\, du \right\} dv \right] ds \right) dt + \cdots .$$

As a specific example consider $f(x) = -e^{-x}$ and

$$y'' + e^{-x} y = 0, \quad y(0) = 1, \quad y'(0) = 1.$$

The exact solution is the complicated expression, given in terms of Bessel functions,

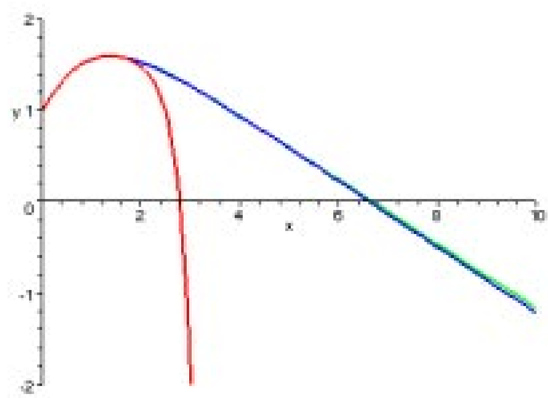
$$y(x) = \frac{[Y_0(2) + Y_0'(2)]\, J_0(2e^{-x/2}) - [J_0(2) + J_0'(2)]\, Y_0(2e^{-x/2})}{(J_0(2) Y_0'(2) - J_0'(2) Y_0(2))}.$$

There is a maple file on the web page that computes the asymptotic expansion of this solution

$$y_P(x) = \frac{257}{108} - \frac{13}{36}\, x - 3/2\, e^{-x} + 1/4\, e^{-x} x + 1/4\, e^{-2\,x} - 1/4\, e^{-2\,x} x - \frac{7}{54}\, e^{-3\,x} - 1/36\, e^{-3\,x} x$$

and compares it with the exact solution and the 10th degree Taylor approximation $y_T(x)$

$$y_T(x) = 1 + x - 1/2\, x^2 + 1/12\, x^4 - 1/24\, x^5 + \frac{7}{720}\, x^6 + \frac{1}{5040}\, x^7 - \frac{47}{40320}\, x^8 + \frac{7}{12960}\, x^9 - \frac{263}{1814400}\, x^{10}$$



Graphs of $y(x)$ (green), y_P (blue), and y_T (red)

5.5.2 Singular Perturbation Methods

We will illustrate several singular perturbation techniques applied to a particular nonlinear oscillator known as Duffing equation.

Example 5.5 (Duffing's Equation).

$$\ddot{y} + \omega^2 y + \epsilon y^3 = 0, \;\; y(0) = A, \;\; \dot{y}(0) = B, \quad 0 \le t < \infty. \tag{5.5.22}$$

Consider, for example, the case $A = 1$, $B = 0$, i.e.,

$$\ddot{y} + \omega^2 y + \epsilon y^3 = 0, \;\; y(0) = 1, \;\; \dot{y}(0) = 0.$$

First we note that the solution to this problem is bounded on $[0, \infty)$. To see this we multiply the equation by $\dot{y}$ to get

$$\dot{y}\ddot{y} + \omega^2 \dot{y} y + \epsilon y^3 \dot{y} = 0.$$

This implies

$$\frac{1}{2}\frac{d}{dt}\left(\dot{y}\right)^2 + \frac{\omega^2}{2}\frac{d}{dt}\left(y^2\right) + \frac{\epsilon}{4}\frac{d}{dt}\left(y^4\right) = 0,$$

or

$$\frac{1}{2}\frac{d}{dt}\left[\left(\dot{y}\right)^2 + \omega^2\left(y^2\right) + \frac{\epsilon}{2}\left(y^4\right)\right] = 0.$$

Integrating this equation we have

$$\left(\dot{y}\right)^2 + \omega^2\left(y^2\right) + \frac{\epsilon}{2}\left(y^4\right) = C.$$

Now employing the initial conditions we find that

$$C = \omega^2 + \frac{\epsilon}{2}.$$

Now using the fact that $\left(\dot{y}\right)^2 \ge 0$ and $\frac{\epsilon}{2}\left(y^4\right) \ge 0$ we can write

$$y^2 \le 1 + \frac{\epsilon}{2\omega^2},$$

or

$$|y(t)| \le \sqrt{1 + \frac{\epsilon}{2\omega^2}}.$$

On the other hand in an attempt to find a regular asymptotic expansion for y we would seek

$$y = y_0 + y_1\epsilon + y_2\epsilon^2 + \cdots .$$

Substitution of this expression into the equation gives

$$\left(\ddot{y}_0 + \ddot{y}_1\epsilon + \ddot{y}_2\epsilon^2 + \cdots\right) + \omega^2\left(y_0 + y_1\epsilon + y_2\epsilon^2 + \cdots\right)$$

$$= -\epsilon\left(y_0 + y_1\epsilon + y_2\epsilon^2 + \cdots\right)^3$$

$$= -\epsilon\left(y_0^3 + 2y_0^2 y_1\epsilon + 2y_0 y_1^2\epsilon^2 + \cdots\right).$$

For the initial conditions we have

$$1 = y(0) = y_0(0) + y_1(0)\epsilon + y_2(0)\epsilon^2 + \cdots ,$$
$$0 = y'(0) = y_0'(0) + y_1'(0)\epsilon + y_2'(0)\epsilon^2 + \cdots .$$

From this we obtain the sequence of problems

$$\epsilon^0 : \ \ddot{y}_0 + \omega^2 y_0 = 0, \ \ y_0(0) = 1, \ \ \dot{y}_0(0) =, \tag{5.5.23}$$

$$\epsilon^1 : \ \ddot{y}_1 + \omega^2 y_1 = -y_0^3, \ \ y_1(0) = 0, \ \ \dot{y}_1(0) = 0, \tag{5.5.24}$$

$$\epsilon^2 : \ \ddot{y}_2 + \omega^2 y_2 = -2y_0^2 y_1, \ \ y_2(0) = 0, \ \ \dot{y}_2(0) = 0, \tag{5.5.25}$$

$$\vdots$$

In what follows we will have several occasions to use the trig identity

$$\cos^3(\omega t) = \frac{3}{4}\cos(\omega t) + \frac{1}{4}\cos(3\omega t). \tag{5.5.26}$$

From Equation (5.5.23) we obtain a periodic solution

$$y_0(t) = \cos(\omega t). \tag{5.5.27}$$

Then, using (5.5.26) and the expression for y_0 in (5.5.27) expression on the right hand side in (5.5.24), we have

$$\ddot{y}_1 + \omega^2 y_1 = -\left(\frac{3}{4}\cos(\omega t) + \frac{1}{4}\cos(3\omega t)\right), \ \ y_1(0) = 0, \ \ y_1'(0) = 0.$$

This second order, linear, nonhomogeneous problem can be solved by the method of undetermined coefficients (or using maple) to obtain

$$y_1(t) = \frac{1}{32\omega^2}\cos(3\omega t) - \frac{1}{32\omega^2}\cos(\omega t) - \frac{3}{8\omega} t\sin(\omega t). \tag{5.5.28}$$

This to second order our asymptotic expnasion gives

$$y(t) \sim \cos(\omega t) + \epsilon\left(\frac{1}{32\omega^2}\cos(3\omega t) - \frac{1}{32\omega^2}\cos(\omega t) - \frac{3}{8\omega} t\sin(\omega t)\right) + \mathcal{O}(\epsilon^2).$$

There is a big problem – this function is not bounded in t. The reason for this is due to the phenomena of resonance. Namely, we are driving a harmonic oscillator with natural frequency ω with a sum of terms, one of which

$$-\frac{1}{32\omega^2}\cos(\omega t)$$

is oscillating at the natural frequency of the system. It is well know that this must lead to terms of the form

$$At\sin(\omega t) \quad \text{or} \quad Bt\cos(\omega t),$$

and these terms become unbounded as $t \to \infty$. In the asymptotic expansion we call such a term a *secular term*. Our objective in the singular perturbation methods will be to remove these secular terms form the asymptotic expansion.

Before beginning to study specific methods let me give a simple example to show how secular terms can arise very naturally and how the can be eliminated (but not in a practical way).

Consider the asymptotic series

$$1+\left[\frac{(-1)^1t^1}{1}\right]\epsilon+\left[\frac{(-1)^2t^2}{2}\right]\epsilon^2+\left[\frac{(-1)^3t^3}{6}\right]\epsilon^3+\cdots+\left[\frac{(-1)^nt^n}{n!}\right]\epsilon^n+\cdots .$$

Every term in this expansion (beyond the first) is a secular term for large t, i.e., for $t\sim 1/\epsilon$ or larger. But the infinite sum of this series is $e^{-\epsilon t}$ which is bounded for all t. For our ODE with $\omega=1$ to simplify the notation, an induction argument can be used to show (cf, [1]) that the sum of the leading order secular terms in the regular expansion, written in complex notation, are given by

$$\frac{t^n}{2n!}\left[\left(\frac{3i}{8}\right)^n e^{it}+\left(\frac{-3i}{8}\right)^n e^{-it}\right].$$

Note that there are many lower order term, i.e., terms involving t^j for $j<n$, but for large t these are dominated by t^n. If we sum only the leading order secular terms the asymptotic expansion from $n=0$ to $n=\infty$ we have

$$\sum_{n=0}^{\infty}\frac{t^n\epsilon^n}{2n!}\left[\left(\frac{3i}{8}\right)^n e^{it}+\left(\frac{-3i}{8}\right)^n e^{-it}\right]=\cos\left[t\left(1+\frac{3}{8}\epsilon\right)\right],$$

which is bounded for all t. In fact, it turns out that the approximation

$$y_\epsilon(t)=\cos\left[t\left(1+\frac{3}{8}\epsilon\right)\right] \tag{5.5.29}$$

is a very good approximation to the actual solution (which, by the way, cannot be obtained using maple) and it is bounded for all t. We will see that it is exactly this expression that we will obtain by employing the methods for singular perturbation problems in what follows.

Lindstedt-Lighthill-Poincare Method (Method of Strained Variables)

In it simplest form we seek a perturbation expansion of both the independent variable t and dependent variable y. Thus we seek

$$t=\tau+\epsilon t_1(\tau)+\epsilon^2t_2(\tau)+\cdots, \tag{5.5.30}$$

$$t_i(0)=0,\quad i=1,2,\cdots,$$

$$y(\tau)=y_0(\tau)+\epsilon y_1(\tau)+\epsilon^2y_2(\tau)+\cdots . \tag{5.5.31}$$

We note that in most cases we can replace the more general expansion (5.5.30) by

$$t=\tau(1+\epsilon b_1+\epsilon^2b_2+\cdots), \tag{5.5.32}$$

where b_j are constants.

In the following calculations we will use $'=\dfrac{d}{d\tau}$ and we liberally use the chain rule, e.g., we have

$$\frac{d}{dt}=\frac{d\tau}{dt}\frac{d}{d\tau}.$$

We also have

$$\frac{d\tau}{dt} = \left(\frac{dt}{d\tau}\right)^{-1} = \frac{1}{1+\epsilon t_1' + \epsilon^2 t_2' + \cdots)} = 1 - \epsilon t_1' + \epsilon^2((t_1')^2 - t_2') + \cdots .$$

As we have mentioned, in most cases it suffices to choose t as in (5.5.32) and in this case we get

$$\frac{d\tau}{dt} = \left(\frac{1}{1+\epsilon b_1 + \epsilon^2 b_2 + \cdots}\right),$$

or

$$\frac{d\tau}{dt} = \left(1 - \epsilon b_1 + \epsilon^2(b_1^2 - b_2) + \epsilon^3(b_3 - 2b_1b_2 + b_1^3) + \cdots)\right).$$

So that

$$\frac{d}{dt} = \left(1 - \epsilon b_1 + \epsilon^2(b_1^2 - b_2) + \epsilon^3(b_3 - 2b_1b_2 + b_1^3) + \cdots\right)\frac{d}{d\tau}. \tag{5.5.33}$$

Similarly,

$$\begin{aligned}\frac{d^2y}{dt^2} &= \frac{d}{dt}\left(\frac{dy}{d\tau}(1 - \epsilon b_1 + \epsilon^2(b_1^2 - b_2) + \cdots)\right)\\ &= \frac{d^2y}{d\tau^2}\left(1 - \epsilon b_1 + \epsilon^2(b_1^2 - b_2) + \cdots\right)^2\\ &= \frac{d^2y}{d\tau^2}\left(1 - 2\epsilon b_1 + \cdots\right)\\ &\equiv \left(1 - 2\epsilon b_1 + \cdots\right) y''. \end{aligned} \tag{5.5.34}$$

We apply this idea to the Duffing equation fron Example 5.5.

Example 5.6 (Duffing Revisited). Substituting (5.5.32) and (5.5.34) into (5.5.22) we get

$$\begin{aligned} 0 =& \ddot{y} + \omega^2 y + \epsilon y^3 \qquad (5.5.35)\\ =& (1 - 2\epsilon b_1 + \cdots)\left(y_0''(\tau) + \epsilon y_1''(\tau) + \epsilon^2 y_2''(\tau) + \cdots\right)\\ &+ \omega^2\left(y_0(\tau) + \epsilon y_1(\tau) + \epsilon^2 y_2(\tau) + \cdots\right)\\ &+ \epsilon\left(y_0(\tau) + \epsilon y_1(\tau) + \epsilon^2 y_2(\tau) + \cdots\right)^3 \end{aligned}$$

Also we have

$$1 = y_0(0) + \epsilon y_1(0) + \cdots , \tag{5.5.36}$$

$$\begin{aligned} 0 &= \dot{y}(t=0) = y'(\tau = 0)\left(1 - 2\epsilon b_1 + \cdots\right) \qquad (5.5.37)\\ &= \left(y_0'(0) + \epsilon y_1'(0) + \cdots\right)\left(1 - 2\epsilon b_1 + \cdots\right)\\ &= y_0'(0) + \epsilon\left(y_1'(0) - b_1 y_0'(0)\right) + \cdots . \end{aligned}$$

Thus we get

$$\epsilon^0 : \; y_0'' + \omega^2 y_0 = 0, \;\; y_0(0) = 1, \;\; y_0(0) = 0, \tag{5.5.38}$$

$$\begin{aligned} \epsilon^1 : \; y_1'' + \omega^2 y_1 &= -y_0^3 + 2b_1 y_0'' \\ &= -2b_1\omega^2 \cos(\omega t) - \cos^3(\omega t) \\ &= -2b_1\omega^2 \cos(\omega t) - \left[\frac{3}{4}\cos(\omega t) + \frac{1}{4}\cos(3\omega t)\right] \end{aligned} \tag{5.5.39}$$

In order to cancel the secular term we must eliminate the resonance that occurs due to the combination of the ω^2 term on the left and the $\cos(\omega t)$ on the right. Thus we need to set

$$\left[-2b_1\omega^2 - \frac{3}{4}\right]\cos(\omega t) = 0.$$

Thus we take

$$b_1 = -\frac{3}{8\omega^2},$$

which implies

$$t = \tau\left[1 - \frac{3\epsilon}{8\omega^2} + \cdots\right] \;\; \Rightarrow \;\; \tau = t\left[1 + \frac{3\epsilon}{8\omega^2} + \cdots\right].$$

With this choice we can write (5.5.39) as

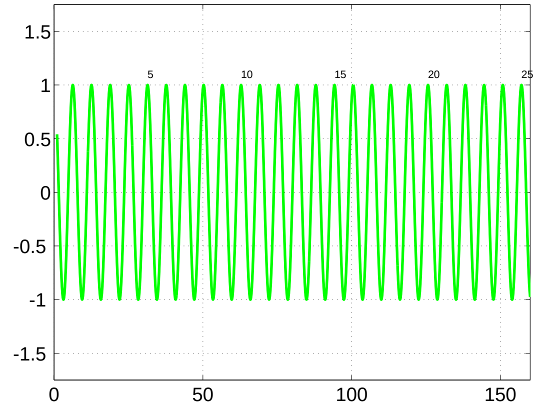
$$y_1'' + \omega^2 y_1 = -\frac{1}{4}\cos(3\omega\tau), \;\; y_1(0) = 0, \;\; y_1'(0) = 0.$$

This implies

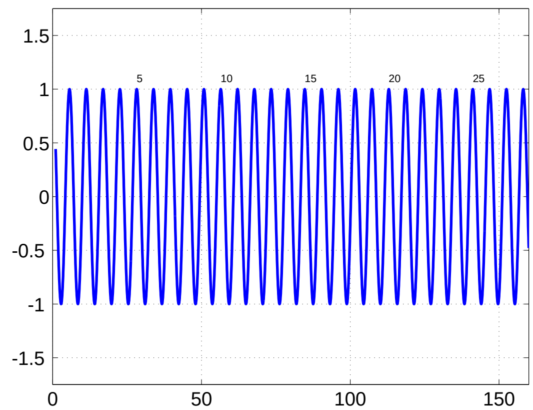
$$y_1(\tau) = -\frac{1}{32\omega^2}\cos(\omega\tau) + \frac{1}{32\omega^2}\cos(3\omega\tau) \tag{*}$$

so our first order approximation is

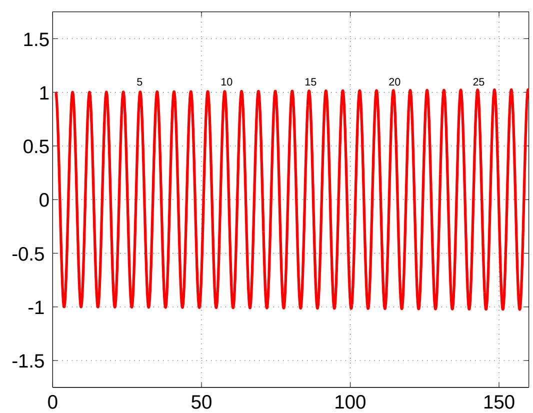
$$y \sim y_0(\tau) = \cos(\omega\tau) \sim \cos\left[\omega\left(1 + \frac{3\epsilon}{8\omega^2}\right)t\right] = \cos\left[\left(\omega + \frac{3\epsilon}{8\omega}\right)t\right].$$



Graph of regular purturbation approximation $y_0(x) = \cos(\omega t)$ with $\epsilon = .3$



Graphs of $y_\epsilon(x)$ with $\epsilon = .3$



Graphs of exact solution $y(x)$ computed numerically with $\epsilon = .3$

Example 5.7. For the more general problem given in (5.5.22)

$$\ddot{y} + \omega^2 y + \epsilon y^3 = 0, \;\; y(0) = A, \;\; \dot{y}(0) = \omega B, \quad 0 \le t < \infty.$$

we could proceed exactly as we did above. But just to give you a sampling of various peoples approaches to these problems we will write things in a slightly differently (when the dust settles it is the same as above).

We seek $\tau = \beta t$ with $\beta = \beta(\epsilon)$. By the chain rule we can write

$$\frac{d}{dt} = \frac{d\tau}{dt}\frac{d}{d\tau} = \beta\frac{d}{d\tau},$$
$$\frac{d^2}{dt^2} = \beta\frac{d^2}{dtd\tau} = \beta\frac{d\tau}{dt}\frac{d^2}{d\tau^2} = \beta^2\frac{d^2}{d\tau^2}.$$

With this change of variables the equations becomes

$$\beta^2 y'' + \omega^2 y + \epsilon y^3 = 0.$$

Now we seek expansions

$$\begin{aligned} y(\tau) &= y_0(\tau) + \epsilon y_1(\tau) + \cdots , \\ \beta(\epsilon) &= 1 + \epsilon\beta_1 + \epsilon^2\beta_2 + \cdots , \end{aligned}$$

where β_j are scalars.

Thus we obtain

$$(1 + \epsilon\beta_1 + \cdots)^2(y_0''(\tau) + \epsilon y_1''(\tau) + \cdots) + \omega^2(y_0(\tau) + \epsilon y_1(\tau) + \cdots)$$

$$+ \epsilon(y_0(\tau) + \epsilon y_1(\tau) + \cdots)^3 = 0.$$

As usual we now group according to powers of ϵ to obtain

$$0 = (y_0'' + \omega^2 y_0) + \epsilon(y_1'' + \omega^2 y_1 + y_0^3 + 2\beta_1 y_0'') + \cdots .$$

For the boundary conditions we have

$$A = y(0) = y_0(0) + \epsilon y_1(0) + \cdots ,$$

and

$$\begin{aligned} \omega B = \dot{y}(0) &= y'(0)(1 + \epsilon\beta_1 + \cdots) \\ &= (y_0'(0) + \epsilon y_1'(0) + \cdots)(1 + \epsilon\beta_1 + \cdots) \\ &= y_0'(0) + \epsilon(y_1'(0) + \beta_1 y_0'(0)) + \cdots . \end{aligned}$$

Thus we have

$$y_0'(0) = \omega B, \quad y_1'(0) = -\omega B\beta_1.$$

Equating powers of ϵ to zero we get

$$\epsilon^0 : \; y_0'' + \omega^2 y_0 = 0, \;\; y_0(0) = A, \;\; y_0'(0) = \omega B,$$

$$\epsilon^1 : \; y_1'' + \omega^2 y_1 = -y_0^3 - 2\beta_1 y_0'', \;\; y_1(0) = 0, \;\; y_1'(0) = -\omega B\beta_1.$$

Now the first equation gives

$$y_0(\tau) = -\frac{B}{\omega}\sin(\omega\,\tau) + A\cos(\omega\,\tau) = a\cos(\omega\tau + b),$$

where

$$a = \sqrt{A^2 + B^2}, \qquad b = \tan^{-1}\left(\frac{-B}{A}\right).$$

For the second equation we need to simplify the right hand side.

$$-y_0^3 - 2\beta_1 y_0'' = -a^3\,(1/4\,\cos(3\,\omega\,\tau + 3\,b) + 3/4\,\cos(\omega\,\tau + b)) + 2\,\beta_1\,a\cos(\omega\,\tau + b)\omega^2.$$

So to cancel the resonance we need to choose β_1 so that

$$2\beta_1\omega^2 a - \frac{3}{4}a^3 = 0,$$

or

$$\beta_1 = \frac{3a^2}{8\omega^2}.$$

Thus we obtain

$$y_1 = \frac{a^3}{32\omega^2}\Big[\cos(\omega\tau - 3b) - 2\cos(\omega\tau + 3b) \\ - 12B\sin(\omega\tau) + \cos(3\omega\tau + 3b)\Big].$$

and

$$\beta = 1 + \frac{3a^2}{8\omega^2}\epsilon + \cdots$$

so we conclude that

$$\begin{aligned} y &\sim a\cos(\omega\tau + b) + \cdots \\ &= a\cos\left(\omega\left(1 + \frac{3a^2}{8\omega^2}\epsilon\right)t + b\right) + + \cdots . \end{aligned}$$

As special cases for y_0, y_1 we have

1. If $A = 0$, $B > 0$ then $a = B$, $b = -\pi/2$ implies

$$y_0(t) = B\sin\left(\omega t + \frac{3B^3\epsilon}{8\omega}t\right),$$

$$y_1(t) = \frac{B^3}{32\omega^2}\big[(3 - 12B)\sin(\omega\tau) - \sin(3\omega\tau)\big].$$

   In the special case $B = 1$ we get

$$y_1(t) = \frac{-1}{32\omega^2}\big[9\sin(\omega\tau) + \sin(3\omega\tau)\big].$$

2. If $B = 0$, $A > 0$, then $a = A$ and $b = 0$ which gives

$$y_0(t) = A\cos\left(\omega t + \frac{3A^3\epsilon}{8\omega}t\right),$$

$$y_1(\tau) = \frac{A^3}{32\omega^2}\big[\cos(3\omega\tau) - \cos(\omega\tau)\big],$$

   in agreement with $(*)$ from the last section.

We note that this expansion is uniform to first order in ϵ since there are no secular terms and the order ϵ term is small compared to the first term (for small ϵ).

Krylov-Bogoliubov Method – Method of Averaging

In this section we present an alternative method for obtaining uniform asymptotic expansions for singular problems. We will consider this method for problems that can be cast in the form

$$\begin{aligned} &\ddot{y} + \omega^2 y = a + \epsilon f(y, \dot{y}), \quad 0 < \epsilon \ll \omega, \\ &y(0) = b, \quad \dot{y}(0) = c, \end{aligned} \tag{5.5.40}$$

where a, b and c are constants.

The following example is taken from [14].

Example 5.8 (Precession of Mercury).

$$f(u, v) = u^2, \ \ , \omega = 1, \ \ a = .98, \ \ b = 1.01, \ \ c = 0.$$

In this case (5.5.40) becomes

$$\ddot{y} + y = a + \epsilon y^2.$$

Example 5.9 (Rayleigh Oscillator).

$$f(u, v) = v - \frac{1}{3}v^3, \ \ , \omega = 1, \ \ a = 0.$$

In this case (5.5.40) becomes

$$\ddot{y} + y = \epsilon \left[\dot{y} - \frac{1}{3}(\dot{y})^2 \right].$$

Example 5.10 (Van der Pol Oscillator).

$$f(u, v) = (1 - u^2)v, \ \ \omega = 1, \ \ a = 0.$$

In this case (5.5.40) becomes

$$\ddot{y} + y = \epsilon(1 - y^2)\dot{y}.$$

Example 5.11 (Duffing Equation).

$$f(u, v) = -y^3, \ \ a = 0.$$

In this case (5.5.40) becomes

$$\ddot{y} + \omega^2 y + \epsilon y^3 = 0.$$

For $\epsilon = 0$ the equation (5.5.40) becomes

$$\ddot{y} + \omega^2 y = a$$

which for suitable A and B (determined by the initial conditions) can be written as

$$y(t) = \frac{a}{\omega^2} + A \sin(\omega t + B). \tag{5.5.41}$$

Namely, we have

$$A = \sqrt{b^2 + c^2}, \quad B = \tan^{-1}\left(\frac{b}{c}\right).$$

For the method of Averaging we assume that $A = A(t)$ and $B = B(t)$ and differentiate (5.5.41) with respect to t to obtain

$$\frac{dy}{dt} = \frac{dA}{dt}\sin(\omega t + B) + A\left(\omega + \frac{dB}{dt}\right)\cos(\omega t + B).$$

We seek A and B so that

$$\frac{dA}{dt}\sin(\omega t + B) + A\frac{dB}{dt}\cos(\omega t + B) = 0. \tag{5.5.42}$$

This implies

$$\frac{dy}{dt} = A\omega\cos(\omega t + B). \tag{5.5.43}$$

Now we use equations (5.5.40) and (5.5.43) to compute $\dfrac{d^2y}{dt^2}$. So on the one hand we obtain

$$\frac{d^2y}{dt^2} = \omega\frac{dA}{dt}\cos(\omega t + B) - \omega A\left[\omega + \frac{dB}{dt}\right]\sin(\omega t + B).$$

On the other hand we have

$$\begin{aligned}\frac{d^2y}{dt^2} &= -\,\omega^2 y + a + \epsilon f(y, \dot{y}) \qquad (5.5.44)\\ &= -\,\omega^2\left[\frac{a}{\omega^2} + A\sin(\omega t + B)\right]\\ &\quad + a + \epsilon f\left(\frac{a}{\omega^2} + A\sin(\omega t + B), A\omega\cos(\omega t + B)\right).\end{aligned}$$

Equating the results from (5.5.43) and (5.5.44) and defining

$$\varphi = \omega t + B, \tag{5.5.45}$$

we obtain

$$\frac{dA}{dt}\omega\cos\varphi - \frac{dB}{dt}A\omega\sin\varphi = \epsilon f\left(\frac{a}{\omega^2} + A\sin\varphi, A\omega\cos\varphi\right). \tag{5.5.46}$$

Using (5.5.45) in (5.5.42) we obtain a 2×2 system

$$\begin{bmatrix}\sin\varphi & A\cos\varphi\\ \omega\cos\varphi & -A\omega\sin\varphi\end{bmatrix}\begin{bmatrix}\dfrac{dA}{dt}\\ \dfrac{dB}{dt}\end{bmatrix} = \begin{bmatrix}0\\ \epsilon f\left(\dfrac{a}{\omega^2} + A\sin\varphi, A\omega\cos\varphi\right)\end{bmatrix} \tag{5.5.47}$$

which, using cramer's rules, gives

$$\frac{d}{dt}\begin{bmatrix} A(t) \\ B(t) \end{bmatrix} = \frac{\epsilon}{\omega} f\left(\frac{a}{\omega^2} + A\sin\varphi, A\omega\cos\varphi\right)\begin{bmatrix} \cos\varphi \\ -\dfrac{\sin\varphi}{A} \end{bmatrix} \tag{5.5.48}$$

This is a complicated first order nonlinear system of ordinary differential equations. But at this point it is still an exact set of equations – that is, no approximations have been made. At this point we notice that dA/dt and dB/dt are proportional to the small quantity ϵ. This means that for ϵ small A and B are slowly varying. If we assume this to be true then the main variation on the right in (5.5.48) is due to the terms involving φ which always occur as the argument of a 2π-periodic function (sine or cosine). Thus the Method of Averaging entails replacing (5.5.48) by the averaged value on the right hand side over one period in φ.

$$\frac{d}{dt}\begin{bmatrix} A_0(t) \\ B_0(t) \end{bmatrix} = \frac{\epsilon}{2\pi\omega}\int_0^{2\pi} f\left(\frac{a}{\omega^2} + A\sin\varphi, A_0\omega\cos\varphi\right)\begin{bmatrix} \cos\varphi \\ -\dfrac{\sin\varphi}{A_0} \end{bmatrix} d\varphi \tag{5.5.49}$$

In this integration A_0 and B_0 are held fixed during the φ integration. Then after computing A_0 and B_0 from (5.5.49) we set

$$y_0(t,\epsilon) = \frac{a}{\omega^2} + A_0(t,\epsilon)\sin(\omega t + B_0(t,\epsilon)). \tag{5.5.50}$$

It has been proved by Bogoliubov in (1958) and Mitropolsky in (1961) that this is an approximation to the exact solution y over any time interval of length $\mathcal{O}(1/\epsilon)$, i.e.,

$$|y(t,\epsilon) - y_0(t,\epsilon)| \le c_1\epsilon$$

uniformly for all t and ϵ satisfying

$$0 \le t \le c_2/\epsilon$$

where c_1 and c_2 do not depend on ϵ.

Example 5.12 (Rayleigh Oscillator continued).

$$f(u,v) = v - \frac{1}{3}v^3, \ \ ,\omega = 1, \ \ a = 0.$$

In this case (5.5.49) becomes

$$\frac{d}{dt}\begin{bmatrix} A_0(t) \\ B_0(t) \end{bmatrix} = -\epsilon\int_0^{2\pi} \cos(\varphi)\left(1 - \frac{1}{3}A_0^2\cos^2(\varphi)\right)\begin{bmatrix} -A_0\cos\varphi \\ \sin\varphi \end{bmatrix} d\varphi \tag{5.5.51}$$

Example 5.13 (Duffing Equation continued).

$$f(u,v) = -y^3, \ \ a = 0.$$

$$\frac{d}{dt}\begin{bmatrix} A_0(t) \\ B_0(t) \end{bmatrix} = -\frac{\epsilon}{\omega}\int_0^{2\pi} \left(\frac{a}{\omega^2} + A_0\sin\varphi\right)^3\begin{bmatrix} \cos\varphi \\ -\dfrac{\sin\varphi}{A_0} \end{bmatrix} d\varphi \tag{5.5.52}$$

Example 5.14 (Precession of Mercury continued).

$$f(u,v)=u^2, \ \ ,\omega=1, \ \ a=.98, \ \ b=1.01, \ \ c=0.$$

In this case (5.5.49) becomes

$$\frac{d}{dt}\begin{bmatrix} A_0(t) \\ B_0(t) \end{bmatrix} = \frac{\epsilon}{2\pi}\int_0^{2\pi}(a+A_0\sin\varphi)^2 \begin{bmatrix} \cos\varphi \\ -\dfrac{\sin\varphi}{A_0} \end{bmatrix} d\varphi \qquad (5.5.53)$$

We will consider this example in more detail. The first equation in (5.5.53) is

$$\frac{dA_0}{dt}=\frac{\epsilon}{2\pi}\int_0^{2\pi}(a+A_0\sin(\varphi))^2\ \cos(\varphi)\,d\varphi$$

with A_0 held fixed during the integration.

If we make the substitution $\eta=a+A_0\sin(\varphi)$ the integral becomes

$$\frac{dA_0}{dt}=\frac{\epsilon}{2\pi A_0}\int_a^a \eta^2\,d\eta=0.$$

or

$$\frac{dA_0}{dt}=0$$

or

$$A_0=\alpha.$$

Also, the second equation in (5.5.53) is

$$\frac{dB_0}{dt}=-\frac{\epsilon}{2\pi A_0}\int_0^{2\pi}(a+A_0\sin(\varphi))^2\ \sin(\varphi)\,d\varphi$$

and a direct integration on expanding the squared term gives

$$\frac{dB_0}{dt}=-\epsilon a$$

or

$$B_0=-\epsilon\alpha t+\beta,$$

where α, β are constants of integration. Thus we obtain

$$y_0(t,\epsilon)=a+\alpha\sin((1-\epsilon a)t+\beta), \qquad (5.5.54)$$

which in turn implies

$$\frac{dy_0}{dt}(t,\epsilon)=\alpha(1-\epsilon a)\cos((1-\epsilon a)t+\beta), \qquad (5.5.55)$$

where α and β are determined from the initial conditions

$$y_0(0)=b, \quad \frac{dy_0}{dt}(0)=0.$$

Using (5.5.54) and (5.5.55) we obtain

$$\alpha\cos(\beta) = 0, \quad \alpha\sin(\beta) = b - a$$

which implies

$$\beta = \pi/2, \quad \alpha = b - a,$$

and

$$y_0(t,\epsilon) = a + (b-a)\cos((1-\epsilon a)t). \tag{5.5.56}$$

Remark 5.1.

1. If we were to seek a regular perturbation expansion for the solution to this problem the result would yield a first order approximation of

   $$y_0(t) = a + (b-a)\cos(t)$$

   which is exactly (5.5.56) with $\epsilon = 0$.

2. For our approximation (5.5.56) it can be shown that

   $$|y(t,\epsilon) - y_0(t,\epsilon)| \le 17\epsilon \approx \frac{1}{10^6}$$

   for all $0 \le t < 1/\epsilon$.

   This follows from the known estimates for the first order approximations with the Method of Averaging due to the value of ϵ. Namely,

   $$a = \frac{GM\overline{r}}{h^2} \approx .98, \;\; b \approx 1.01, \;\; \epsilon = \frac{3GM}{c^2\overline{r}} \approx 10^{-7},$$

   where M is the mass of the sun, G is Newton graviational constant, h is the angular momentum of Mercury, $\overline{r} = 5.83 \times 10^{12}$ cm is the typical distance from Mercury to the Sum, c is the speed of light in a vacuum.

   From this it can be determined that the perihelion (direction of the major axis of the elliptical orbit of the planet) advances by an amount equal to $2\pi a\epsilon$ which interprets to approximately 40secs of arc per century. This is almost exactly the amount by which there was a deviation from the expected value due to Newtonian mechanics.

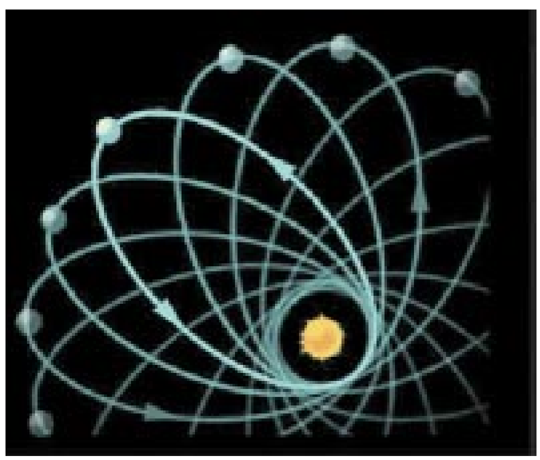
   Jose Wudka member of the Physics Department at UC Riverside:

   > To understand what the problem is let me describe the way Mercury's orbit looks. As it orbits the Sun, this planet follows an ellipse...but only approximately: it is found that the point of closest approach of Mercury to the sun does not always occur at the same place but that it slowly moves around the sun (see the figure). This rotation of the orbit is called a precession.
   >
   > The precession of the orbit is not peculiar to Mercury, all the planetary orbits precess. In fact, Newton's theory predicts these effects, as being produced by the pull of the planets on one another. The question is whether Newton's

predictions agree with the amount an orbit precesses; it is not enough to understand qualitatively what is the origin of an effect, such arguments must be backed by hard numbers to give them credence. The precession of the orbits of all planets except for Mercury's can, in fact, be understood using Newton;s equations. But Mercury seemed to be an exception.

As seen from Earth the precession of Mercury's orbit is measured to be 5600 seconds of arc per century (one second of arc=1/3600 degrees). Newton's equations, taking into account all the effects from the other planets (as well as a very slight deformation of the sun due to its rotation) and the fact that the Earth is not an inertial frame of reference, predicts a precession of 5557 seconds of arc per century. There is a discrepancy of 43 seconds of arc per century.

This discrepancy cannot be accounted for using Newton's formalism. Many adhoc fixes were devised (such as assuming there was a certain amount of dust between the Sun and Mercury) but none were consistent with other observations (for example, no evidence of dust was found when the region between Mercury and the Sun was carefully scrutinized). In contrast, Einstein was able to predict, without any adjustments whatsoever, that the orbit of Mercury should precess by an extra 43 seconds of arc per century should the General Theory of Relativity be correct.



Method of Multiple Scales

Just as a clock has different time scales, T_0 for seconds, T_1 for minutes and T_2 for hours, differential equations can have different time scales, such as, $T_0 = t$, $T_0 = \epsilon t$ and $T_2 = \epsilon^2 t$. These time scales vary from fast, to slower to even slower. In a problem we imagine that these time scales are independent variable (even though they are not actually). Thus we consider

$$y(t, \epsilon) \quad \text{replaced by} \quad y(T_0, T_1, T_2, \cdots, \epsilon)$$

with $T_0 = t$, $T_0 = \epsilon t$ and $T_2 = \epsilon^2 t$, etc.

Using the chain rule we have

$$\frac{d}{dt} = \frac{\partial}{\partial T_0} + \epsilon\frac{\partial}{\partial T_1} + \epsilon^2\frac{\partial}{\partial T_2} + \cdots ,$$

$$\begin{aligned}
\frac{d^2}{dt^2} =& \frac{d}{dt}\left[\frac{\partial}{\partial T_0} + \epsilon\frac{\partial}{\partial T_1} + \epsilon^2\frac{\partial}{\partial T_2} + \cdots\right] \\
=& \frac{\partial}{\partial T_0}\left[\frac{\partial}{\partial T_0} + \epsilon\frac{\partial}{\partial T_1} + \epsilon^2\frac{\partial}{\partial T_2} + \cdots\right] \\
&+ \epsilon\frac{\partial}{\partial T_1}\left[\frac{\partial}{\partial T_0} + \epsilon\frac{\partial}{\partial T_1} + \epsilon^2\frac{\partial}{\partial T_2} + \cdots\right] \\
&+ \epsilon^2\frac{\partial}{\partial T_2}\left[\frac{\partial}{\partial T_0} + \epsilon\frac{\partial}{\partial T_1} + \epsilon^2\frac{\partial}{\partial T_2} + \cdots\right] \\
&+ \cdots \\
=& \frac{\partial^2}{\partial T_0^2} + 2\epsilon\frac{\partial^2}{\partial T_0\partial T_1} + \epsilon^2\left(2\frac{\partial^2}{\partial T_0\partial T_2} + \frac{\partial^2}{\partial T_2^2}\right) + \cdots .
\end{aligned}$$

As an example consider the Duffing equation

$$\ddot{y} + y + \epsilon y^3 = 0. \tag{5.5.57}$$

We obtain

$$\frac{\partial^2 y}{\partial T_0^2} + 2\epsilon\frac{\partial^2 y}{\partial T_0\partial T_1} + \epsilon^2\left(2\frac{\partial^2 y}{\partial T_0\partial T_2} + \frac{\partial^2 y}{\partial T_2^2}\right) + y + \epsilon y^3 = 0. \tag{5.5.58}$$

Thus an ODE has been turned into a PDE. Normally this would not be a good idea but in the present case we now seek a uniform approximation to the solution of (5.5.57) in the form

$$y = y_0(T_0, T_1, T_2, \cdots) + \epsilon y_1(T_0, T_1, T_2, \cdots) + \epsilon^2 y_2(T_0, T_1, T_2, \cdots) + \cdots .$$

Subtituting this expression into (5.5.58), collecting powers of ϵ and equating to zero, we obtain

$$\begin{aligned}
0 =& \left(\frac{\partial^2 y_0}{\partial T_0^2} + \epsilon\frac{\partial^2 y_1}{\partial T_0^2} + \cdots\right) \\
&+ 2\epsilon\frac{\partial^2}{\partial T_0\partial T_1}\left(y_0 + \epsilon y_1 + \cdots\right) \\
&+ \left(y_0 + \epsilon y_1 + \cdots\right) \\
&+ \epsilon(y_0 + \epsilon y_1 + \cdots)^3.
\end{aligned}$$

$$\frac{\partial^2 y_0}{\partial T_0^2} + y_0 = 0, \tag{5.5.59}$$

$$\frac{\partial^2 y_1}{\partial T_0^2} + y_1 = -y_0^3 - 2\frac{\partial^2 y_0}{\partial T_0\partial T_1} \tag{5.5.60}$$

$$\vdots$$

The general solution of (5.5.59) is

$$y_0 = a(T_1, T_2, \cdots) \cos(T_0 + b(T_1, T_2, \cdots)). \tag{5.5.61}$$

Note that a and b are functions of T_1, T_2, etc. The functional dependence on these variables will be determined as we eliminate the secular terms.

Now we substitute our value for y_0 into (5.5.60) to obtain

$$\begin{aligned}
&\frac{\partial^2 y_1}{\partial T_0^2} + y_1 = -y_0^3 - 2\frac{\partial^2 y_0}{\partial T_0 \partial T_1} \\
=& -2\frac{\partial^2}{\partial T_0 \partial T_1}\left(a\cos(T_0 + b)\right) - a^3\cos^3(T_0 + b) \\
=& 2\frac{\partial a}{\partial T_1}\sin(T_0 + b) + 2a\frac{\partial b}{\partial T_1}\cos(T_0 + b) \\
&- \frac{3}{4}a^3\cos(T_0 + b) - \frac{1}{4}a^3\cos(3T_0 + 3b).
\end{aligned}$$

Thus we have

$$\begin{aligned}
\frac{\partial^2 y_1}{\partial T_0^2} + y_1 = 2\frac{\partial a}{\partial T_1}\sin(T_0 + b) + \left(2a\frac{\partial b}{\partial T_1} - \frac{3}{4}a^3\right)\cos(T_0 + b) \\
- \frac{1}{4}a^3\cos(3T_0 + 3b).
\end{aligned}$$

We need to remove the secular terms which arise due to the terms $\sin(T_0 + b)$ and $\cos(T_0 + b)$ which introduce resonance. Thus we see that we need

$$\frac{\partial a}{\partial T_1} = 0 \quad \text{and} \quad 2a\frac{\partial b}{\partial T_1} - \frac{3}{4}a^3 = 0.$$

This implies that a is independent of T_1 so we have

$$a = a(T_2, t_3, \cdots),$$

and, using this result, we also have

$$\frac{\partial b}{\partial T_1} = \frac{3}{8}a^2, \quad \Rightarrow \quad b = \frac{3}{8}a^2 T_1 + b_0(T_2, T_3, \cdots). \tag{5.5.62}$$

We can solve for y_1 and we get

$$y_1 = \frac{1}{32}a^3\cos(3T_0 + 3b). \tag{5.5.63}$$

Now substitute our values for a and b to get the first two terms of our asymptotic expansion for y

$$\begin{aligned}
y \sim& a(T_2, T_3, \cdots)\cos\left(T_0 + \frac{3}{8}T_1 a^2(T_2, T_3, \cdots) + b_0(T_2, T_3, \cdots)\right) \\
&+ \frac{\epsilon}{32}a^3(T_2, T_3, \cdots)\cos\left(3T_0 + \frac{9}{8}T_1 a^2(T_2, T_3, \cdots) + 3b_0(T_2, T_3, \cdots)\right).
\end{aligned}$$

If we stop the expansion at this step then we can consider a and b to be constants (to within the order of the error indicated), i.e.,

$$\begin{aligned} a(T_2, T_3, \cdots) &= a(\epsilon^2 t, \epsilon^3 t, \cdots) \\ &= a(0,0,\cdots) + \frac{\partial a}{\partial T_2}(0,0,\cdots)\epsilon^2 t + \cdots \qquad = \widehat{a} + \mathcal{O}(\epsilon^2 t), \end{aligned}$$

and

$$\begin{aligned} b_0(T_2, T_3, \cdots) &= b_0(\epsilon^2 t, \epsilon^3 t, \cdots) \\ &= b_0(0,0,\cdots) + \frac{\partial b_0}{\partial T_2}(0,0,\cdots)\epsilon^2 t + \cdots \qquad = \widehat{b_0} + \mathcal{O}(\epsilon^2 t), \end{aligned}$$

where $\widehat{a}$ and $\widehat{b_0}$ are constants. Replacing a and b_0 by $\widehat{a}$ and $\widehat{b_0}$, respectively, we have

$$\begin{aligned} y \sim& \widehat{a}\cos\left(T_0 + \frac{3}{8}T_1\widehat{a} + \widehat{b_0}\right) \\ &+ \frac{\epsilon}{32}\widehat{a}^3\cos\left(3T_0 + \frac{9}{8}T_1\widehat{a}^2 + 3\widehat{b_0}\right) + \mathcal{O}(\epsilon^2 t). \end{aligned}$$

Boundary Layer methods

We begin this topic with an example

Example 5.15. Consider the equation

$$\epsilon y'' + y' = 2, \quad y(0) = 0, \; y(1) = 1, \quad 0 \le x \le 1.$$

with exact solution

$$y_\epsilon(x) = 2x - \frac{(1 - \exp(-x/\epsilon))}{(1 - \exp(-1/\epsilon))}.$$

Note that as $\epsilon \to 0$ this function approaches

$$y_{\text{outer}}(x) = 2x - 1$$

which is exactly the first term in an asympotic expansion for this problem, i.e., it is the solution of $y'(x) = 2$ which satisfies the boundary condition at $x = 1$. Notice that it cannot satisfy the boundary condition at $x = 0$. Also notice that

$$y_\epsilon(x) - y_{\text{outer}}(x) = 1 - \frac{(1 - \exp(-x/\epsilon))}{(1 - \exp(-1/\epsilon))} = \frac{(\exp(-x/\epsilon) - \exp(-1/\epsilon))}{(1 - \exp(-1/\epsilon))}.$$

For $\epsilon \approx 0$ we have the right hand side is exponentially small.

At this point we seek to find a change of variables $\xi = \epsilon^p x$ so that the terms $\epsilon y''$ and y' are of the same order for small ϵ. Applying the chain rule we have

$$\frac{dy}{dx} = \frac{dy}{d\xi}\frac{d\xi}{dx} = \epsilon^p \frac{dy}{d\xi}, \quad \frac{d^2y}{dx^2} = \epsilon^{2p}\frac{d^2y}{d\xi^2}.$$

So we get

$$\epsilon^{1+2p}\frac{d^2y}{d\xi^2} + \epsilon^p\frac{dy}{d\xi} = 2.$$

For the terms on the left to be of the same order we need

$$1 + 2p = p$$

which implies that $p = -1$. In this case we can write

$$\frac{d^2y}{d\xi^2} + \frac{dy}{d\xi} = 2\epsilon.$$

If we consider the problem with $\epsilon = 0$ we have

$$\frac{d^2y}{d\xi^2} + \frac{dy}{d\xi} = 0,$$

with general solution

$$B + Ce^{-\xi}.$$

In order that this solution satisfy the boundary condition (of the original problem) at $x = 0$ we take $B = -1$ and $C = 1$,

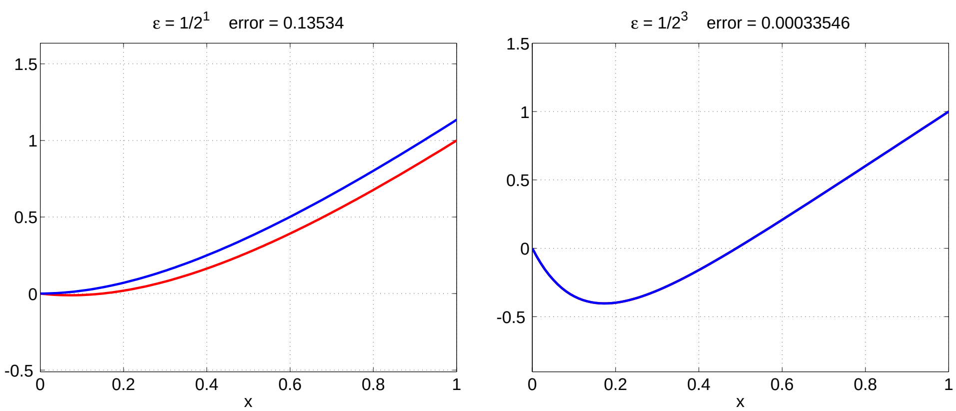
$$y_{\text{inner}}(\xi) = e^{-\xi} - 1.$$

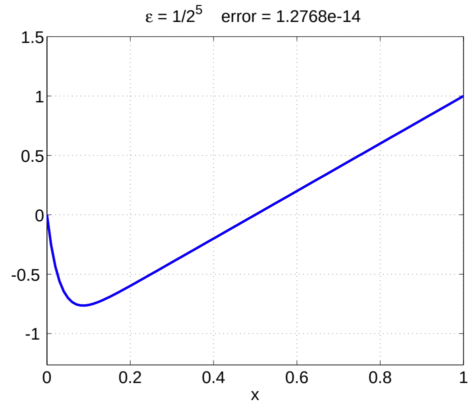
or

$$y_{\text{inner}}(x) = e^{-x/\epsilon} - 1.$$

At this point we add our *inner* and *outer* solutions and subtract the common term (-1) to get

$$y \sim y_{\text{outer}}(x) + y_{\text{inner}}(x) = (2x-1) + (e^{-x/\epsilon} - 1) - 1 = 2x - 1 + e^{-x/\epsilon}.$$





Example 5.16. Consider the equation

$$\epsilon y'' + y' + y = 0, \quad y(0) = 0, \quad y(1) = 1, \quad 0 \le x \le 1. \tag{5.5.64}$$

This is clearly a singular perturbation problem since when we set $\epsilon = 0$ we have a reduced equation of order one, namely,

$$Y' + Y = 0, \tag{5.5.65}$$

with solution

$$Y(x) = Ae^{-x}, \quad A \text{ (constant)}. \tag{5.5.66}$$

It is clear that we cannot possible satisfy both boundary conditions in (5.5.64). The exact solution to this problem is

$$y(x) = \frac{(e^{r_+ x} - e^{r_- x})}{(e^{r_+} - e^{r_-})},$$

where

$$r_{\pm} = \frac{\left(-1 \pm \sqrt{1 - 4\,\epsilon}\right)}{2\epsilon}.$$

Without using this exact solution for motivation let us proceed using a hueristic argument

Note that, in general, the solution of the constant coefficient linear homogeneous ODE in (5.5.64) is a sum of two linearly independent solutions. The solution we have in (5.5.66) which can satisfy the BC at $x = 1$ also has the following properties:

1. $\epsilon Y''$ is uniformly small compared to Y' and Y (Not to their difference which is zero) ,

2. the domain of x is finite, $[0, 1]$.

Thus we might expect that any other independent solution y might have the property that $\epsilon y''$ have the same order of at least one of y' or y.

Motivated by what we would do with a polynomial of the form

$$\epsilon z^2 + z + 1$$

(namely, introduce $z = \epsilon^{-1}w$) we proceed by introducing new independent variable $\xi = \epsilon^{-1}x$. By the chain rule we have

$$\frac{dy}{dx} = \frac{dy}{d\xi}\frac{d\xi}{x} = \frac{1}{\epsilon}\frac{dy}{d\xi}, \quad \text{and} \quad \frac{d^2y}{dx^2} = \frac{1}{\epsilon^2}\frac{d^2y}{d\xi^2},$$

and we can write (5.5.64) as

$$\ddot{y} + \dot{y} + \epsilon y = 0, \quad y(0) = 0, \;\; y(\epsilon^{-1}) = 1, \quad 0 \le \xi \le \epsilon^{-1}. \tag{5.5.67}$$

We seek a solution of the transformed equation such that ϵy is small compared to $\ddot{y}$ and $\dot{y}$. This leads to the equation

$$\ddot{W} + \dot{W} = 0$$

whose general ,solution is

$$W(\xi) = B + Ce^{-\xi} = B + Ce^{-x/\epsilon} \tag{5.5.68}$$

We need to set $B = 0$ since otherwise ϵB would not be small (as we assumed) compared to $\ddot{B}$ and $\dot{B}$ which are zero. If we now add our two solutions Y and W we have

$$y \sim Ae^{-x} + Ce^{-x/\epsilon}$$

which we hope will lead to a good approximation of our solution. Imposing the BC's we have

$$\begin{array}{l} A + C = 0 \\ Ae^{-1} + Ce^{-1/\epsilon} = 1 \end{array}.$$

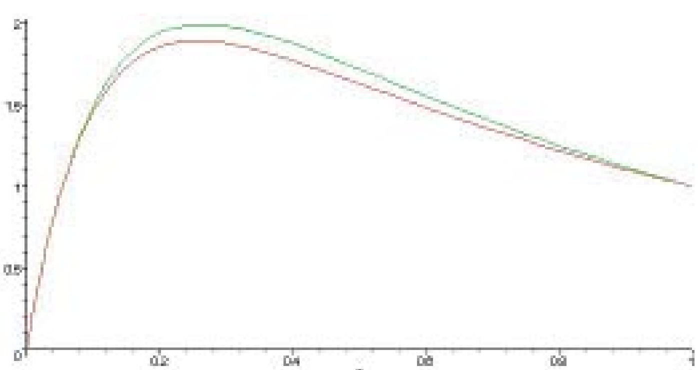
The first equation implies $C = -A$ and then from the second we get

$$A = \frac{1}{(e^{-1} + e^{-1/\epsilon})} = \frac{e}{(1 + e^{1-1/\epsilon})}.$$

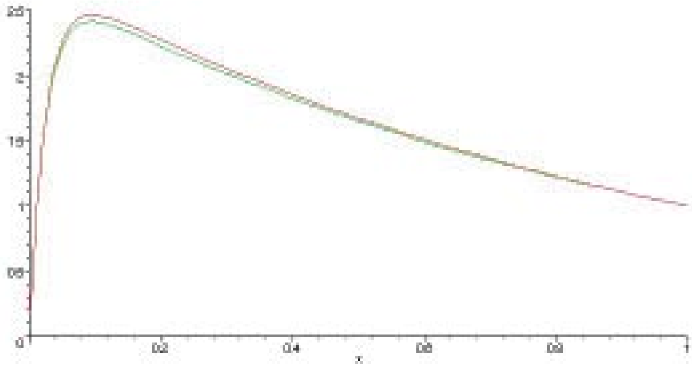
Thus we obtain

$$y(x) \sim \frac{\left(e^{1-x} - e^{1-x/\epsilon}\right)}{(1 - e^{1-1/\epsilon})} = \left(e^{1-x} - e^{1-x/\epsilon}\right) + \mathcal{O}(\epsilon^{-1/\epsilon}). \tag{5.5.69}$$

The term $e^{-x/\epsilon}$ is called a **Boundary Layer** because it is significant only in a very narrow layer of width $\mathcal{O}(\epsilon)$ near $x = 0$. The other function e^{-x} approximates the solution outside the boundary layer.



Exact and Asymptotic solutions for $\epsilon = .1$



Exact and Asymptotic solutions for $\epsilon = .025$

It turns out that boundary value problems are, in some sense, more difficult to study then initial value problems. namely there is not such a simple fundamental existence, uniqueness theory for BVPs as there is for IVPs. At this point we digress and present some results concerning existence and uniqueness of solutions to BVPs.

Existence Theorems for Boundary Value Problems

First define a nonlinear two-point boundary value problem:

$$\begin{aligned} y'' &= f(x, y, y') \qquad , \quad x \in (a,b) &(5.5.70)\\ a_0 y(a) - a_1 y'(a) &= \alpha \qquad , |a_0| + |a_1| \neq 0 &(5.5.71)\\ b_0 y(b) - b_1 y'(b) &= \alpha \qquad , |b_0| + |b_1| \neq 0 &(5.5.72)\end{aligned}$$

Definition 5.1. *$f(x, \vec{u})$ is Lipschitz in u_j, $\vec{u} = (u_1, u_2, \ldots, u_n)$, with Lipschitz constant K_j if*

$$|f(x, u_1, \ldots, u_j, \ldots, u_n) - f(x, u_1, \ldots, v_j, \ldots, u_n)| < K_j |u_j - v_j| \qquad (5.5.73)$$

If (5.5.73) is satisfied uniformly in u_j for $j = 1, 2, \ldots n$, then $f(x, \vec{u})$ is Lipschitz in $\vec{u}$ with Lipschitz constant

$$K = \left(\sum_{j=1}^{n} K_j^2 \right)^{1/2} \qquad (5.5.74)$$

when the Euclidean norm is chosen.

Theorem 5.3. *Let $f(x, u_1, u_2)$ be continuous on $D = [a, b] \times \mathbb{R}^2$ and be Lipschitz in $\vec{u}$ uniformly with Lipschitz constant K. Also, assume f_{u_1}, f_{u_2} are continuous on D, $f_{u_1} > 0$ on D, and there exists a constant $M > 0$ such that*

$$\left| \frac{\partial f}{\partial u_2} \right| \le M \qquad , \quad (x, \vec{u}) \in D \qquad (5.5.75)$$

If $a_0 a_1 \ge 0$,$b_0 b_1 \ge 0$ and $|a_0| + |b_0| \neq 0$ then there exists a unique solution to (5.5.70)-(5.5.72).

Corollary 5.1. *Let $p, q, r \in C_0[a, b]$, $q(x) > 0$ on $[a, b]$. Then there exists a unique solution $y(x)$ of the boundary value problem*

$$\begin{aligned} L[y] = -y'' + p(x)y' + q(x)y &= r(x) \qquad , \quad x \in (a, b) && (5.5.76) \\ a_0 y(a) - a_1 y'(a) &= \alpha \qquad , |a_0| + |a_1| \neq 0 && (5.5.77) \\ b_0 y(b) - b_1 y'(b) &= \alpha \qquad , |b_0| + |b_1| \neq 0 && (5.5.78) \end{aligned}$$

providing $a_0 a_1 \geq 0$,$b_0 b_1 \geq 0$ and $|a_0| + |b_0| \neq 0$.

Theorem 5.4. *(Alternative) Let p, q, r and L be as in the previous corollary and define the two problems:*

$$\begin{aligned} L[y] &= r(x) \quad , y(a) = \alpha \quad , y(b) = \beta && (5.5.79) \\ L[y] &= 0 \quad , y(a) = 0 \quad , y(b) = 0 && (5.5.80) \end{aligned}$$

then (5.5.79) has unique solution if and only if the only solution to (5.5.80) is $y \equiv 0$.

The text here was taken from the homepage of Mark C. Pernarowski, Department of Mathematics, Montana State University, where the results were attributed to H. Keller [8].

Matching: Theory, Definition and Issues

Let $D = [0, 1]$, $I = (0, \epsilon_1)$ and $y(x, \epsilon)$ be continuous on $D \times I$. Furthermore, suppose that there are $y_k(x)$ such that the outer expansion

$$y \sim y_0(x) + \epsilon y_1(x) + \epsilon^2 y_2(x) + \cdots \qquad (5.5.81)$$

is uniformly valid on $[\bar{x}, 1], \bar{x} > 0$, as $\epsilon \to 0^+$. Also, for the inner variable

$$X = \frac{x}{\epsilon} \qquad (5.5.82)$$

suppose there exist $Y_k(X)$ such that

$$y \sim Y_0(X) + \epsilon Y_1(X) + \epsilon^2 Y_2(X) + \cdots \qquad (5.5.83)$$

uniformly on $[0, \bar{X}]$, for some $\bar{X} < 1/\epsilon_1$, as $\epsilon \to 0^+$. That is to say, we are supposing that we have both an inner and outer expansion for the same function $y(x, \epsilon)$. The function $y(x, \epsilon)$ should be viewed as a solution of the algebraic problem

$$f(x, y, \epsilon) = 0 \qquad (5.5.84)$$

or a boundary-value problem like

$$\begin{aligned} L_\epsilon[y] \equiv \epsilon \frac{d^2 y}{dx^2} + a(x)\frac{dy}{dx} + b(x) &= f(x, \epsilon) \qquad x \in (0, 1) \quad , && (5.5.85) \\ y(0, \epsilon) = A \qquad , \qquad y(1, \epsilon) &= B && (5.5.86) \end{aligned}$$

For the "outer limit" in (5.5.81), x is fixed. For the "inner limit" in (5.5.83), X is fixed.

For the purpose of clarity we will let $D_o(\bar{x})$ and $D_i(\bar{X})$ denote the regions of uniform validity for the outer and inner expansions, respectively:

$$D_o(\bar{x}) = \{(x, \epsilon) : x \in [\bar{x}, 1], \epsilon \in I\} \tag{5.5.87}$$

$$D_i(\bar{X}) = \{(x, \epsilon) : x \in [0, \bar{X}\epsilon], \epsilon \in I\} \tag{5.5.88}$$

Though, as defined, $D_o(\bar{x})$ and $D_i(\bar{X})$ depend on the (ϵ-independent) fixed values $\bar{x}$ and $\bar{X}$, these values will be seen to be irrelevant to the latter discussions of overlap regions and matching. Henceforth, we will denote these regions simply as D_o and D_i. *Extension theorems* are theorems which extend the region of uniformity of asymptotic statements like (5.5.81). One early (and relatively simple) theorem is due to Kaplan (1967):

Theorem 5.5. *Let* $D = [0, 1]$, $I = (0, \epsilon_1)$ *and* $y(x, \epsilon)$ *be continuous on* $D \times I$. *Also, let* $y_0(x)$ *be some continuous function on* $(0, 1]$ *such that*

$$\lim_{\epsilon \to 0^+} [y(x, \epsilon) - y_0(x)] = 0 \tag{5.5.89}$$

uniformly on $[\bar{x}, 1]$*, for every* $\bar{x} > 0$*. Then there exists a function* $0 < \delta(\epsilon) \ll 1$ *such that*

$$\lim_{\epsilon \to 0^+} [y(x, \epsilon) - y_0(x)] = 0 \tag{5.5.90}$$

uniformly on $[\delta(\epsilon), 1]$.

(see Eckhaus (1979) for more theorems). There are clearly examples of functions satisfying the hypothesis of this theorem. For example,

$$y(x, \epsilon) = x + e^{x/\epsilon} + \epsilon \qquad , \qquad y_0(x) = x \tag{5.5.91}$$

Moreover, the limit (5.5.89) implies $y(x, \epsilon) \sim y_0(x) + o(1)$ uniformly on $[\bar{x}, 1]$.

What this theorem does is effectively extend the region of uniform validity D_o to one like $\hat{D}_o$. To more carefully define $\hat{D}_o$, *intermediate variables* need to be introduced. Let $\eta(\epsilon)$ be any function with $0 < \eta(\epsilon) \ll 1$. We define the intermediate variable x_η by

$$x = \eta(\epsilon) x_\eta \tag{5.5.92}$$

Then, the conclusion of the theorem may be stated

$$\lim_{\epsilon \to 0^+, x_\eta \ fixed} [y(\eta x_\eta, \epsilon) - y_0(\eta x_\eta)] = 0 \tag{5.5.93}$$

uniformly on $x_\eta \in [\bar{x}_\eta, 1]$, for all η with $\delta = O(\eta)$. Generally, when introducing intermediate variables we view η as satisfying $\delta \ll \eta \ll 1$, though to clearly define $\hat{D}_o$ we can set η equal to δ or 1:

$$\hat{D}_o(\bar{x}_\eta) = \{(x, \epsilon) : x \in [\bar{x}_\eta \delta(\epsilon), 1], \epsilon \in I\} \tag{5.5.94}$$

For the example in (5.5.91), we have for some intermediate variable x_η:

$$y(x, \epsilon) - y_0(x) = e^{\frac{-x_\eta \eta}{\epsilon}} + \epsilon = o(1)$$

uniformly on $[\bar{x}_\eta, 1]$ providing $\bar{x}_\eta > 0$ and $\epsilon \ll \eta$. For instance, one could choose $\delta(\epsilon) = \epsilon^{1/2}$ in the theorem.

In an analogous fashion, one can construct an extended domain of validity $\hat{D}_i$ for the inner expansion (5.5.83) noting the inner variable

$$X = \frac{\eta x_\eta}{\epsilon} \tag{5.5.95}$$

For some (x, ϵ) near $(0, 0)$ the non-extended domains D_o and D_i do not *overlap* (do not intersect regardless of the choices of $\bar{x}, \bar{X}$). Similarly, one can have nonoverlapping extended domains and overlapping extended domains. If there is an overlapping extended domain, there are functions $\eta_i(\epsilon)$ and $\eta_o(\epsilon)$ such that for any intermediate variable x_η with $\eta_i(\epsilon) \ll \eta(\epsilon) \ll \eta_o(\epsilon)$ both the inner and outer expansions are uniformly valid. That is to say, given any η with $\eta_i(\epsilon) \ll \eta(\epsilon) \ll \eta_o(\epsilon)$, there is an ϵ-independent interval I_η such that both

$$\lim_{\epsilon \to 0^+, x_\eta \; fixed} [y(\eta x_\eta, \epsilon) - y_0(\eta x_\eta)] = 0 \tag{5.5.96}$$

$$\lim_{\epsilon \to 0^+, x_\eta \; fixed} \left[y(\eta x_\eta, \epsilon) - Y_0\left(\frac{\eta x_\eta}{\epsilon}\right)\right] = 0 \tag{5.5.97}$$

uniformly on $x_\eta \in I_\eta$, $x_\eta > 0$. Subtracting these expressions we have obtain a matching condition:

$$\lim_{\epsilon \to 0^+, x_\eta \; fixed} \left[y_0(\eta x_\eta) - Y_0\left(\frac{\eta x_\eta}{\epsilon}\right)\right] = 0 \tag{5.5.98}$$

And, if $y_0(0^+)$ and $Y_0(\infty)$ exist, since $\epsilon \ll \eta \ll 1$,

$$\lim_{x \to 0^+} y_0(x) = \lim_{X \to \infty} Y_0(X) \tag{5.5.99}$$

which is the *Prandtl matching condition*. If (5.5.98) can be satisfied, then one would say that the leading outer expansion $y_0(x)$ can be matched to the leading inner expansion $Y_0(X)$ on an overlap domain

$$\bar{D}_0 = \{(x, \epsilon) : x_\eta = x\eta \in I_\eta, \eta_i(\epsilon) \ll \eta(\epsilon) \ll \eta_o(\epsilon)\} \tag{5.5.100}$$

At this stage, we need to make a few points. Firstly, $y_0(0^+)$ or $Y_0(\infty)$ may not exist in which case the inner and outer expansions cannot be matched to leading-order using the Prandtl matching condition. However, it may still be possible to match the expansions by demonstrating the existence of an overlap domain for which (5.5.98) is satisfied. Secondly, even if the matching condition (5.5.98) cannot be satisfied that does not preclude the possibility of a P term outer expansion matching a Q term inner expansion. That is to say, there may be some overlap domain where

$$\lim_{\epsilon \to 0^+, x_\eta \; fixed} \left[\sum_{n=0}^{P} \epsilon^n y_n(x_\eta \eta) - \sum_{n=0}^{Q} \epsilon^n Y_n\left(\frac{x_\eta \eta}{\epsilon}\right)\right] = 0 \tag{5.5.101}$$

At this point we are in a position to define matching.

Definition *Choose and fix* $x_\eta = \frac{x}{\eta(\epsilon)} \in \mathbb{R}$ *and let* R *be any nonnegative integer. We say that the outer and inner expansions defined in (5.5.81)-(5.5.83) match to* $O(\epsilon^R)$ *on a common domain of validity* $\bar{D}_R(x_\eta)$ *if there exist functions* η_1 *and* η_2 *with* $\eta_1 \ll \eta_2$ *and integers* P, Q *such that*

$$\lim_{\epsilon \to 0^+, x_\eta \ fixed} \frac{M_{PQ}}{\epsilon^R} = \lim_{\epsilon \to 0^+, x_\eta \ fixed} \left[\frac{\displaystyle\sum_{n=0}^{P} \epsilon^n y_n(x_\eta \eta) - \sum_{n=0}^{Q} \epsilon^n Y_n\left(\frac{x_\eta \eta}{\epsilon}\right)}{\epsilon^R} \right] = 0 \qquad (5.5.102)$$

for any function η *satisfying* $\eta_1 \ll \eta \ll \eta_2$ *and*

$$\bar{D}_R(x_\eta) = \{(x, \epsilon) : x_\eta = x\eta, \eta_1(\epsilon) \ll \eta(\epsilon) \ll \eta_2(\epsilon)\} \qquad (5.5.103)$$

We conclude with a few remarks:

1) General theorems showing the existence of overlap domains have not been found (Lagerstrom 1988). In practice, the existence of overlap domains where inner and outer solutions can be matched is done on a case by case basis.

2) For boundary value problems where the method of matched asymptotics is applied, matching conditions are used to find integration constants occurring in the inner expansion. Typically, inner and outer expansions can be matched only if those constants are chosen equal to specific values.

3) Prandtl matching corresponds to leading-order matching with $P = Q = R = 0$.

4) In some problems, P and Q may not be known apriori. Moreover, P may not equal Q.

5) Some expansions cannot be matched. The matching defined in (5.5.102) is with respect to the guage functions $\phi_n(\epsilon) = \epsilon^n, n \geq 0$. Clearly, some functions y may have more general outer expansions:

$$y(x, \epsilon) \sim \sum_{n \geq 0} \phi_n(\epsilon) y_n(x) \qquad (5.5.104)$$

Indeed, the inner variable could be defined in a more general way, $X = x/\delta(\epsilon), 0 < \delta \ll 1$, and the inner expansion may be with respect to different guage functions. These sorts of generalizations are not normally considered.

Example:

In this section we consider a single example which illustrates all of the features discussed in the previous section. We will use the following facts throughout the discussion: If $0 < \delta(\epsilon)$, $x > 0$

$$|\log(\epsilon)| \ll \delta \;\Rightarrow\; e^{-\delta} \ll \epsilon^n \quad \forall n > 0 \tag{5.5.105}$$

$$\delta = O_s(|\log(\epsilon)|) \;\Rightarrow\; e^{-\delta} = O_s(1) \tag{5.5.106}$$

$$x \ll \epsilon|\log(\epsilon)| \;\Rightarrow\; e^{-x/\epsilon} \ll \epsilon^n \quad \forall n > 0 \tag{5.5.107}$$

where $\phi = O_s(\psi)$ means $\phi = O(\psi)$ and $\psi = O(\phi)$.

Specifically we will consider matching of inner and outer expansions of the function

$$y(t, \epsilon) = \frac{1}{\sqrt{1-4\epsilon}} \left\{ \exp\left[-(1 - \sqrt{1-4\epsilon}) \frac{t}{2\epsilon} \right] - \exp\left[-(1 + \sqrt{1-4\epsilon}) \frac{t}{2\epsilon} \right] \right\} \tag{5.5.108}$$

which is the solution of the singular initial value problem

$$\epsilon y'' + y' + y = 0 \quad , \tag{5.5.109}$$

$$y(0, \epsilon) = 0 \quad , \quad y'(0, \epsilon) = \frac{1}{\epsilon} \tag{5.5.110}$$

The first two terms of the outer expansion

$$y(t, \epsilon) \sim y_0(t) + \epsilon y_1(t) + \epsilon^2 y_2(t) + \cdots \tag{5.5.111}$$

can easily be determined from (5.5.108). Fixing t and expanding in ϵ one finds

$$y = (1 + 2\epsilon + O(\epsilon^2)) \left[e^{-t-\epsilon t + O(\epsilon^2)} - e^{-t/\epsilon + t + O(\epsilon)} \right] \tag{5.5.112}$$

from which we deduce

$$y_0(t) = e^{-t} \quad , \quad y_1(t) = (2 - t)e^{-t} \tag{5.5.113}$$

Similarly, to compute the inner expansion

$$y(t, \epsilon) = Y(T, \epsilon) \sim Y_0(T) + \epsilon Y_1(T) + \epsilon^2 Y_2(T) + \cdots \quad , \quad T = \frac{t}{\epsilon} \tag{5.5.114}$$

reexpress (5.5.108) in terms of T, fix T and then expand in ϵ:

$$Y = (1 + 2\epsilon + O(\epsilon^2)) \left[e^{-\epsilon T - \epsilon^2 T + O(\epsilon^3)} - e^{-T + \epsilon T + O(\epsilon^2)} \right] \tag{5.5.115}$$

From this one finds:

$$Y_0(T) = 1 - e^{-T} \quad , \quad Y_1(T) = (2 - T) - (2 + T)e^{-T} \tag{5.5.116}$$

Before we find the overlap domains where the outer and inner expansions match to $O(1)$ and $O(\epsilon)$, we will discuss how these expansions would arise had we not know the exact solution *apriori*.

By substituting the expansion (5.5.111) into (5.5.109) we obtain the problems:

$$O(1) \quad : \quad y_0' + y_0 = 0 \tag{5.5.117}$$
$$O(\epsilon) \quad : \quad y_1' + y_1 = -y_0'' \tag{5.5.118}$$

whose general solutions are (for a_0, b_0 constant)

$$y_0(t) = a_0 e^{-t} \tag{5.5.119}$$
$$y_1(t) = (b_0 - a_0 t)e^{-t} \tag{5.5.120}$$

Clearly, a_0 cannot be chosen so that $y_0(t)$ satisfy both initial conditions. Therefore, there must be a layer at $t = 0$. In terms of Y and T the initial value problem (5.5.109)-(5.5.110) can be written

$$Y'' + Y' + \epsilon Y = 0 \quad , \tag{5.5.121}$$

$$Y(0,\epsilon) = 0 \quad , \quad Y'(0,\epsilon) = 1 \tag{5.5.122}$$

from which we obtain the inner problems

$$O(1) \quad : \quad Y_0'' + Y_0' = 0 \quad , \quad Y_0(0) = 0 \quad , \quad Y_0'(0) = 1 \tag{5.5.123}$$
$$O(\epsilon) \quad : \quad Y_1'' + Y_1' = -Y_0 \quad , \quad Y_1(0) = 0 \quad , \quad Y_1'(0) = 0 \tag{5.5.124}$$

whose solutions are that given in (5.5.116). In contrast to boundary value problems, the unknown constants of integration to be determined from matching are part of the outer solution. If we apply Prandtl matching to match y_0 and Y_0 we find

$$\lim_{t \to 0^+} y_0(t) = a_0 = 1 = \lim_{T \to \infty} Y_0(T) \tag{5.5.125}$$

and recover $y_0(t)$ in (5.5.113).

Demonstrating extended outer domains to $O(1)$

To find an extended domain for the outer expansion one assumes $\eta(\epsilon) \ll 1$ and seeks an $\eta_1(\epsilon)$ such that $\eta_1(\epsilon) \ll \eta(\epsilon)$ implies

$$\lim_{\epsilon \to 0^+, t_\eta \ fixed} [y(\eta t_\eta, \epsilon) - y_0(\eta t_\eta)] = 0 \tag{5.5.126}$$

for the intermediate variable

$$t_\eta = \frac{t}{\eta} > 0 \tag{5.5.127}$$

Given (5.5.112), this limit holds providing $e^{-t_\eta \eta/\epsilon} \ll 1$. To assure this, we choose $\eta_1(\epsilon) = \epsilon|log(\epsilon)|$. Now let the notation $\phi \ll= \psi$ mean that either $\phi \ll \psi$ or $\phi = O_s(\psi)$. Then we can conclude that $\bar{D}_o$ will be an extended domain for the outer expansion so long as η satisfies

$$\eta_{1,0} \equiv \epsilon|log(\epsilon)| \ll \eta \ll= 1 \tag{5.5.128}$$

Though each η defines a different region in the (x,ϵ)-plane for $t_\eta \in I_\eta$, all that really matters for the limit to vanish is that η satisfy (5.5.128). So it is common practice to say that the extended domain for the single term outer expansion $y_0(t)$. "is" (5.5.128).

Demonstrating extended outer domains to $O(\epsilon)$

To find the extended domain for the two term outer expansion $y_0(t) + \epsilon y_1(t)$ one assumes $\eta(\epsilon) \ll 1$ and seeks an $\eta_{1,1}(\epsilon)$ such that $\eta_{1,1}(\epsilon) \ll \eta(\epsilon)$ implies

$$\lim_{\epsilon\to 0^+, t_\eta\ fixed} \frac{[y(\eta t_\eta,\epsilon) - y_0(\eta t_\eta) - \epsilon y_1(\eta t_\eta)]}{\epsilon} = 0 \tag{5.5.129}$$

Again from (5.5.112), we find that if η satisfies (5.5.128) the above limit holds. That is to say the choice $\eta_{1,1} = \eta_{1,0}$ works. If we continue this process of extending the domain in an R term outer expansion to find $\eta_{1,R}$ it is often the case that $\eta_{1,R} \ll \eta_{1,R+1}$ since adding more terms to the limit places more restrictions on η. For this particular example the extended outer domains at $O(1)$ and $O(\epsilon)$ turned out to be the same.

Demonstrating extended inner domains to $O(1)$ and $O(\epsilon)$

To find an extended domain for the single term inner expansion one assumes $\epsilon \ll \eta(\epsilon)$ and seeks an $\eta_2(\epsilon)$ such that $\eta \ll \eta_2(\epsilon)$ implies

$$\lim_{\epsilon\to 0^+, t_\eta\ fixed} [y(\eta t_\eta,\epsilon) - Y_0(\eta t_\eta/\epsilon)] = 0 \tag{5.5.130}$$

Again from (5.5.112) it is easy to verify that the extended domain for the single term inner expansion is defined by

$$\epsilon \ll= \eta \ll \eta_{2,0} \equiv 1 \tag{5.5.131}$$

Finding the extended inner domain to $O(\epsilon)$ is more delicate. In terms of the intermediate variables

$$\frac{y(\eta t_\eta,\epsilon)}{\epsilon} = \frac{1}{\epsilon} - \frac{e^{-t_\eta \eta/\epsilon}}{\epsilon} - \frac{\eta}{\epsilon} t_\eta - \frac{\eta}{\epsilon} t_\eta e^{-t_\eta\eta/\epsilon} + 2 - 2e^{-t_\eta\eta/\epsilon}$$
$$+ O(\eta) + O\left(\frac{\eta^2}{\epsilon}\right) + O(\epsilon)$$

and in terms of the intermediate variables

$$\frac{1}{\epsilon} Y_0 + Y_1 = \frac{y(\eta t_\eta,\epsilon)}{\epsilon} = \frac{1}{\epsilon} - \frac{e^{-t_\eta \eta/\epsilon}}{\epsilon} - \frac{\eta}{\epsilon} t_\eta - \frac{\eta}{\epsilon} t_\eta e^{-t_\eta\eta/\epsilon} + 2 - 2e^{-t_\eta\eta/\epsilon} \tag{5.5.132}$$

Subtracting these two expressions we see that

$$\lim_{\epsilon\to 0^+, t_\eta\ fixed} \frac{[y(\eta t_\eta,\epsilon) - Y_0(\eta t_\eta/\epsilon) - \epsilon Y_1(\eta t_\eta/\epsilon)]}{\epsilon} = 0 \tag{5.5.133}$$

provided $\eta^2/\epsilon \ll 1$. That is to say the choice $\eta_{2,1} = \epsilon^{1/2}$ ensures the limit vanishes and the extended inner domain to $O(\epsilon)$ is

$$\epsilon \ll= \eta \ll \eta_{2,1} \equiv \epsilon^{1/2} \tag{5.5.134}$$

Here we note the extended domain to $O(\epsilon)$ is "smaller" than the domain to $O(1)$, i.e. $\eta_{2,1} \ll \eta_{2,0}$.

Demonstrating overlap to $O(1)$ and $O(\epsilon)$

Considering the previous discussions it is clear to see that the overlap domains to $O(1)$ and $O(\epsilon)$ are, respectively,

$$\eta_{1,0} \ll \eta \ll \eta_{2,0} \tag{5.5.135}$$
$$\eta_{1,1} \ll \eta \ll \eta_{2,1} \tag{5.5.136}$$

or

$$\epsilon|\log(\epsilon)| \ll \eta \ll 1 \tag{5.5.137}$$
$$\epsilon|\log(\epsilon)| \ll \eta \ll \epsilon^{1/2} \tag{5.5.138}$$

If η satisfies these asymptotic relations, the outer and inner expansions match to $O(1)$ and $O(\epsilon)$, respectively. Explicitly, if η satisfies (5.5.135) then

$$\lim_{\epsilon \to 0^+, t_\eta\ fixed} [y_0(\eta t_\eta) - Y_0(\eta t_\eta/\epsilon)] = 0 \tag{5.5.139}$$

And, if η satisfies the more stringent requirement (5.5.136)

$$\lim_{\epsilon \to 0^+, t_\eta\ fixed} \frac{[y_0(\eta t_\eta) + \epsilon y_1(\eta t_\eta) - Y_0(\eta t_\eta/\epsilon) - \epsilon Y_1(\eta t_\eta/\epsilon)]}{\epsilon} = 0 \tag{5.5.140}$$

If the exact solution y was not known *apriori* then one would choose a_0 in the incomplete outer solution $y_0(t) = a_0 e^{-t}$ and find $\eta_{1,0}, \eta_{2,0}$ so that (5.5.139) is satisfied.

1. Consider the problem

$$y'' + \epsilon y' + y = 0, \quad y(0) = 1, \quad y'(0) = 0. \tag{*}$$

   (a) Find the exact solution to this initial value problem.

   (b) Obtain a two term regular asymptotic expansion for (*).

   (c) Compare graphically your answer for $\epsilon = .25$ and $\epsilon = .1$ with the exact answer for $0 \le x \le 2$.

2. Obtain a two term regular expansion for $y'' + 2y = e^{\epsilon x}$, $y(0) = y(1) = 0$.

3. Obtain a two term regular expansion for $(1 + \epsilon x^2)y'' + y = x^2$, $y(0) = \epsilon$, $y(1) = 1$.

4. Find a two term expansion for $y'' = \sin(x)\, y$, $y(0) = 1$, $y'(0) = 1$ using the method of successive integration from Example 5.4.

5. Find a regular expansion for the system $\begin{cases} \dot{x} = x - 2y + \epsilon xy \\ \dot{y} = x - 3y - \epsilon xy \end{cases}$ Is the expansion valid for all $t \ge 0$? Give a reason.

6. Use the method of strained variables to obtain a two term expansion for

$$(x + \epsilon y)\frac{dy}{dx} + y = 0, \quad y(1) = 1.$$

Also find the exact solution and compare graphically your results and the exact solution on the interval $[0, 2]$ for $\epsilon = .1$.

7. Use the method of averaging to find an approximate periodic solution to the Van der Pol oscillator

$$\ddot{y} + y + \epsilon(y^2 - 1)\,\dot{y} = 0,$$

i.e., find an approximation $y(t) \sim a(t)\cos(t + \theta(t)), \quad a(0) = a_0, \;\; \Theta(0) = \Theta_0$.

8. Find a uniform asymptotic approximation to the boundary layer problem, i.e., find an inner, outer and matched solution.

$$\epsilon y'' + (1 + \epsilon)y' + y = 0, \quad y(0) = 0, \quad y(1) = 1.$$

Also compute the exact solution and graphically compare your answers for $\epsilon = .1$ and $\epsilon = .025$.

9. Find a uniform asymptotic approximation to the boundary layer problem, i.e., find an inner, outer and matched solution.

$$\epsilon y'' + 2y' + e^y = 0, \quad y(0) = 0, \quad y(1) = 0$$

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