Perturbation Theory

Perturbation theory is a problem-solving method which is applicable in situations in which we know the solution to a certain problem and now want to solve a new problem which is very close to the first—specifically, which is obtained from the first by making a small change in some parameter. In this case the original problem is called the *unperturbed* problem and the small change is a *perturbation*.

Section 1: Roots of polynomials

In this section we take up one of the simplest perturbation problems: we want to determine how the roots of a polynomial change when the coefficients of the polynomial are perturbed.

1.1 Introduction Let us consider first a simple example.

Example 1: Suppose that we want to find the roots of the polynomial

$$x^3 - 3x^2 + 2x + 0.01. \tag{1}$$

We think of this polynomial as obtained by a small change of the simpler polynomial $x^3 - 3x^2 + 2x$, whose roots we can find easily: the constant term is changed from 0 to $\varepsilon_0 = 0.01$. The idea of perturbation theory is to consider this change as arising from the introduction of a new *variable* ε, to study the problem for general ε, and then to specialize to $\varepsilon = \varepsilon_0$.

We therefore study the roots of

$$P_\varepsilon(x) = x^3 - 3x^2 + 2x + \varepsilon.$$

The unperturbed polynomial $P_0(x) == x^3 - 3x^2 + 2x$ has roots $x_1 = 0$, $x_2 = 1$, and $x_3 = 2$, and it can be proved that the roots of the perturbed polynomial $P_\varepsilon(x)$ will, for small values of ε, be analytic functions of ε which approach x_1, x_2, and x_3 as ε goes to 0:

$$x_1(\varepsilon) = x_1 + a_1\varepsilon + a_2\varepsilon^2 + a_3\varepsilon^3 + a_4\varepsilon^4 + \cdots = a_1\varepsilon + a_2\varepsilon + \cdots \tag{2a}$$
$$x_2(\varepsilon) = x_2 + b_1\varepsilon + b_2\varepsilon^2 + b_3\varepsilon^3 + b_4\varepsilon^4 + \cdots = 1 + b_1\varepsilon + b_2\varepsilon + \cdots \tag{2b}$$
$$x_3(\varepsilon) = x_3 + c_1\varepsilon + c_2\varepsilon^2 + c_3\varepsilon^3 + c_4\varepsilon^4 + \cdots = 2 + c_1\varepsilon + c_2\varepsilon + \cdot \tag{2c}$$

To obtain the coefficients in these series we substitute them into $P_\varepsilon(x)$ and collect powers of ε. For example

$$\begin{aligned} P_\varepsilon(x_1(\varepsilon)) &= (a_1\varepsilon + a_2\varepsilon^2 + a_3\varepsilon^3 + a_4\varepsilon^4 + \ldots)^3 \\ &\qquad - 3(a_1\varepsilon + a_2\varepsilon^2 + a_3\varepsilon^3 + a_4\varepsilon^4 + \ldots)^2 \\ &\qquad + 2(a_1\varepsilon + a_2\varepsilon^2 + a_3\varepsilon^3 + a_4\varepsilon^4 + \ldots) + \varepsilon \\ &= (2a_1 + 1)\varepsilon + (2a_2 - 3a_1^2)\varepsilon^2 + (2a_3 + a_1^3 - 6a_1a_2)e^3 \\ &\qquad + (2a_4 - 3a_2^2 - 6a_1a_3 + 3a_1^2a_2)\varepsilon^4 + \cdots \end{aligned} \tag{3}$$

Notice that there is no constant term in the second line of (3); this is because the constant term in (2a), $x_1 = 0$, was already a root of P_0. If (3) is to vanish for all ε then the coefficient of each power of ε must vanish:

$$\begin{array}{llcl}
\varepsilon: & 2a_1 + 1 = 0 & \Rightarrow & a_1 = -\frac{1}{2} \\
\varepsilon^2: & 2a_2 - 3a_1 = 0 & \Rightarrow & a_2 = \frac{3}{2}a_1^2 = \frac{3}{8} \\
\varepsilon^3: & 2a_3 + a_1^3 - 6a_1a_2 = 0 & \Rightarrow & a_3 = \frac{1}{2}(6a_1a_2 - a_1^3) = -\frac{1}{2} \\
\varepsilon^4: & 2a_4 - 3a_2^2 - 6a_1a_3 + 3a_1^2a_2 = 0 & \Rightarrow & a_4 = \frac{1}{2}(6a_1a_3 + 3a_2^2 - 3a_1^2a_2) = \frac{105}{128}
\end{array}$$

Obviously one can continue to find as many terms as one likes; Maple tells us that

$$x_1(\varepsilon) = -\frac{1}{2}\varepsilon + \frac{3}{8}\varepsilon^2 - \frac{1}{2}\varepsilon^3 + \frac{105}{128}\varepsilon^4 - \frac{3}{2}\varepsilon^5 + \frac{3003}{1024}\varepsilon^6 - 6\varepsilon^7 + \frac{415701}{32768}\varepsilon^8 - \frac{55}{2}\varepsilon^9 + \cdots \quad \text{(4a)}$$

One determines the series for the other roots similarly:

$$x_2(\varepsilon) = 1 + \varepsilon + \varepsilon^3 + 3\varepsilon^5 + 12\varepsilon^7 + 55\varepsilon^9 + \cdots \quad \text{(4b)}$$
$$x_3(\varepsilon) = -\frac{1}{2}\varepsilon - \frac{3}{8}\varepsilon^2 - \frac{1}{2}\varepsilon^3 - \frac{105}{128}\varepsilon^4 - \frac{3}{2}\varepsilon^5 - \frac{3003}{1024}\varepsilon^6 - 6\varepsilon^7 - \frac{415701}{32768}\varepsilon^8 - \frac{55}{2}\varepsilon^9 - \cdots \quad \text{(4c)}$$

As we will see below, what is important here is that the roots of the unperturbed polynomial were *simple* and *finite*. In the perturbed polynomial, such roots will always change slightly in a way which is given by a power series of the form (2), that is, they will be analytic functions of the perturbation parameter ε. The situation for a multiple root of the unperturbed polynomial (that is, a root x_0 for which $P_0(x)$ has a factor $(x - x_0)^k$ with $k \geq 2$), or for an infinite root (to be explained below) is more complicated, and we take it up in the next subsections.

Remark 1: (a) If we had originally wanted to study the roots of a polynomial in which several coefficients were perturbed, say of $x^3 - 3.04x^2 + 1.98x + 0.01$ rather than of (1), we still could do so with the introduction of only one perturbation parameter, by studying $x^3 - (3 + 4\varepsilon)x^2 + (2 - 2\varepsilon)x + \varepsilon$.

(b) There is an alternative way of treating a non-zero root x_0 (such as x_2 and x_3 in the example above): one may make a change of variable $y = x - x_0$, thus moving the root to $y = y_0 = 0$, The series for the root $y_0(\varepsilon)$ will then have no constant term, as in (2a).

(c) One can, in fact, determine the radii of convergence of the series (2). Consider $x_1(\varepsilon)$, as ε varies, this root will move around in the complex plane. Of course, the roots $x_2(\varepsilon)$ and $x_3(\varepsilon)$ will also be moving; at some value(s) of ε, $x_1(\varepsilon)$ will collide with one of these others. If ε_1^* is the value of such a collision ε for which $|\varepsilon_1^*|$ is the smallest, then the series for $x_1(\varepsilon)$ will have radius of convergence $|\varepsilon_1^*|$. For the simple example here we can calculate that ε_1^* is a root of $4 - 27\varepsilon^2 = 0$, so that $|\varepsilon_1^*| = 2\sqrt{3}/9 \approx 0.3849\cdots$.

1.2 Regular and singular perturbations. We will follow Bender and Orszag [1] in classifying perturbation problems (of all types, not just root finding) as *regular* or *singular*. A *regular* problem has two characteristics:

(i) The solution of the perturbed problem has the same general character as the solution of the unperturbed problem.

(ii) The solution of the perturbed problem is an analytic function of ε, for small ε, and thus has a representation as a convergent power series in ε.

Bender and Orszag suggest that these two characteristics are generally found together. A problem which does not have both these characteristics is called *singular*.

How does this classification apply to our current problem of finding the roots of perturbed polynomials? If we look at Example 1 we see the two characteristics of a regular problem: (i) the perturbed problem, like the unperturbed one, has three distinct roots, and (ii) the perturbed roots are given as convergent power series in ε. Thus this is a regular perturbation problem. In our current context, singular problems can occur in two distinctly different ways, as illustrated by the next two examples.

Example 2: Consider

$$P_\varepsilon(x) = x^2 - \varepsilon.$$

The unperturbed polynomial $P_0(x) = x^2$ has a double root at $x = 0$, but for $\varepsilon \neq 0$, $P_\varepsilon(x)$ has two distinct roots, at $\sqrt{\varepsilon}$ and $-\sqrt{\varepsilon}$. Thus neither characteristic of a regular perturbation problem holds here: the character of the solution has changed as we pass from $\varepsilon = 0$ to $\varepsilon \neq 0$ (since one double root has become two separate roots) and the roots are not analytic functions of ε (since $\sqrt{\varepsilon}$ is not analytic at $\varepsilon = 0$). This then is clearly a singular perturbation problem. We note a property of the solution which, as we will see, is typical for singular perturbations of polynomial roots: the roots behave for small ε like ε^p for some power p other than $p = 1$ (here $p = 1/2$).

Example 3: Consider

$$P_\varepsilon(x) = \varepsilon x^2 + 2x - 3.$$

The unperturbed polynomial $P_0(x) = 2x - 3$ has just one root, $x_1 = 3/2$, but for $\varepsilon \neq 0$, $P_\varepsilon(x)$, as a quadratic polynomial, has two roots, which may of course be found from the quadratic formula. From this, and with the Taylor series $\sqrt{1+z} = 1 + \frac{1}{2}z - \frac{1}{8}z^2 + \frac{1}{16}z^3 + \cdots$ we then have

$$\begin{aligned} x_+(\varepsilon) &= \frac{-1 + \sqrt{1 - 3\varepsilon}}{\varepsilon} \\ &= \frac{1}{\varepsilon}\left[-1 + \left(1 + \frac{1}{2}(3\varepsilon) - \frac{1}{8}(3\varepsilon)^2 + \frac{1}{16}(3\varepsilon)^3 - \cdots\right)\right] \\ &= \frac{3}{2} - \frac{9}{8}\varepsilon + \frac{27}{16}\varepsilon^2 - \cdots; \end{aligned} \tag{5a}$$

this is the perturbation expansion of the unperturbed root $x_1 = 3/2$. The second root is

$$x_-(\varepsilon) = \frac{-1 - \sqrt{1 - 3\varepsilon}}{\varepsilon}$$

$$= \frac{1}{\varepsilon}\left[-1 - \left(1 + \frac{1}{2}(3\varepsilon) - \frac{1}{8}(3\varepsilon)^2 + \frac{1}{16}(3\varepsilon)^3 - \cdots\right)\right]$$
$$= -\frac{2}{\varepsilon} - \frac{3}{2} + \frac{9}{8}\varepsilon - \frac{27}{16}\varepsilon^2 - \cdots; \tag{5b}$$

this shows that the second root $x_-(\varepsilon)$, present for $\varepsilon \neq 0$, is for small ε approximately $-2/\varepsilon$. Thus this root travels off to ∞ as ε approaches zero, which is why for $\varepsilon = 0$ we see only one root. This is certainly a singular perturbation problem: (i) the character of the problem changes from having one root to having two roots as ε becomes nonzero, and (ii) the root $x_-(\varepsilon)$ is certainly not analytic in ε for ε small.

For both the singular problems Example 1 and Example 2 we thus encounter roots which, for small ε, behave as ε^p for some power $p \neq 1$: $p = 1/2$ in Example 2 and $p = -1$ in Example 3. This is the typical pattern, and we now turn to discussing this in full generality. Incidentlly, we will see that the two phenomena of Example 1 and Example 2 can occur simultaneously: $P_\varepsilon(x)$ can have a multiple root at ∞.

1.3 Roots of a general polynomial. We now consider the general problem of finding the solutions of $P_\varepsilon(x) = 0$, where $P_\varepsilon(x)$ is a polynomial in x, of degree n, whose coefficients depend on the parameter ε:

$$P_\varepsilon(x) = a_0(\varepsilon) + a_1(\varepsilon)x + \cdots + a_n(\varepsilon)x^n = \sum_{k=0}^{n} a_k(\varepsilon)x^k. \tag{6}$$

Here each coefficient $a_k(\varepsilon)$ is itself a polynomial in ε (so that in fact $P_\varepsilon(x)$ is a polynomial in two variables, although we do not emphasize this since the roles played by x and by ε are so different). For simplicity, and without loss of generality:

- We assume that $a_n(\varepsilon)$ is not identically zero, that is, it is a polynomial in ε with at least one nonzero coefficient. For if this were not true, then $P_\varepsilon(x)$ would really be a polynomial of degree $n-1$, and we could treat it that way.
- Similarly, we assume that $a_0(\varepsilon)$ is not identically zero. For if it were, then we could write $P_\varepsilon(x) = xQ_\varepsilon(x)$ with $Q_\varepsilon(x)$ a polynomial of degree $n-1$; thus 0 would be a root of P_ε for all ε, and we could simply note this fact and proceed to study the roots of Q_ε.

When we set $\varepsilon = 0$, some of the coefficients of P_ε may vanish; in particular, let us suppose that $a_n(0) = a_{n-1}(0) = \cdots = a_{m+1}(0) = 0$ but that $a_m(0) \neq 0$. The unperturbed polynomial $P_0(x)$ will then be of degree m and have m roots $x_1, \ldots, x_m$, and $P_\varepsilon(x)$ will have n roots $x_1(\varepsilon), \ldots, x_n(\varepsilon)$, with $x_{m+1}(\varepsilon), \ldots, x_n(\varepsilon)$ approaching infinity as ε approaches 0. Note that if a root of a polynomial has multiplicity j then when we count the roots of a polynomial we count that root j times.

Now we turn to developing the perturbation series for these roots. *We concentrate on the behavior of a zero root or an infinite root; see Remark 2 immediately below for the general case.* Our general method, based on our experience in Examples 1 through 3, will be to make a substitution of the form $x = \varepsilon^p w$, and then to choose p in such a way that we can study the behavior of certain roots. We expect $p = 1$ for a simple, finite root, and this will always be the case (see Example 1); for such roots one may start directly with a substitution as in (2), whether the root is zero or not. We also expect that p may be fractional for a multiple root (see Example 2), and that p will be negative for an infinite root (see Example 3), that is, for the roots $x_{m+1}(\varepsilon), \ldots, x_n(\varepsilon)$.

Remark 2: As mentioned immediately above, the method we will describe here is directly applicable to the study of either a root at 0 or a root at ∞. To study the perturbation of a nonzero finite root x_0 we can use one of two methods:

- As in Remark 1(b) we can make the change of variable $y = x - x_0$, thus moving the root to $y = 0$, and then use the substitution $y = \varepsilon^p w$, following the methods described below, or
- We can study the root directly by a substitution $x = x_0 + \varepsilon^p w$. Once we have simplified the resulting expression, however, the result will be just that which we would have obtained using the method of substitution described immediately above.

See Example 6 below.

We now return to the problem of determining the perturbation expansion of the roots of $P_\varepsilon(x)$ which arise from roots at $x = 0$, or at $x = \infty$, of the unperturbed polynomial $P_0(x)$. The key idea is to look for numbers p such that some root or roots of $P_\varepsilon(x) = 0$ behave, when $\varepsilon \to 0$, as $\varepsilon^p w$, with $w \neq 0$. To find such values of p, and more information about the behavior of the corresponding roots, we follow steps 1–3 below.

Example 4: We will illustrate our process as we go along with the model polynomial

$$P_\varepsilon(x) = \varepsilon^2 x^5 + (2\varepsilon + 3\varepsilon^2)x^3 + x^2 + \varepsilon x - 3\varepsilon + 4\varepsilon^3. \tag{7}$$

Note that $P_0(x) = x^2$ has two zero roots and three infinite roots.

Step 1. We make the substitution $x = \varepsilon^p w$ in $P_\varepsilon(x)$, and ask how each term $a_k(\varepsilon)x^k$ in (6) will behave under this substitution. $a_k(\varepsilon)$ is a polynomial; suppose that the smallest power of ε appearing there is r_k, that is, that $a_k(\varepsilon) = \varepsilon^{r_k}\widetilde{a}_k(\varepsilon)$, with $\widetilde{a}_k(\varepsilon)$ a polynomial satisfying $\tilde{a}_k(0) = A_k \neq 0$. Then

$$a_k(\varepsilon)x^k \quad \xrightarrow{x=\varepsilon^p w} \quad \varepsilon^{r_k}\widetilde{a}_k(\varepsilon)(\varepsilon w)^p \sim \varepsilon^{r_k+kp}A_k w^k,$$

where in the last expression we have replaced factors $\widetilde{a}(\varepsilon)$ by its $\varepsilon = 0$ value A_k. Thus under this substitution we have

$$P_\varepsilon(x) \sim \sum_{k=0}^{n} \varepsilon^{r_k+kp} A_k w^k. \tag{8}$$

Example 4 (continued): For the model polynomial P_ε^* of (7) the polynomial of (8) is

$$\varepsilon^{2+5p}w^5 + \varepsilon^{1+3p}2w^3 + \varepsilon^{2p}w^2 - \varepsilon^{1+p}w - 3\varepsilon.$$

Step 2. We now are interested nonzero roots, as $\varepsilon \to 0$, of the polynomial (8) obtained in the previous step. As ε becomes very small only the terms in this polynomial with the

j	k	$r_j + pj = r_k + pk$	p	Exponent $r_i + pi$ $i=5$ $2+5p$	$i=3$ $1+3p$	$i=2$ $2p$	$i=1$ $1+p$	$i=0$ 1
0	1	$1 = 1 + p$	0	2	1	0	1	1
0	2	$1 = 2p$	$1/2$	$9/2$	$5/2$	1	$3/2$	1
0	3	$1 = 3p + 1$	0	2	1	0	1	1
0	5	$1 = 5p + 2$	$-1/5$	1	$2/5$	$-2/5$	$3/5$	1
1	2	$1 + p = 2p$	1	7	4	2	2	1
1	3	$1 + p = 1 + 3p$	0	2	1	0	1	1
1	5	$1 + p = 5 + 2p$	$-1/4$	$3/4$	$1/4$	$-1/2$	$3/4$	1
2	3	$2p = 1 + 3p$	-1	-3	-2	-2	0	1
2	5	$2p = 2 + 5p$	$-2/3$	$-4/3$	-1	$-4/3$	$1/3$	1
3	5	$1 + 3p = 2 + 5p$	$-1/2$	$-1/2$	$-1/2$	-1	$1/2$	1

Table 1

smallest power of ε will be relevant, and a nonzero root will exist only if there are at least two of these relevant terms. This leads to our criterion for the possible values of p:

> For at least two indices j, k, with $0 \le j < k \le n$, the exponents of ε in (8) must agree, and must further be the smallest among all the exponents:
>
> $$r_j + pj = r_k + pk \le r_i + pi \quad \text{for all } i,\ 0 \le i \le n.$$

Example 4 (continued): Let us see what this means for our model polynomial (7) We tabulate the relevant data in Table 1: for each possible index pair j, k we give the equation $r_j + pj = r_k + pk$, the value of p thus determined, and the values of all the exponents $r_i + ip$ for this value of p. The values of p which determine the behavior of roots are those in which the exponents for the indices j and k are the smallest among all these exponents. By inspection of the table we see that there are two such values, $p = 1/2$, from the second row of the table, and $p = -2/3$, from the ninth row. We will also need the corresponding exponents $e = r_j + pj = r_k + pk$, which for these two rows are $e = 1$ and $e = -4/3$; when we make the substation $x = \varepsilon^p w$ the resulting polynomial will contain an overall factor ε^e. (The notation e for this exponent is taken from [3] and used also in [2].) In summary, the values we will need as we continue are

$$\begin{array}{ccl} p & e & \\ 1/2 & 1 & (\text{from } j=0,\ k=2); \\ -2/3 & -4/3 & (\text{from } j=2,\ k=5). \end{array} \tag{9}$$

Step 3. We now fix one of the (p, e) pairs found in Step 2 and study the behavior of those roots of $P_\varepsilon(x)$ which behave as ε^p as $\varepsilon \to 0$. It is convenient to make two more changes of variable. First, if p is not an integer then our scaling will involve fractional powers of

ε, and this is inconvenient for several reasons. Therefore if $p = \mu/\nu$ with μ and ν integers with no common factor and $\nu \geq 2$, we replace ε with a new variable β defined by $\varepsilon = \beta^\nu$; then the replacement we make, corresponding to $x = \varepsilon^p w = \varepsilon^{\mu/\nu} w$, is $x = \beta^\mu w$. Second, since we know that under this substitution , P_ε will acquire an overall factor $\varepsilon^e = \beta^{e\nu}$, we remove this factor by multiplying P_ε by $\beta^{-e\nu}$. In summary: we study the nonzero roots of

$$Q_\beta(w) = \beta^{-e\nu} P_{\beta^\nu}(\beta^\mu w).$$

It turns out that these nonzero roots, say $w_1, \ldots, w_l$, will be simple and will be analytic functions of β, so that their perturbation expansion $w_i(\beta) = a_0 + a_1 w + a_2 w^2 + \cdots$ can be studied as were the roots in Example 1.

Example 4 (continued): Again we work this out for our model problem, using the (p, e) pairs given in (9). For $p = 1/2$ we have (by comparison with $p = \mu/\nu$) that $\mu = 1$, $\nu = 2$, and so must introduce β by $\varepsilon = \beta^2$ and make the substitution $x = \varepsilon^p w = \beta w$. Then, since $e = 1$, we study the polynomial

$$\begin{aligned} Q_\beta(w) &= \beta^{-2} P_{\beta^2}(\beta w) \\ &= \beta^{-2}\bigl(\beta^4(\beta w)^5 + (2\beta^2 + 3\beta^4)(\beta w)^3 + (\beta w)^2 + \beta^2(\beta w) - 3\beta^2 + 4\beta^6\bigr) \\ &= \beta^7 w^5 + (2\beta^3 + 3\beta^5) w^3 + w^2 + \beta w - 3 + 4\beta^4. \end{aligned} \tag{10}$$

The unperturbed polynomial here, $Q_0(w)$, is $w^2 - 3$, with roots $w_1 = \sqrt{3}$ and $w_2 = -\sqrt{3}$, and so $Q_\beta(w)$ will have roots of the form (compare (2))

$$w_1(\beta) = \sqrt{3} + c_1\beta + c_2\beta^2 + c_3\beta^3 + c_4\beta^4 + \cdots, \tag{11a}$$
$$w_2(\beta) = -\sqrt{3} + d_1\beta + d_2\beta^2 + d_3\beta^3 + d_4\beta^4 + \cdots. \tag{11b}$$

To determine the coefficients in these expansion we substitute (11) into (10), set the result equal to zero, and solve. For example, for $w_1(\beta)$ this gives

$$\begin{aligned} \beta^7(\sqrt{3} + c_1\beta + c_2\beta^2 + \cdots)^5 + (2\beta^3 + 3\beta^5)(\sqrt{3} + c_1\beta + c_2\beta^2 + \cdots)^3 \\ + (\sqrt{3} + c_1\beta + c_2\beta^2 + \cdots)^2 + \beta(\sqrt{3} + c_1\beta + c_2\beta^2 + \cdots) - 3 + 4\beta^4 = 0 \end{aligned}$$

from which, setting the coefficients of various powers of β to 0, we have

$$\begin{array}{rccc} \beta^0: & 3 - 3 = 0 & & \\ \beta: & 2\sqrt{3}c_1 + \sqrt{3} = 0 & \Rightarrow & c_1 = -\dfrac{1}{2} \\ \beta^2: & c_1 + c_1^2 + 2\sqrt{3}c_2 = 0 & \Rightarrow & c_2 = \dfrac{\sqrt{3}}{24} \\ \beta^3: & c_2 + 2\sqrt{3}c_3 + 2c_1c_2 + 6\sqrt{3} = 0 & \Rightarrow & c_3 = -3 \\ \beta^4: & 4 + c_3 + 2\sqrt{3}c_4 + 2c_1c_3 + c_2^2 + 18c_1 = 0 & \Rightarrow & c_4 = \dfrac{959\sqrt{3}}{1152} \end{array}$$

The coefficients d_i for $w_2(\beta)$ are obtained from the c_i by changing $\sqrt{3}$ to $-\sqrt{3}$. Thus we have series for two of the roots of the original polynomial $P_\varepsilon(x)$:

$$x_1(\varepsilon) = \sqrt{\varepsilon}\left(\sqrt{3} - \frac{1}{2}\varepsilon^{1/2} + \frac{\sqrt{3}}{24}\varepsilon - 3\,\varepsilon^{3/2} + \frac{959\sqrt{3}}{1152}\varepsilon^2 + \cdots\right)$$
$$x_2(\varepsilon) = \sqrt{\varepsilon}\left(-\sqrt{3} - \frac{1}{2}\varepsilon^{1/2} - \frac{\sqrt{3}}{24}\varepsilon - 3\,\varepsilon^{3/2} - \frac{959\sqrt{3}}{1152}\varepsilon^2 + \cdots\right)$$

We now consider the second (p, e) pair: $p = -2/3$, $e = -4/3$. Now $\mu = -4$, $\nu = 3$; we write $\varepsilon = \beta^3$ and make the substitution $x = \varepsilon^p w = \beta^{-2} w$. Then

$$\begin{aligned} Q_\beta(w) &= \beta^4 P_{\beta^3}(\beta^{-2}w) \\ &= \beta^4\big(\beta^6(\beta^{-2}w)^5 + (2\beta^3 + 3\beta^6)(\beta^{-2}w)^3 + (\beta^{-2}w)^2 + \beta^3(\beta^{-\beta^{-2}w2}w) - 3\beta^3 + 4\beta^9\big) \\ &= w^5 + (2\beta + 3\beta^4)w^3 + w^2 + \beta^5 w - 3\beta^7 + 4\beta^{13}. \end{aligned} \tag{12}$$

The unperturbed polynomial $Q_0(w)$ is $w^5 + w^2$, with roots $w_1 = w_2 = 0$, $w_3 = -1$, $w_4 = e^{\pi i/3}$, and $w_5 = e^{-\pi i/3}$. The two zero roots are the ones whose perturbation expansions were obtained above, and here we are interested in the nonzero roots. They show that $Q_\beta(w)$ will have roots of the form

$$w_3(\beta) = -1 + b_1\beta + b_2\beta^2 + b_3\beta^3 + b_4\beta^4 + \cdots, \tag{13a}$$
$$w_4(\beta) = e^{\pi i/3} + c_1\beta + c_2\beta^2 + c_3\beta^3 + c_4\beta^4 + \cdots, \tag{13b}$$
$$w_5(\beta) = e^{-\pi i/3} + d_1\beta + d_2\beta^2 + d_3\beta^3 + d_4\beta^4 + \cdots \tag{13c}$$

(of course, the coefficients c_i and d_i here are not the same as those in (11).) We determine the coefficients as above; for $w_3(\beta)$ the first step leads to

$$\begin{aligned} &(-1 + b_1\beta + b_2\beta^2 + \cdots)^5 + (2\beta + 3\beta^4)(-1 + b_1\beta + b_2\beta^2 + \cdots)^3 \\ &\qquad + (-1 + b_1\beta + b_2\beta^2 + \cdots)^2 + \beta^5(-1 + b_1\beta + b_2\beta^2 + \cdots) - 3\beta^7 + 4\beta^{13} = 0. \end{aligned}$$

with similar formulas for w_4 and w_5. We omit details of the calculation; the final answers are

$$x_3(\varepsilon) = \varepsilon^{-2/3}\left(-1 + \frac{2}{3}\varepsilon^{1/3} - \frac{8}{81}\varepsilon + \frac{227}{243}\varepsilon^{4/3} + \frac{1}{3}\varepsilon^{5/3} + \cdots\right)$$
$$x_4(\varepsilon) = \varepsilon^{-2/3}\left(-1 + \frac{2}{3}e^{2\pi i/3}\varepsilon^{1/3} - \frac{8}{81}e^{\pi i/3}\varepsilon + \frac{227}{243}e^{2\pi i/3}\varepsilon^{4/3} + \frac{1}{3}\varepsilon^{5/3} + \cdots\right)$$
$$x_5(\varepsilon) = \varepsilon^{-2/3}\left(-1 + \frac{2}{3}e^{\pi i/3}\varepsilon^{1/3} - \frac{8}{81}e^{2\pi i/3}\varepsilon + \frac{227}{243}e^{\pi i/3}\varepsilon^{4/3} + \frac{1}{3}\varepsilon^{5/3} + \cdots\right)$$

j	k	$r_j + pj = r_k + pk$	p	Exponent $r_i + pi$ $i=2$ $1+2p$	$i=1$ p	$i=0$ 0
0	1	$0 = p$	0	1	0	0
0	2	$0 = 1 + 2p$	$-1/2$	0	$-1/2$	0
1	2	$p = 1 + 2p$	-1	-1	-1	0

Table 2

1.4 Further examples We treat briefly several other examples. In most cases we will be quite brief, finding the scaling behavior of the roots but not calculating the higher terms in the perturbation series.

Example 3 revisited: Earlier we studied the roots of the polynomial

$$P_\varepsilon(x) = \varepsilon x^2 + 2x - 3.$$

via the quadratic formula; here we use the method of the previous section. Substituting $x = \varepsilon^p w$ into $P_\varepsilon(x)$ yields

$$\varepsilon^{1+2p} w^2 + \varepsilon^p 2w - 3.$$

Analyzing this as for Example 4 above leads to the results summarized in Table 2, from which it is clear that the relevant (p, e) pairs are

$$\begin{array}{rrl} p & e & \\ 0 & 0 & (\text{from } j=0,\ k=1); \\ -1 & -1 & (\text{from } j=1,\ k=2). \end{array} \tag{14}$$

For the case $p = e = 0$ we have $x = \varepsilon^0 w = w$ and $Q_\varepsilon(w) = \varepsilon^0 P_\varepsilon(x) = P_\varepsilon(x)$, that is, we are simply looking at the original polynomial. Then $P_0(x) = 2x - 3$ has one (simple) root $x_1 = 3/2$ and thus $P_\varepsilon(x)$ has root

$$x_1(\varepsilon) = \frac{3}{2} + b_1\varepsilon + b_2\varepsilon^2 + \cdots, \tag{15a}$$

and the coefficients b_i may be determined in the usual way. For $p = e = -1$, $x = \varepsilon^{-1} w$ and $Q_\varepsilon(w) = \varepsilon P_\varepsilon(\varepsilon^{-1} w) = w^2 + 2w - 3\varepsilon$. Thus $Q_0(w) = w^2 + 2w$ has the simple root $w = -2$, and the second root of $P_\varepsilon(x)$ will have the form

$$x_2 = \varepsilon^{-1} w(\varepsilon) = -\frac{2}{\varepsilon} + c_1 + c_2\varepsilon + \cdots. \tag{15b}$$

Of course, we have just found again the roots (5) found earlier.

Example 5: Consider the polynomial

$$P_\varepsilon(x) = x^4 + \varepsilon^2 x^3 - \varepsilon x^2 + \varepsilon^3.$$

				Exponent $r_i + pi$			
				$i=4$	$i=3$	$i=2$	$i=0$
j	k	$r_j + pj = r_k + pk$	p	$4p$	$2+3p$	$1+2p$	3
0	2	$3 = 1 + 2p$	1	4	5	3	3
0	3	$3 = 2 + 3p$	1/3	4/3	3	5/3	3
0	4	$3 = 4p$	3/4	3	17/4	5/2	3
2	3	$1 + 2p = 2 + 3p$	−1	−4	−1	−1	3
2	4	$1 + 2p = 4p$	1/2	2	7/2	2	3
3	4	$2 + 3p = 4p$	2	8	8	7	3

Table 3

The unperturbed version $P_0(x) = x^4$ has root $x = 0$, of multiplicity 4. The substitution $x = \varepsilon^p w$ leads to

$$\varepsilon^{4p}w^4 + \varepsilon^{2+3p}w^3 + \varepsilon^{1+2p}w^2 + \varepsilon^3,$$

and this leads to the data in Table 3. Recall now that the p values in the table were chosen so that $r_j + jp = r_k + kp$; what remains is to determine the rows in which this common value is the minimum of all the values of $r_i + ip$ in that row. This criterion is satisfied by rows 1, in which $r_0 = r_1 + 1 = 3$ and the other $r_i + ip$ values are 4 and 5, and by row 5, in which $r_2 + 2p = r_4 + 4p = 2$ and the other values are 7/2 and 3. Thus the (p, e) pairs which must be considered are

$$\begin{array}{rrl} p & e & \\ 1 & 3 & \text{(from } j=0,\ k=2\text{);} \\ 1/2 & 2 & \text{(from } j=2,\ k=4\text{).} \end{array} \tag{16}$$

For the case $p = 1$, $e = 3$ we have $x = \varepsilon w$ and $Q_\varepsilon(w) = \varepsilon^{-3}P_\varepsilon(\varepsilon w) = \varepsilon w^4 + \varepsilon^2 w^3 - w^2 + 1$; $Q_0(w) = -w^2 + 1$ has roots ± 1 and so $Q_\varepsilon(w)$ has roots

$$x_1^\pm(\varepsilon) = \varepsilon\bigl(\pm 1 + b_1^\pm \varepsilon + b_2^\pm \varepsilon^2 + \cdots\bigr). \tag{17a}$$

The coefficients $b_i^\pm$ are determined in the usual way. For $p = 1/2$, $e = 2$, $x = \varepsilon^{1/2}w$, $\beta = \sqrt{\varepsilon}$, and $Q_\beta(w) = \beta^4 P_{\beta^2}(\beta w) = w^4 + \beta^3 w^3 - w^2 + \beta^6$. Now $Q_0(w) = w^4 - w^2$ has roots $w = \pm 1$ and $P_\varepsilon(x)$ has roots

$$x_2^\pm = \varepsilon^{1/2}\bigl(\pm 1 + c_1^\pm \varepsilon^{1/2} + c_2^\pm \varepsilon + \cdots. \tag{17b}$$

In this example the order-four root of the unperturbed polynomial $P_0(x)$ has split into two pairs of roots, scaling in different ways: one pair as ε, one as $\sqrt{\varepsilon}$.

Example 6: We consider briefly

$$P_\varepsilon(x) = \varepsilon x^3 + x^2 - (4 + \varepsilon)x + 4 + 2\varepsilon.$$

Now $P_0(x) = x^2 - 4x + 4$ has a double root at $x = 2$. We could investigate the behavior of this root by the substitution $x = 2 + \varepsilon^p w$. Let us rather shift the root to the origin, by the substitution $y = x - 2$, leading to the polynomial

$$P_\varepsilon^*(y) = P_\varepsilon(y + 2) = \varepsilon y^3 + (6\varepsilon + 1)y^2 + 11\varepsilon y + 8\varepsilon.$$

Analysis of $P_\varepsilon^*(y)$ then proceeds as in our earlier examples; we find two roots with leading behavior $y \sim \pm 2\sqrt{2}\sqrt{\varepsilon}$ (corresponding to $x \sim 2 \pm 2\sqrt{2}\sqrt{\varepsilon}$) and one with leading behavior $y \sim -1/\varepsilon$ (so that also $x \sim -1/\varepsilon$).

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