Renormalization and Ward identities using complex space–time dimension

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Complex-dimensional renormalization is defined for an arbitrary Feynman amplitude and shown to be equivalent to BPH renormalization. Using quantum electrodynamics as an example, Ward identities are proved; here Carlson's theorem extends the identities from integer to complex dimension. Both complex dimensional and analytic regularization are necessary at intermediate stages.

In Ref. 1, t'Hooft and Veltman propose a renormalization method based on the generalization of the dimension of space–time to a complex number. (Such a generalization was previously proposed by Regge[2-5] to discuss analytic properties of Feynman amplitudes.) In this paper we verify that the method indeed gives a renormalization (proved in Ref. 1 for graphs continuing up to two loops; see also Ref. 6) and use Carlson's theorem to show that it maintains Ward identities. (We will actually discuss Ward identities in the case of QED, but the methods used are in no way special.) We define an intermediate regularization which includes both complex dimension and the usual $\boldsymbol{\lambda}$ regularization of analytic renormalization, to give amplitudes which are well defined in all integer dimensions; when all the λ parameters are set to 1, the regularization of Ref. 1 is recovered.

1. REGULARIZATION

We suppose given a connected Feynman graph G with m vertices $V_1, \ldots, V_m$ and L lines $\{l_1, \ldots, l_L\} = \mathcal{L}; h = L - m + 1$ will denote the number of loops of G. [When other graphs enter the discussion we write $m(G), \mathcal{L}(G')$, etc.] The line $l \in \mathcal{L}$ has initial vertex V_{i_l} and final vertex V_{f_l}, and we associate with l a complex variable λ_l; the point $\lambda^\circ \in \mathbb{C}^L$ is specified by $\lambda_l = 1$, all l. (Multiples of variables are denoted by boldface.) Finally, n-dimensional Minkowski space M_n uses Lorentz inner product

$$p \cdot q = p^\mu q^\nu g_{\mu\nu} = p^\circ q^\circ - \sum_1^{n-1} p^i x^i$$

and Fourier transform

$$\tilde{f}(p) = \int_{M_n} \frac{d^n p}{(2\pi)^{n/2}} f(x) e^{ip\cdot x};$$

if $\mathbf{p} = (p_1, \ldots p_m)$, we let $s_{ij} = p_i \cdot p_j, \quad 1 \le i \le j \le m$.

We begin by discussing scalar particles, so that each line l has propagator $\Delta^l(\lambda_l) \in \mathcal{S}'(\mathbb{R}^n)$

$$\tilde{\Delta}^l(\lambda_l) = \frac{i}{(m^2 - p^2 - i0)^{\lambda_l}}.$$

The physical propagator is obtained by setting $\lambda_l = 1$. For $\mathrm{Re}\lambda_l > n/2$, Δ^l is a continuous function of[7] $x \in \mathbb{R}^n$, and hence the Feynman amplitude $\mathcal{T}(\boldsymbol{\lambda}, n) \in \mathcal{S}'(\mathbb{R}^{nm})$, given by

$$\mathcal{T}(\boldsymbol{\lambda}, n)(\mathbf{x}) = \Pi_{\mathcal{L}} \Delta^l(\lambda_l)(x_{f_l} - x_{i_l}),$$

is well defined. Its Fourier transform is easily calculated[7]:

$$\tilde{\mathcal{T}}(\boldsymbol{\lambda}, n)(\mathbf{p}) = f_G(\boldsymbol{\lambda}, n)\Gamma\left(\sum\lambda_l - \frac{hn}{2}\right) \times \delta\left(\sum_1^m p_i\right) \int_0^1 \cdots \int_0^1 \left(\Pi_{\mathcal{L}} \alpha_l^{\lambda_l - 1} d\alpha_l\right) \times \delta\left(1 - \sum\alpha_l\right) d(\alpha)^{-n/2} \left(\sum \alpha_l m_l^2 - \frac{D(\alpha, s)}{d(\alpha)}\right)^{(hn/2 - \Sigma\lambda_l)}, \tag{1.1}$$

where

$$f_G(\boldsymbol{\lambda}, n) = ((2\pi)^{n/2}(-i)^{m-1}/2^{(nh/2)}\Pi_{\mathcal{L}}\Gamma(\lambda_l)),$$

and $d(\alpha), D(\alpha, s)$ are the Symanzik polynomials for G. We sometimes write $\mathcal{T} = \lim_{\epsilon\to 0+} \mathcal{T}_\epsilon$, where $\mathcal{T}_\epsilon$ is defined by replacing each m_l^2 in (1.1) by $m_l^2 - i\epsilon$.

Definition 1.1: The (analytically and complex-dimensionally) regularized amplitude $\mathcal{T}(\boldsymbol{\lambda}, \nu)$ for the graph G is obtained from (1.1) by replacing n by the complex variable ν.

We remark that this definition is equivalent to that of Ref. 1 (except for the presence of the λ's); in particular, it is obtained by applying the formulas of Ref. 1, Appendix A, to a p-space Feynman integral. $\mathcal{T}(\boldsymbol{\lambda}, \nu)$ may be considered as an element of $\mathcal{S}'(R^{n\cdot m})$ for any n, since it depends only on the invariants s_{ij}.

Theorem 1.2: $\mathcal{T}(\boldsymbol{\lambda}, \nu)$ may be analytically continued to a meromorphic function of $(\boldsymbol{\lambda}, \nu) \in \mathbb{C}^{L+1}$, having simple poles on the linear varieties

$$\nu_H = \sum_{\mathcal{L}(H)} \lambda_l - h(H)\frac{\nu}{2} = -k, \tag{1.2}$$

for each irreducible subgraph H of G and positive integer k (H is irreducible if it is connected and cannot be disconnected by removing a line or vertex).

Proof: We introduce into (1.1) the scaling transformations of the α variables used in the theory of analytic renormalization.[7] Then $\mathcal{T}(\boldsymbol{\lambda}, \nu)$ becomes a sum of integrals of the form

$$\delta\left(\sum p_i\right)\Gamma(\nu_G) f_G(\boldsymbol{\lambda}, \nu) \int_0^1 \cdots \int_0^1 \prod_H t_H^{\nu_H - 1} dt_H E(t)^{-\nu/2} \times \left(\sum_{\mathcal{L}} m_l^2 \beta_l(t) - \frac{F(t, s)}{E(t)}\right)^{-\nu_G}, \tag{1.3}$$

where Π_H is over an "s family" of subgraphs H which are irreducible or consist of a single line. In (1.3), β_l is a monomial in the t_H with $\beta_l = 1$ for some l, E and F are polynomials with E strictly positive in the region of integration. When the factors $t_H^{\nu_H - 1}$ in (1.3) are regarded as distributions,[7,8] (1.3) is well defined for all $\boldsymbol{\lambda}$, ν, and the singularity structure of (1.2) emerges. [Actually, in

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(1.3) we have $H \neq G$; the singularity (1.2) for $H = G$ arises from the factor $\Gamma(\nu_G)$. Similarly the apparent poles of (1.3) arising from the cases where $H = \{l\}$ are cancelled by the factors $\Gamma(\lambda_l)^{-1}$ in $f_G(\boldsymbol{\lambda}, \nu)$.]

We now extend the definition to the case of particles with spin 1 or $\frac{1}{2}$; we assume that a particular representation $\{\gamma_\mu \mid \mu = 0, 1, \ldots, n-1\}$ of the Clifford algebra $C(M_n)$ has been chosen in each dimension, such that the trace of any product of an odd number of γ's vanishes. We will actually normalize the trace to satisfy $\mathrm{Tr}(1) = 4$ in all dimensions.[1] Now the amplitude for any process, calculated in n-dimensions, is a linear combination of certain tensor forms

$$p_i{}^\mu,\; g^{\mu\nu},\; \not{p}_i,\; \not{p}_i\gamma^\mu, \text{etc.} \tag{1.4}$$

(with distributions similar to 1.1 as coefficients). How are we to interpret these tensors when n is replaced by the complex variable ν?

The solution proposed in Ref. 1 is threefold:
(a) external momenta are always from M_4; and all γ matrices are eliminated before introducing the complex dimension by (b) evaluating the trace for closed loops of spinor lines and (c) inserting projection operators, and then taking the trace, for open spinor lines. For simple graphs this procedure is adequate; however, when recursive subtractions are necessary, difficulty is encountered, particularly with procedure (c). That is, it is unclear how the amplitude (or vertex part) for a subgraph can be inserted recursively into the amplitude for the graph if we have defined only its traces when multiplied by various γ matrices.

For this reason we will treat tensors such as (1.4) as symbolic quantities, which may be interpreted as existing in whichever dimension is necessary at any time (4 for physical renormalization, arbitrary n for recursive subtraction). To regularize an amplitude we therefore express it as a linear combination of these symbolic forms in dimension n, then replace n by ν in the coefficient distributions. As in Ref. 1, an additional polynomial ν dependence of these coefficients is generated by contractions via the relation $g^\mu{}_\mu = n$.

Finally, we define similarly regularized amplitudes for generalized graphs. Let $Q = \{U_1, \ldots U_n\}$ be a partition of $\{V_1, \ldots V_m\}$, with $U_i = \{V_{i1}, \ldots V_{im(i)}\}$; let $G(U_i)$ be the subgraph of G formed by all lines joining vertices in U_i; and let $\tilde{G}$ be the graph obtained from G by contracting the subgraphs $G(U_i)$. Suppose we are given vertex parts

$$\mathfrak{X}(U_i) = \begin{cases} 1, & \text{if } m_i = 1, \\ 0, & \text{if } G(U_i) \text{ not IPI}, \\ \delta(\sum p_{ia}) D_i(p_{ia}) & \text{otherwise}, \end{cases} \tag{1.5}$$

with D_i a Lorentz-invariant polynomial, of degree at most equal to

$$\mu(G(U_i)) = \sum_{\mathcal{L}(G(U_i))} (r_l + 2) - 4(m_i - 1), \tag{1.6}$$

where r_l is the degree of the polynomial in the numerator of the lth propagator. (Note that this superficial divergence is computed in dimension 4.) The coefficients of D may depend on $\boldsymbol{\lambda}, \nu$. $\mathfrak{X}(U_i)$ may be interpreted as defining a vertex part in any particular dimension, according to the interpretation above.

In dimension n we form the amplitude

$$\mathcal{T}_{Q,\mathfrak{X}}(\boldsymbol{\lambda}, n) = \prod_{\mathcal{L}(\tilde{G})} \Delta^l(\lambda_l) \prod_{i=1}^{M} \mathfrak{X}(U_i). \tag{1.7}$$

Equation (1.7) is again a linear combination of invariant tensors with coefficients similar to (1.1), so that we may define $\mathcal{T}(\lambda, \nu)$ for any ν as above. Symanzik rules for these amplitudes are worked out in Appendix C; for future use we note that each coefficient is a sum of terms of the form (compare 1.3)

$$\begin{aligned} f_G(\boldsymbol{\lambda}, \nu)\Gamma(\nu_G - j_G)\delta(\textstyle\sum p_i)\int_0^1 \cdots \int_0^1 \prod_H t_H{}^{\nu_H - j_H - 1} dt_H E_{\tilde{G}}(t)^{-\nu/2 - j} \\ \times P(t, s, \nu)[\textstyle\sum m_l^2 \beta_l(t) - F_{\tilde{G}}(t, s)/E_{\tilde{G}}(t)]^{-(\nu_G - j_G)}, \end{aligned} \tag{1.8}$$

where j, j_H are positive integers and P is a polynomial.

2. RENORMALIZATION

The regularized amplitudes are renormalized[1] by a slight variation of the standard BPH scheme of recursive subtractions. In the α-space context of this paper we can use the (minimal) counterterm structure associated with generalized vertices of the graph, i.e., we make subtractions only for those divergent subgraphs consisting of *all* lines connecting a given subset of the vertices. (In a p-space formulation as in Ref. 1 subtractions for additional divergent loop integrations are necessary.)

In this section we use dimensional regularization only, i.e., we set $\boldsymbol{\lambda} = \boldsymbol{\lambda}^\circ$ at all times, and will therefore omit the λ dependence of the amplitudes.

Definition 2.1: If $f(\nu)$ has an isolated singularity at $\nu = 4$, let Kf be the singular part, defined by

$$Kf(\nu) = \int_{|\nu' - 4| = r} \frac{f(\nu')}{\nu' - \nu}\, d\nu'$$

for $|\nu - 4| < r$.

Definition 2.2: Let $\mathcal{Y}(\nu)(V_i) = 1$, and suppose inductively that we have defined $\mathcal{Y}(\nu)(V_1', \ldots, V_{r'}')$ for all generalized vertices $\{V_1', \ldots, V_{r'}'\} \subset \{V_1, \ldots, V_m\}$, with $r' < r$. Then

$$\bar{\mathcal{P}}(\nu)(V_1', \ldots, V_r') = \sum_Q \mathcal{T}_{Q,\mathcal{Y}}(\nu)(V_1', \ldots, V_r'), \tag{2.1}$$

$$\mathcal{Y}(\nu)(V_1', \ldots, V_r') = -K\bar{\mathcal{P}}(\nu)(V_1', \ldots, V_r'), \tag{2.2}$$

$$\mathcal{P}(\nu)(V_1', \ldots, V_r') = \bar{\mathcal{P}}(\nu)(V_1, \ldots, V_r) + \mathcal{Y}_\nu(V_1', \ldots, V_r'). \tag{2.3}$$

In (2.1) the sum is over all partitions Q of $\{V_1', \ldots, V_r'\}$ into at least two generalized vertices. $\mathcal{P}(\nu)(V_1, \ldots, V_m)$ *is then the renormalized amplitude for the graph G.*

We prove below that $\mathcal{Y}(\nu)(V_1', \ldots, V_r')$ is in fact a vertex part; this was shown in Ref. 1 for graphs containing up to two loops. [Actually, it is necessary to know this inductively for $r' < r$ in order that (2.1) be well defined.] Formulas (2.1)–(2.3) exactly parallel the BPH scheme except that renormalization is effected by discarding a pole in the complex dimension ν rather than discarding low order terms of a Taylor series.

We wish to compare (2.1)–(2.3) with the BPH $\mathcal{R}$ operation, and will follow the notation of Ref. 9; in particular, if $\mathcal{W}(\nu)$ is an amplitude associated with some generalized vertex $\{V_1', \ldots, V_r'\}$, and $\mathcal{W}(\nu) = \delta(\sum p_i')F(\nu, \mathbf{p})$, then $M\mathcal{W} = \delta(\sum p_i')G(\nu, p)$, with G the Maclaurin series for F in $\mathbf{p}$ up to order $\mu(V_1', \ldots, V_r')$. Define finite vertex parts by

$$\hat{\mathfrak{X}}(\nu)(V_1', \ldots, V_r') = \begin{cases} 1 & \text{if } r = 1 \\ 0 & \text{if } G(V_1', \ldots, V_r') \text{IPR} \\ (1 - K)M\mathcal{P}(\nu)(V_1', \ldots, V_r') & \text{otherwise}. \end{cases} \tag{2.4}$$

[$\hat{\mathfrak{X}}$ is certainly a vertex part, by definition of M; it is "finite" because the $(1-K)$ factor removes the singularity at $\nu=4$.] Let $\mathcal{R}', \mathfrak{X}', \overline{\mathcal{R}}'$ be the BPH quantities defined using this finite renormalization and complex dimensional regularization:

$$\overline{\mathcal{R}}'(\nu)(V_1',\ldots,V_r') = \sum_Q \mathcal{T}_{Q,\mathfrak{X}'}(\nu)(V_1',\ldots,V_r'), \tag{2.5}$$

$$\begin{aligned}\mathfrak{X}'(\nu)(V_1',\ldots,V_r') = &-M\overline{\mathcal{R}}'(\nu)(V_1',\ldots,V_r') \\ &+ \hat{\mathfrak{X}}(\nu)(V_1',\ldots,V_r'),\end{aligned} \tag{2.6}$$

$$\mathcal{R}(\nu)(V_1',\ldots,V_r') = \overline{\mathcal{R}}(\nu)(V_1',\ldots,V_r') + \mathfrak{X}'(\nu)(V_1',\ldots,V_r'), \tag{2.7}$$

with $\sum_Q$ as in (2.1).

Theorem 2.3: For any $\{V_1',\ldots,V_r'\}\subset\{V_1,\ldots,V_m\}$,

$$\mathfrak{X}'(\nu)(V_1',\ldots,V_r') = \mathcal{Y}(\nu)(V_1',\ldots,V_r'), \tag{2.8}$$

$$\overline{\mathcal{R}}'(\nu)(V_1',\ldots,V_r') = \overline{\mathcal{P}}(\nu)(V_1',\ldots,V_r'), \tag{2.9}$$

$$\mathcal{R}'(\nu)(V_1',\ldots,V_r') = \mathcal{P}(\nu)(V_1',\ldots,V_r'). \tag{2.10}$$

Corollary 2.4: The complex-dimensional renormalization of Definition 2.2 belongs to the class of BPH renormalizations.

Proof: This is precisely the content of (2.10).

Corollary 2.5: $\mathcal{Y}(\nu)$ is a vertex part.

Proof: This follows from (2.8), since $\mathfrak{X}'(\nu)$ is a vertex part.

The crucial lemma is

Lemma 2.6: $\mathcal{R}'(\nu)(V_1',\ldots,V_r')$ is analytic at $\nu=4$.

Since we expect the $\mathcal{R}$ operation to remove all divergences, Lemma 2.6 is intuitively reasonable. The proof, however, is complicated by the complex-dimensional regularizations; we relegate it to Appendix A.

Proof of Theorem 2.3: Formulas (2.8)–(2.10) certainly hold if $r=1$; suppose inductively that they hold for all $r<r_0$. Then from (2.1) and (2.5), using (2.8) for $r<r_0$, (2.9) holds for $r=r_0$. Thus $\hat{\mathfrak{X}}(\nu)(V_1',\ldots,V_{r_0}') = (1-K)M\overline{\mathcal{R}}'(\nu)(V_1',\ldots,V_{r_0}')$, and from (2.6), $\mathfrak{X}'(\nu)(V_1',\ldots,V_{r_0}') = -KM\overline{\mathcal{R}}'(\nu)(V_1',\ldots,V_{r_0}')$. Since $K^2=K$, $K\mathfrak{X}'(\nu)(V_1',\ldots,V_{r_0}') = \mathfrak{X}'(\nu)(V_1',\ldots,V_{r_0}')$. But from Lemma 2.6,

$$\begin{aligned}0 &= K\mathcal{R}'(\nu)(V_1',\ldots,V_{r_0}') \\ &= K\overline{\mathcal{R}}(\nu)(V_1',\ldots,V_{r_0}') + K\mathfrak{X}'(\nu)(V_1',\ldots,V_{r_0}') \\ &= -\mathcal{Y}(\nu)(V_1',\ldots,V_{r_0}') + \mathfrak{X}'(\nu)(V_1',\ldots,V_{r_0}'),\end{aligned}$$

proving (2.8). Equation (2.10) follows immediately from (2.8) and (2.9).

3. WARD IDENTITIES

We will use QED as an example in this section, but the arguments given are quite general. Consider then a particular Ward identity, e.g., for the vacuum polarization tensor:

$$k_\mu \Pi^{\mu\sigma}(k) = 0. \tag{3.1}$$

We wish to prove a regularized version

$$k_\mu \Pi^{\mu\sigma}(\nu;k) = 0. \tag{3.2}$$

Then the recursive subtractions of (2.1) preserve (3.2) (the proof is the same as for any gauge-invariant regularization), and the $(1-K)$ operation in (2.3) yields a renormalized amplitude which also satisfies the Ward identity.

There are three difficulties in establishing (3.2): (a) in complex dimension the contraction over the index μ is meaningless (as explained in Sec. 1 we do not take external vectors as four-dimensional); (b) even in integer dimensions divergent quantities are involved, and the formal proofs are therefore suspect; and (c) the usual manipulations to establish (3.1) are based on the p-space integral form, which is not available to us when ν is complex. We will treat these difficulties in turn.

The problem of contractions in nonintegral dimension is handled by regarding Ward identities as relations between the coefficients of various tensors. In (3.2), for example, write $\Pi^{\mu\sigma}(\nu;k) = A(\nu,k^2)g^{\mu\sigma} + B(\nu;k^2)k^\mu k^\sigma$, so that the Ward identity becomes

$$A(\nu,k^2) + k^2 B(\nu,k^2) = 0. \tag{3.3}$$

Now (3.3) makes sense for all values of ν (and we prove it below). All Ward identities may be interpreted in this sense. (It is necessary to first choose a linear basis for all the tensor forms, and express all amplitudes in terms of this basis.) We will usually not mention this explicitly in what follows.

Before proceeding we introduce the following notation. For any QED graph G, with l a Fermion line incident on an external photon vertex V_i, let $G_{l,i}$ be the graph obtained from G by replacing l with a scalar particle, and removing the γ matrix associated with V_i; let $\tilde{G}_{l,i}$ be obtained from $G_{l,i}$ by contracting l. Then the amplitudes for these graphs are related by

Lemma 3.1:

$$\mathcal{T}_{G_{l,i}}(\boldsymbol{\lambda},\nu)|_{\lambda_l=0} = i\mathcal{T}_{\tilde{G}_{l,i}}(\boldsymbol{\lambda},\nu).$$

Proof: With ν an integer, the lemma follows immediately from the p-space Feynman integral, using $\tilde{\Delta}^l|_{\lambda_l=0} = i$. For nonintegral ν we argue directly from (1.1), treating the factor $\alpha_l^{\lambda_l-1}$ [$=(\alpha_l)_+^{\lambda_l-1}$] as a distribution and using[8] $\{\alpha_l^{\lambda_l-1}/\Gamma(\lambda_l)\}|_{\lambda_l=0} = \delta(\alpha_l)$. By holding $\mathrm{Re}\,\lambda_{l'} \gg 0$, $l'\neq l$, divergence difficulties are avoided; the result extends to all $\boldsymbol{\lambda}$ by analytic continuation.

To treat difficulty (b) we prove a modified identity involving the λ's. The propagator $\tilde{S}(\lambda,p) = i(\not{p}+m)(m^2-p^2-i0)^{-\lambda}$ satisfies a generalization of the usual Ward–Takahashi identity:

$$\begin{aligned}\tilde{S}(\lambda_a,p)\not{k}\tilde{S}(\lambda_b,p+k) = &-\tilde{S}(\lambda_a,p)\tilde{\Delta}(\lambda_b-1,p+k) \\ &+\tilde{\Delta}(\lambda_a-1,p)\tilde{S}(\lambda_b,p+k),\end{aligned} \tag{3.4}$$

where Δ is the scalar propagator of mass m. Inserting (3.4) into the p-space Feynman integral for $\mathcal{T}_G$ immediately proves the integer dimension case of

Theorem 3.2: Suppose that G is a QED graph with V_1 an external photon vertex and a, b the fermion lines incident on V_1. Then

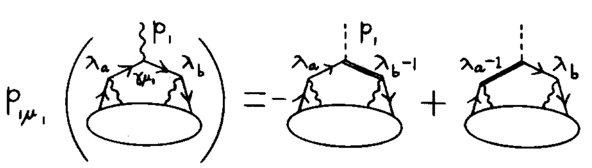

FIG. 1. A λ-regularized identity.

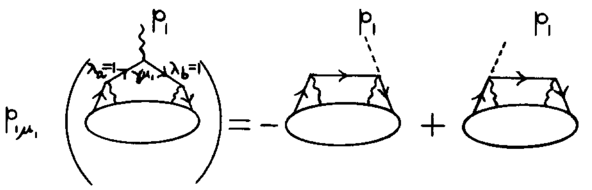

FIG. 2. The identity with λ regularization removed.

$$(p_1)_{\mu_1}\mathcal{T}_G^{\mu_1\mu_2\cdots}(\boldsymbol{\lambda},\nu)(p_1,\ldots,p_m)$$
$$-\mathcal{T}_{G_{b,1}}^{\mu_2\cdots}(\lambda_a,\lambda_b-1,\ldots;\nu)(\mathbf{p})$$
$$+\mathcal{T}_{G_{a,1}}^{\mu_2\cdots}(\lambda_a-1,\lambda_b,\ldots;\nu)(\mathbf{p}). \qquad (3.5)$$

Remark: This identity is indicated pictorially in Fig. 1, where the double line denotes a scalar particle, and the dotted line an external momentum (no longer particularly a photon). If in (3.5) we set $\lambda_l = 1$ for all l, and use Lemma 3.1, we obtain

$$(p_1)_{\mu_1}\mathcal{T}_G^{\mu_1\cdots}(\nu)(\mathbf{p}) = -\mathcal{T}_{G_{b,1}}^{\mu_2\cdots}(\nu)(\mathbf{p}) + \mathcal{T}_{G_{a,1}}^{\mu_2\cdots}(\nu)(\mathbf{p}) \qquad (3.6)$$

(see Fig. 2). But (3.6) is precisely the relation needed to establish Ward identities (after inserting an external photon vertex into a diagram in all possible ways). Thus there remains only to prove Theorem 3.2 for noninteger ν.

We wish to apply Carlson's theorem[10,11] to a suitable function; to avoid complicated analytic continuations we work in a region of $(\boldsymbol{\lambda},\nu)$ space in which there is no ultraviolet divergence. (It is here that we use critically the λ regularization.) The necessary estimate comes from

Lemma 3.3: Let G be an arbitrary Feynman graph, $\mathcal{T}_\epsilon'(\boldsymbol{\lambda},\nu,\mathbf{s})$ the coefficient of some tensor form in the Feynman amplitude for G, and m_0 the minimal mass occurring in G. Then there exist positive constants a, b, k such that for $|s_{ij}| < a$ and $\mathrm{Re}\lambda_l > \frac{1}{2}\mathrm{Re}\nu > b$,

$$\left|f_G^{-1}(\boldsymbol{\lambda},\nu)\Gamma(\nu_G)^{-1}\left(\frac{m_0^2}{2}\right)^{\nu_G}\mathcal{T}_\epsilon(\boldsymbol{\lambda},\nu,\mathbf{s})\right| \le K\exp\frac{(2\epsilon L|\nu_G|)}{m_0^2}. \qquad (3.7)$$

Proof: According to (1.8) the function to be estimated is a sum of terms of the form

$$\left(\prod_{i=1}^{j_G}(\nu_G-i)^{-1}\right)\left(\frac{m_0^2}{2}\right)^{j_G}\int_0^1\cdots\int_0^1\prod t_H^{\nu_H-j_H-1}dt_H P(\mathbf{t},\mathbf{s},\nu)$$
$$\times E(\mathbf{t})^{-\nu/2-j}X^{-\nu_G+j_G}, \qquad (3.8)$$

where

$$X = \left[\left(\frac{F(\mathbf{t},\mathbf{s})}{E(\mathbf{t})} + \sum_l (m_l^2 - i\epsilon)\beta_l\right)\frac{2}{m_0^2}\right],$$

with P a polynomial and j_H, j fixed positive integers.

Take $b = \max\{j_H, j\} + 1$ ($H = G$ is included). Then $|\nu_G - i|^{-1} \le 1$, for $i \le j_G$; using (1.2), $\mathrm{Re}\nu_H \ge \mathrm{Re}\nu/2 > j_H + 1$, so $|t_H^{\nu_H - j_H - 1}| \le 1$; and, since $E(t) \ge 1$, $|E(t)^{-\nu/2-j}| < 1$. The polynomial ν dependence is dominated by the exponential in (3.7). Finally, for a sufficiently small, $\mathrm{Re}X > 1$ (since $\beta_l \ge 0$ and $\beta_{l_0} = 1$ for some l_0) and $\mathrm{Im}X \le 2L\epsilon/m_0^2$. Hence

$$0 \ge \arg X \ge \tan^{-1}\left(-\frac{2\epsilon L}{m_0^2}\right) \ge \frac{-2\epsilon L}{m_0^2},$$

and

$$|X^{-(\nu_G - j)_G}| = |X|^{j_G}\exp(-\mathrm{Re}\nu_G\ln|X| + \mathrm{Im}\nu_G\arg X)$$
$$\le K'\exp(2\epsilon L|\nu_G|/m_0^2).$$

Inserting these estimates into (3.8) yields (3.7) immediately.

Proof of Theorem 3.2: Let the coefficient of some tensor form in (3.5) (with ϵ dependence added) be $g_\epsilon(\boldsymbol{\lambda},\nu,\mathbf{s})$; we wish to show

$$g_\epsilon(\boldsymbol{\lambda},\nu,\mathbf{s}) = 0. \qquad (3.9)$$

Fixing real numbers η_l, with $\eta_l > \frac{1}{2}$, and setting $\lambda_l = \eta_l\nu$, we have $\nu_G = \beta\nu$ with $\beta = \beta(\eta) > 1$. For ϵ sufficiently small and $|s_i| < a$, we may by Lemma 3.5 apply Carlson's theorem to

$$h(\nu) = f(\boldsymbol{\lambda},\nu)^{-1}\Gamma(\nu_G)^{-1}\left(\frac{m_0^2}{2}\right)^{\nu_G} g_\epsilon(\boldsymbol{\lambda},\nu,\mathbf{s})\Big|_{\lambda_l = \eta_l\nu},$$

to find that $h(\nu) = 0$. Since g_ϵ is analytic in $\boldsymbol{\lambda}$ and real analytic in $\mathbf{s}$, (3.9) follows for all $(\boldsymbol{\lambda},\mathbf{s})$; the $\epsilon \to 0$ limit gives (3.5).

APPENDIX A: THE BPH $\mathcal{R}$ OPERATION WITH COMPLEX-DIMENSIONAL REGULARIZATION

Our purpose is to sketch the proof of Lemma 2.6. We wish to use, with as little additional machinery as possible, the convergence estimates which Hepp[9] has given for the $\mathcal{R}$ operation. Complications arise because Hepp's methods rely heavily on the product structure of the regularized Feynman amplitude, a structure which is destroyed by the complex-dimensional regularization.

We first rewrite the $\mathcal{R}'$ operation in terms of the $\mathcal{R}$ operation (which involves no finite renormalization) for generalized graphs:

$$\mathcal{R}'\mathcal{T}(\boldsymbol{\lambda},\nu) = \sum_Q \mathcal{R}\mathcal{T}_{Q,\hat{\mathfrak{X}}}(\boldsymbol{\lambda},\nu), \qquad (A1)$$

where the sum is over all partitions Q of $\{V_1,\ldots,V_m\}$ and $\mathcal{T}_{Q,\hat{\mathfrak{X}}}$ is defined as in (1.7), but starting from the vertex parts $\hat{\mathfrak{X}}$ of (2.4). The usual proof[12] of (A1) does not involve the product structure of $\mathcal{T}$, but only the multilinearily in the vertex parts of the amplitude for a generalized graph; this continues to hold for complex-dimensional regularization. It therefore suffices to prove

Lemma A1: For any finite vertex parts $\hat{\mathfrak{X}}(\boldsymbol{\lambda},\nu)$, the amplitude

$$\mathcal{R}\mathcal{T}_{Q,\hat{\mathfrak{X}}}(\boldsymbol{\lambda},\nu) \qquad (A2)$$

is analytic at $(\boldsymbol{\lambda}^0, 4)$.

Proof: For simplicity we discuss only the partition

$Q = \{\{V_1\}, \ldots, \{V_m\}\}$. Hepp (Lemma 3.1 of Ref. 9) gives a representation of (A2) which extends immediately to n dimensions and the inclusion of $\boldsymbol{\lambda}$ regularization:

$$\mathcal{R}\mathcal{T}(\boldsymbol{\lambda}, n) = \sum_{\pi}\sum_{T} \mathfrak{F}_T^{\pi}(\boldsymbol{\lambda}, n). \tag{A3}$$

Here $\sum_\pi$ runs over permutations π of $\mathfrak{L}$, and $\sum_T$ over certain "trees"; $\mathfrak{F}_T^\pi$ is the Feynman amplitude for the tree T integrated over the region $\mathfrak{D}^\pi = \{\alpha \mid \alpha_{l_{\pi(1)}} \leq \cdots \leq \alpha_{l_{\pi(L)}}\}$. Moreover, $\mathfrak{F}_T^\pi$ is shown to be given a Feynman-like integral; following through the proof, we find that the $\boldsymbol{\lambda}, n$ dependence enters in four ways: an overall factor $f(\boldsymbol{\lambda}, n)\Gamma(\nu_G - j_G)^{-1}$, as in (1.8); possible polynomial n-dependence (from $g^\mu{}_\mu = n$); factors $\alpha_l^{\lambda_l - 1}$ in the integrand, and a replacement of Hepp's $D^{I'} = \prod_{l\in\mathfrak{L}(I')-\mathfrak{M}} D_l^{-2}$ by $\prod_{l\in\mathfrak{L}(I')-\mathfrak{M}} D_l^{-n/2}$. Thus we may define $\mathfrak{F}_T^\pi(\boldsymbol{\lambda}, \nu)$ for noninteger ν; (A3) for general ν now follows from Carlson's theorem (the argument is similar to those of Sec. 3). Finally, the estimates given by Hepp in Lemma 3.4 show that $\mathfrak{F}_T^\pi(\boldsymbol{\lambda}, \nu)$ is analytic at $(\boldsymbol{\lambda}_0, 4)$.

APPENDIX B: SYMANZIK RULES FOR GENERALIZED GRAPHS

In Ref. 7 (see also Ref. 13) we have given "Symanzik" rules for arbitrary Feynman graphs, which allow the α-space Feynman integral to be written down directly. We here record the corresponding rules for generalized graphs.

Thus let G be a Feynman graph as in Sec. 1, with propagators

$$\tilde{\Delta}_l(\lambda_l)(p) = \frac{iZ_l(p)}{(m^2 - p^2 - i0)^{\lambda_l}}, \tag{B1}$$

where Z_l is a polynomial. Let Q be a partition of $\{V_1, \ldots, V_m\}$ into generalized vertices $U_i = \{V'_{i1}, \ldots, V'_{im(i)}\}$, $i = 1, \ldots, M$, and $\tilde{G}$ the corresponding contracted graph. The incidence matrix for G is written

$$e^l_{ia} = \begin{cases} 1, & \text{if } V_{f_l} = V'_{ia} \\ -1, & \text{if } V_{i_l} = V'_{ia} \\ 0, & \text{otherwise,}\end{cases}$$

so that $e^l_i = \sum_a e^l_{ia}$ is the incidence matrix of $\tilde{G}$. Given vertex parts $\mathfrak{X}(U_i)$ [(1.5)], we wish to calculate $\mathcal{T}_{Q,\mathfrak{X}}$ as given by (1.7).

Writing

$$Z_l(p) = Z_l\left(\frac{1}{i}\frac{\partial}{\partial u_l}\right) e^{ip\cdot u_l}\Big|_{u_l=0},$$

(B1) becomes $\Delta_l = \lim_{\epsilon\to 0^+}\Delta_{l,\epsilon}$, where

$$\Delta_{l,\epsilon}(x) = \frac{\exp[\frac{1}{2}\pi i(\lambda_l - n/2)]}{2^{n/2}\Gamma(\lambda_l)} Z_l\left(\frac{1}{i}\frac{\partial}{\partial u_l}\right)\int_0^\infty \alpha_l^{\lambda_l - 1 - n/2} d\alpha_l$$

$$\times \exp\{i[-(1/4\alpha_l)(e^l_{ia}x_{ia} + u_l)^2 - (m^2 - i\epsilon)\alpha_l]\}. \tag{B2}$$

Similarly,

$$\hat{\mathfrak{X}}(\mathbf{x}_i) = (2\pi)^{(n/2)(m_i - 2)} D_i\left(\frac{1}{i}\frac{\partial}{\partial s_{ia}}\right)$$

$$\times \prod_{a=1}^{m_i - 1}\delta(x_{ia} + s_{ia} - x_{i,a+1} - s_{i,a+1})\Big|_{s_{ia}=0}. \tag{B3}$$

We insert (B2) and (B3) into (1.7) and take the Fourier transform. It is most convenient to change integration variables to $y_{ia} = x_{ia} + s_{ia}$. The $y_{i2}, \ldots, y_{im(i)}$ integrations are then done using the δ functions; the remaining Gaussian integration is virtually identical with that encountered in calculating an amplitude for $\tilde{G}$. Evaluating this as in Ref. 7 (or Ref. 12) gives

$$\tilde{\mathcal{T}}_{Q,\hat{\mathfrak{X}}}(\boldsymbol{\lambda}, n) = g_Q(\boldsymbol{\lambda}, n)\delta(\textstyle\sum p_{ia})\displaystyle\int_0^\infty\cdots\int_0^\infty(\prod \alpha_l^{\lambda_l - 1}d\alpha_l)\, d(\alpha)^{-n/2}$$

$$\times \prod D_i\left(\frac{1}{i}\frac{\partial}{\partial s_{ia}}\right)\prod Z_l\left(\frac{1}{i}\frac{\partial}{\partial u_l}\right)\exp\left[i\left(\sum_{i,j=1}^{M} r_i r_j\frac{D^k_{ij}(\alpha)}{d(\alpha)}\right.\right.$$

$$\left.\left. + \sum p_{ia}s_{ia} - \sum_l[t_l^2/4\alpha_l + (m^2 - i\epsilon)\alpha_l]\right)\right]\Big|_{\mathbf{s}=\mathbf{u}=0}. \tag{B4}$$

Here

$$g_Q(\boldsymbol{\lambda}, n) = \prod_{\mathfrak{L}(\tilde{G})}\left(\frac{\exp[(\pi i/2)(\lambda_l - n/2)]}{2\Gamma(\lambda_l)}\right)\left(\frac{e^{n\pi i/4}}{\pi i}\right)^{M-1},$$

$$t_l = u_l - \sum_{i,a} e^l_{ia}s_{ia},$$

$$r_i = q_i + \sum_l \alpha_l^{-1}e^l_i t_l$$

$$q_i = \sum_{a=1}^{m(i)} p_{ia},$$

and $d(\alpha), D^k_{ij}(\alpha)$ are the Symanzik functions for[7] $\tilde{G}$:

$$d(\alpha) = \sum_T \prod_{l\notin T}\alpha_l,$$

$$D^k_{ij} = \sum_{T_2}\prod_{l\notin T_2}\alpha_l;$$

the sums are respectively over all trees in $\tilde{G}$ and over all 2-trees in $\tilde{G}$ which separate U_k from U_i and U_j. $k \in \{1, \ldots, M\}$ is arbitrary.

Now recall[12] the following definition: Given a set of quantities $\{X_i\}$ and associated pairwise contractions $\{\overline{X_iX_j}\}$, the τ *product* of a monomial in $\mathbf{X}$ is defined by summing over all contractions, precisely as in Wick's theorem for a T product; the τ product is extended to polynomials in $\mathbf{X}$ by linearity. If we evaluate the u, s derivatives in (B4) and set $\mathbf{u} = \mathbf{s} = 0$, the integrand will contain a factor (see Ref. 12)

$$\tau[\prod Z_l(X^l)\prod D_i(Y_{ia})],$$

where

$$X^l = \sum_{i,j} e^l_i D^k_{ij}\, q_j/\alpha_l\, d(\alpha), \tag{B5}$$

$$Y_{ia} = p_{ia} - \sum_l e^l_{ia}X^l, \tag{B6}$$

$$\overline{X^l_\mu X^{l'}_\nu} = \tfrac{1}{2}\left(\sum_{i,j}\frac{e^l_i e^{l'}_j D^k_{ij}}{\alpha_l\alpha_{l'}d(\alpha)} - \frac{\delta_{l,l'}}{\alpha_l}\right)g_{\mu\nu}, \tag{B7}$$

and other contractions are calculated from (B6), (B7), and the relation $\overline{X^l p_{ia}} = 0$.

[The momenta X^l have an intrinsic characterization. For fixed $\{\alpha_l\}$ and $\{q_i\}$, let $\{k^l \mid l \in \mathfrak{L}(\tilde{G})\}$ be arbitrary n-vectors satisfying momentum conservation in $\tilde{G}$. $k^l = X^l$ is then a stationary point for the function $\sum \alpha_l(k^l)^2$;

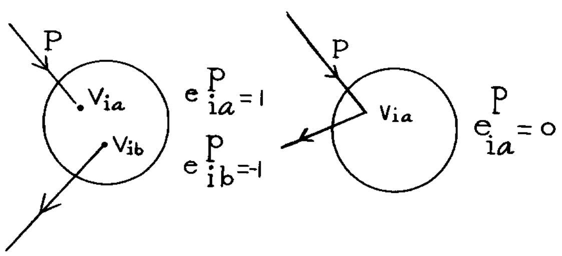

FIG. 3. Incidence of paths on generalized vertices.

Y_{ia} is the total momentum flowing out of V_{ia} in this momentum configuration. See Ref. 5, Sec. 4.2.]

More explicit Symanzik rules depend on the idea of incidence of an oriented path or circuit P in $\tilde{G}$ on lines of $\tilde{G}$ and on vertices of G. For $l \in \mathcal{L}(\tilde{G})$,

$$e_l^P = \begin{cases} 1, & \text{if } l \in P, \text{ and the orientations of } l \text{ and } P \text{ coincide,} \\ 0, & \text{if } l \notin P, \\ -1, & \text{otherwise.} \end{cases}$$

For a vertex V_{ia}, $e_{ia}^P = \sum_l e_{ia}^l e_l^P$, i.e.,

$$e_{ia}^P = \begin{cases} 1, & \text{if precisely one line of } P \text{ is incident on } V_{ia}, \text{ and } P \text{ is oriented into } V_{ia}, \\ -1, & \text{if one line of } P \text{ is incident, oriented out,} \\ 0, & \text{otherwise.} \end{cases}$$

See Fig. 3. For each tree T of $\tilde{G}$ let $P_{kj}(T)$ be the path in T joining U_k to U_j, oriented from U_k to U_j. Then

$$X^l = d(\alpha)^{-1}\sum_T\sum_j e^{P_{kj}(T)}\Big(\prod_{l\notin T}\alpha_l\Big)\, q_j, \tag{B8}$$

$$Y_{ia} = p_{ia} - d(\alpha)^{-1}\sum_T\sum_j e_{ia}^{P_{kj}(T)}\Big(\prod_{l\notin T}\alpha_l\Big)\, q_j \tag{B9}$$

(the result is independent of k). Let T^* denote a set of M lines in $\tilde{G}$ containing precisely one circuit $C(T^*)$, which is given an arbitrary orientation. Then

$$\overline{X_\mu^l X_\nu^{l'}} = d(\alpha)^{-1}\sum_{T^*}\Big(\prod_{l''\notin T^*}\alpha_{l''}\Big)\, e_l^{C(T^*)}e_{l'}^{C(T^*)}g_{\mu\nu}, \tag{B10}$$

$$\overline{X_\mu^l Y_{ia\nu}} = -\, d(\alpha)^{-1}\sum_{T^*}\Big(\prod_{l''\notin T^*}\alpha_{l''}\Big)\, e_l^{C(T^*)}e_{ia}^{C(T^*)}g_{\mu\nu}, \tag{B11}$$

$$\overline{Y_{ia\mu}Y_{jb\nu}} = d(\alpha)^{-1}\sum_{T^*}\Big(\prod_{l\notin T^*}\alpha_l\Big)\, e_{ia}^{C(T^*)}e_{jb}^{C(T^*)}g_{\hat{\mu}\nu}. \tag{B12}$$

(B8)–(B12) are the desired rules.

The final form of (B4) is thus

$$\tilde{I}_{Q,\mathfrak{X}}(\boldsymbol{\lambda}, n) = g_Q(\boldsymbol{\lambda}, n)\delta(\textstyle\sum p_{ia})\int_0^\infty \cdots \int_0^\infty (\prod \alpha_l^{\lambda_l - 1}\, d\alpha_l)$$
$$\times\, d_{\tilde{G}}(\alpha)^{-n/2}\tau[\prod D_i(Y_{ia})\prod Z_l(X^l)]$$
$$\times \exp\Big[i\Big(D_{\tilde{G}}(\alpha, s)/d_{\tilde{G}}(\alpha) - \sum_l (m_l^2 - i\epsilon)\alpha_l\Big)\Big]. \tag{B13}$$

At this point the variable scalings corresponding to s families in $\tilde{G}$ may be introduced. The result is of the form (1.8).

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