Light-cone finite normal products

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A graphical subtraction procedure for constructing the perturbative Green functions of light-cone finite, multiply localized products of fields is proposed. The existence of the Green functions as tempered distributions is proved, together with the properties of light-cone finiteness and localization on a line segment. The derivation of light cone expansions is sketched, but not treated in detail.

1. INTRODUCTION

A convenient operator expansion for displaying in concise form the light-cone singularities of products of fields would be one of the from[1]

$$A(x)B(y)=N^*[A(x)B(y)]+\sum_i F_i((x-y)^2)O_i(x,y), \qquad (1.1)$$

where $N^*[A(x)B(y)]$ is a light-cone finite bilocal field (normal product), the $O_i(x, y)$ are smeared out N^* normal products, and the $F_i(z^2)$ are complex valued functions which are singular for z^2 tending to zero. In Ref. 1, an expansion of the form (1.1) has been shown to exist in a relatively simple example, that of the product $A(x)A(y)$ in the perturbative A^4 model. In that case, the F_i are all powers of logarithms, and the O_i can all be expressed in terms of smeared-out light-cone finite normal products

$$N^*[\partial_{\mu_1}\cdots\partial_{\mu_a}A(x)\partial_{\nu_1}\cdots\partial_{\nu_b}A(y)].$$

In certain respects, the construction of Ref. 1 falls short of a completely satisfactory realization of the program sketched above. To motivate the present work, it is useful to review the major deficiencies of that construction.

(1) It depends on a rather unwieldy subtraction procedure, based on iterated application of Zimmermann's identities[2] relating short-distance finite normal products. Although the subtractions have a recursive structure reminiscent of renormalization, no prescription for removing light-cone singularities on a graph-by-graph basis is given.

(2) It does not establish uniform localization of the formally bilocal fields appearing in the light-cone expansion. A reasonable definition (presumably not the only one) of "bilocal" would require $O_i(x, y)$ to be localized on the line segment joining x to y, with $O_i(x, y)$ commuting with all z which are spacelike with respect to all points of that line segment. In the construction of Ref. 1, the formal bilocals are indeed localized on a segment of the line passing through x and y, but unbounded growth of the localization segment with increasing perturbative order is not excluded.

(3) The construction cannot be generalized to other products of fields, such as products of currents in a charged scalar theory, without encountering formidable complications. In Ref. 1, the analysis was considerably simplified by the limitations to logarithmic light-cone singularities and bilinear products of the basic fields and their derivatives. Such simplifications could not be expected to persist for general field products.

(4) There is no uniform (in all orders in the coupling constant) polynomial bound on the momentum-space growth at infinity of the vertex functions of the N^* products defined in Ref. 1. Hand in hand with this is the necessity of increasingly many subtractions as one proceeds to higher orders, just as one finds for ordinary vertex functions in a nonrenormalizable theory. This suggests that outside of perturbation theory, the N^* product may not be well-defined as a tempered distribution, requiring stronger large-momentum cut-offs than are provided by Schwartz class test functions. Moreover, there will be no renormalization group or Callan—Symanzik equations for the vertex functions of such normal products, making expansions such as (1.1) of only limited usefulness in the phenomenology of theories with asymptotic freedom.

In this paper we define light-cone finite normal products which avoid the first three of the enumerated difficulties. We develop a graph-by-graph subtraction scheme which allows one to define a quite general light-cone finite normal product $N^*[\prod_i\phi_i(x+\theta_i\xi)]$ localized on the minimal line segment containing all $x+\theta_i\xi$. Our method comes tantalizingly close to complete success. For low order diagrams, we are able to maintain the desired control over the numbers of subtractions. We are unable, however, to establish convergence of the Feynman-parameter integral for graphs of arbitrary complexity without making addition-

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al subtractions. It is to be hoped that future investigations will yield the key to pushing the full program through to completion.

This article is organized as follows. In Sec. 2 the formal outlines of our subtraction scheme are developed in conjunction with a study of certain one-loop, two-loop and many-loop graphs. Later in Sec. 2, a precise formulation of the subtraction procedure is presented, and the corresponding convergence theorem is stated The proof of the theorem is given in Sec. 3. We conclude, in Sec. 4, with a brief discussion of light-cone expansions involving our normal products.

2. DEFINITION OF N^* PRODUCTS

We wish to define vacuum expectation values

$$\langle 0|T[N^*(\prod_{i=1}^{n}\phi_i(x+\theta_i\xi))\prod_{j=1}^{m}\tilde{\psi}_j(p_j)]|0\rangle, \qquad (2.1)$$

where ϕ_i and ψ_j are interacting fields, and (2.1) is to be a tempered distribution in x, $p_1, \cdots, p_m \in \mathbb{R}^4$ which is a continuous function of $\xi \in \mathbb{R}$ and $\theta_i \in \mathbb{R}$. Note that, by taking various θ_i equal, we may define light-cone finite products of several currents.) Let $\hat{\Gamma}$ be a Feynman graph contributing to (2.1), with Γ the graph obtained by identifying all vertices $x+\theta_i\xi$, and let $\Delta_l(q)$ be the propagator for a line l of $\hat{\Gamma}$, $\{k_i\}$ be loop momenta for Γ, and

$$q_l = \sum_i a_{li}k_i + \sum_j b_{lj}p_j$$

be line momenta for Γ. Then the unrenormalized amplitude for $\hat{\Gamma}$ is formally

$$\exp[ix\cdot(\sum_j p_j)]\int\prod_i dk_i\prod_l \Delta_l(q_l)\prod_{i=1}^{n}\exp(i\theta_i\xi\cdot q_{l_i}), \qquad (2.2)$$

where l_i is the line of Γ which is incident on $x+\theta_i\xi$ in $\hat{\Gamma}$. We must introduce subtractions into (2.2).

A. A one loop example

Let A be a scalar field of mass m with A^4 interaction, and consider

$$\langle 0|T[A(x+\xi)A(x-\xi)\tilde{A}(p_1)\tilde{A}(p_2)]|0\rangle. \qquad (2.3)$$

If Γ is the graph of Fig. 1 (the vertex V represents the coalescence of $x+\xi$ and $x-\xi$), the integral in (2.2) becomes formally

$$\int d^4k\exp(2ik\cdot\xi)\,[(k+\tfrac{1}{2}p)^2-m^2]^{-1}[(k-\tfrac{1}{2}p)^2-m^2]^{-1}$$
$$=\int_0^\infty\int_0^\infty d\alpha_1 d\alpha_2(\alpha_1+\alpha_2)^{-2}\exp i\Big(\frac{\alpha_1\alpha_2}{\alpha_1+\alpha_2}p^2$$
$$+\frac{\alpha_1-\alpha_2}{\alpha_1+\alpha_2}\,p\cdot\xi-\frac{\xi^2}{\alpha_1+\alpha_2}-(\alpha_1+\alpha_2)m^2\Big). \qquad (2.4)$$

For $\xi^2\neq 0$, (2.4) is regularized by the ξ^2 term in the exponential, but the integral diverges (at $\alpha_1=\alpha_2=0$) for $\xi^2=0$. Zimmermann's short distance finite normal product $N_2[A(x+\xi)A(x-\xi)]$ of Ref. 2 is defined for this diagram by subtracting from the integrand its value at $p=0$; using

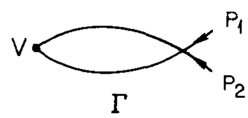

FIG. 1. One loop graph.

$$F(p)-F(0)=\int_0^1 d\tau\,\frac{d}{d\tau}F(\tau p)$$

we obtain

$$i\int d\alpha_1 d\alpha_2(\alpha_1+\alpha_2)^{-3}[2\tau\alpha_1\alpha_2p^2+(\alpha_1-\alpha_2)p\cdot\xi]$$
$$\times\exp i\{\cdots\}. \qquad (2.5)$$

The first term of (2.5) is convergent for all ξ, but the second still diverges for $\xi^2=0$ (although of course it is finite—in fact, zero, —when $\xi\to 0$ in nonlightlike directions). The example suggests a method for defining light-cone finite products: Working in terms of invariants p^2, $p\cdot\xi$, and ξ^2, one subtracts at $p^2=0$; here this gives only the first term of (2.5). Of course, in more complicated graphs, similar subtractions will be necessary for subgraphs as well.

B. Preliminary definition of N^* products

We now give a preliminary version of the renormalization operation needed to define N^* products; a final, precise version will be given below. Let Γ be a 1PI Feynman graph for which each line l has propagator

$$\Delta_l(q)=Z_l(q)\exp(i\xi_l\cdot q)(q^2-m_l^2+i0)^{-1},$$

with Z_l an invariant polynomial of degree ρ_l; we write $n(\Gamma)$, $N(\Gamma)$, and $m(\Gamma)$ for the number of loops, lines, and vertices, respectively. Suppose further that to each 1PI $\gamma\subset\Gamma$ we have assigned a subtraction index $\delta(\gamma)$. Let γ be a 1PI subgraph of Γ, let $\{k_i\}$ and $\{q_l\}$ be loop and line momenta for γ, and let $\mathcal{J}$ be forest of 1PI subgraphs of γ, with $\lambda_1,\cdots\lambda_R$ the maximal proper subgraphs of γ in $\mathcal{J}$ and $\bar{\gamma}=\gamma/\lambda_1\lambda_2\cdots\lambda_R$. Then define recursively

$$\bar{\mathcal{Y}}_{\mathcal{J}}^{\gamma}=\int d^4k_1\cdots d^4k_{n(\bar{\gamma})}\prod_{i=1}^{R}\mathcal{Y}_{\mathcal{J}}^{\lambda_i}(\tilde{\mathrm{p}})\prod_{l\in\bar{\gamma}}[Z_l(q_l)$$
$$\times\exp i\,\xi_l\cdot q_l\exp i\,\alpha_l(q_{l0}^2-(1-i\epsilon)\vec{q}_l^{\,2}-m_l^2(1-i\epsilon))],$$
$$(2.6)$$

$$\mathcal{Y}_{\mathcal{J}}^{\gamma}=\begin{cases}\bar{\mathcal{Y}}_{\mathcal{J}}^{\gamma}, & \text{if } \gamma\notin\mathcal{J},\\ \\ -t^{\delta(\gamma)}\bar{\mathcal{Y}}_{\mathcal{J}}^{\gamma}, & \text{if } \gamma\in\mathcal{J}.\end{cases} \qquad (2.7)$$

Here

$$t^{\delta}\prod_r p_{i_r}^{\mu_r}F(p_i\cdot p_j,\,p_i\cdot\xi_l,\,\xi_l\cdot\xi_m)$$

$$=\sum_{j=0}^{\delta}\Big(\frac{d}{d\tau}\Big)^j\prod_r(\tau p_{i_r}^{\mu_r})F(\tau^2p_i\cdot p_j,\,p_i\cdot\xi_l,\,\xi_l\cdot\xi_m)\big|_{\tau=0} \qquad (2.8)$$

for $\delta > 0$, $t^\delta = 0$ for $\delta < 0$. In (2.6) if j is a vertex of λ_i, $\tilde{p}_j$ denotes the total momentum entering this vertex from external and internal lines of γ. Finally

$$\mathcal{R}^*(p,\ \xi) = \lim_{\epsilon\to 0}\int_0^{\infty}\cdots\int_0^{\infty} \prod_{l\in\Gamma} d\alpha_l \sum_{\mathcal{J}} \mathcal{Y}_{\mathcal{J}}^{\Gamma}.$$

Remark: (a) t^δ is in fact not well defined since its operand is not a covariant function for $\epsilon > 0$, and even if it were, it could not in general be written uniquely as a function of the invariants. Our refined definition below, however, gives an explicit formula for $\mathcal{Y}_j^{\gamma}$ which avoids the difficulty.

(b) Ideally we would like to take $\delta(\gamma) = d(\gamma)$ where $d(\gamma) = 4n(\gamma) - 2N(\gamma) + \sum_{i\in\gamma}\rho_i$ is the superficial divergence. This choice, however, does not appear to give convergence; we try to illuminate the problems in the next section.

C. Two-loop and multi-loop examples

Consider first the contribution to (2.3) from the graph of Fig. 2. Here the forest formula reduces to iterated $(1\text{-}t)$ operations for γ and Γ; since γ does not involve V and $d(\gamma) = 0$ it is natural to choose $\delta(\gamma) = 0$. Then $\mathcal{R}^*$ has the form (ignoring ϵ dependence)

$$\int \prod d\alpha_l (1 - t^{\delta(\Gamma)})[f_1(\alpha) + f_2(\alpha)\xi^2 + \sum_i f_{3i}(\alpha) p_i\cdot\xi + \sum_{i,j} f_{4ij}(\alpha) p_i\cdot p_j] \exp i\ [\sum_{i,j} V_{ij}(\alpha) p_i\cdot p_j + \sum_i Y_i(\alpha)\xi\cdot p_i + W(\alpha)\xi^2 \quad - \sum_l \alpha_l m_l^2]. \tag{2.9}$$

The α_1, α_2 subintegration is convergent because of the γ subtraction, but the overall integration of the $f_1, \cdots, f_4$ terms is respectively logarithmically, linearly, and logarithmically divergent, and convergent. If we take $\delta(\Gamma) = 0$ [note $d(\Gamma) = 0$], the first and third terms are rendered convergent (V is homogeneous of degree 1); the f_2 term appears to be logarithmically divergent but for $\xi^2 \neq 0$ it is regulated by the $W(\alpha)\xi^2$ term in the exponential, and for $\xi^2 \to 0$ it vanishes [compare $\int_0 \xi^2 t^{-1}\exp(-\xi^2/t)dt$]. Thus subtractions of minimal degree suffice to give a light-cone finite normal product here.

Observe, however, that the subtraction of the subgraph γ has led to $p\cdot\xi$ factors in (2.9). Our subtraction procedure (2.8) ignores these, thus oversubtracting as far as momentum power counting is concerned. In general, oversubtraction at one level

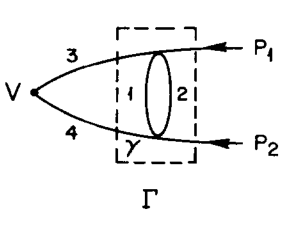

FIG. 2. Two loop graph.

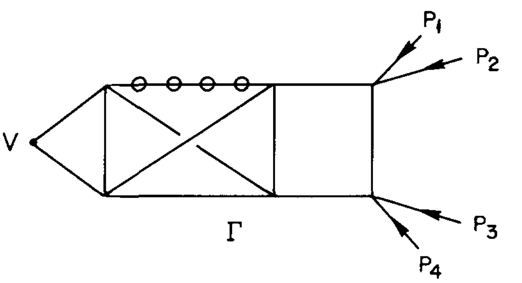

FIG. 3. Many loop graph.

necessitates higher subtraction degrees at higher levels.

This can happen to us. The contribution of Fig. 3 to

$$\langle 0|T[A(x+\xi)A\ x-\xi)]\prod_{i=1}^{4}\tilde{A}(p_i)|0\rangle$$

does not appear to yield a convergent amplitude if $\delta(\Gamma)$ is chosen to be minimal (i.e., negative). Another way to view this difficulty is as follows: the α-space integral for $\mathcal{R}^*$ in this case is similar to (2.9), but the function corresponding to $W(\alpha)$ vanishes in the interior of the integration region, and hence $\exp iW(\alpha)\xi^2$ no longer regulates the term corresponding to $f_2(\alpha)\xi^2$. We conclude that we cannot systematically take $\delta(\gamma) = d(\gamma)$, in fact, we will need subtraction degrees which increase without bound with the order in perturbation theory.

D. Final definition of N^* products

We will complete the definition of the $\mathcal{R}^*$ operation by giving an explicit formula similar to that of Appelquist[3] and Bergere–Zuber[4] for the BPH $\mathcal{R}$ operation. Take Γ as in Sec. 2B, with $\gamma\subset\Gamma$ a 1PI subgraph.

We introduce the standard combinatoric functions for γ, i.e., fixing a vertex k and letting i, j denote vertices and s, t lines of γ, we define

$$U^{\gamma}(\alpha) = \sum_T \prod_{l\notin T}\alpha_l,$$

$$V_{ij}^{\gamma}(\alpha) = (U^{\gamma})^{-1}\sum_{T_2}\prod_{l\notin T_2}\alpha_l,$$

$$Y_{is}^{\gamma}(\alpha) = (U^{\gamma})^{-1}\sum_T(\pm)\ \prod\alpha_l$$

$$X_{st}^{\gamma}(\alpha) = (U^{\gamma})^{-1}\sum_{T^*}(\pm)\prod_{l\notin T^*}\alpha_l,$$

the sums running respectively over all trees T of γ, all two trees T_2 of γ disconnecting i and j from k, all trees T of γ for which the path in T from k to i passes through s in the same (+) or opposite (−) direction as s, and all sets T^* formed by adding one line to a tree, such that the circuit in T^* contains both s and t, oriented coherently (+) or incoherently (−). If $\mathcal{J}$ is a forest for γ, we follow Appelquist by introducing variables τ_λ, $\lambda\in\mathcal{J}$, and writing $\bar{\alpha}_l = (\prod_{l\in\lambda\in\mathcal{J}}\tau_\lambda^2)\alpha_l$,

$\overline{U}(\alpha, \tau) = \Pi_{\lambda\in\mathcal{J}} \tau_\lambda^{-2n(\lambda)} U(\overline{\alpha})$, and $\overline{T}(\alpha, \tau) = T(\overline{\alpha})$ for $T = V$, Y, or X. Finally,

$$\overline{W}^\lambda = \tau_\lambda^2 \overline{X}^\lambda - \sum_\mu \overline{X}^\mu$$

for any $\lambda \in \mathcal{J} \cup \{\gamma\}$; the sum is over the maximal proper subgraphs μ of λ with $\mu \in \mathcal{J}$. Then we define

$$\mathcal{Y}_{\mathcal{J}}^\gamma = \prod_{\lambda\in\mathcal{J}} (-t_{\tau_\lambda}^{\delta(\lambda)}) \prod_{l\in\gamma} Z_l\left(\frac{1}{i}\frac{\partial}{\partial r_l}\right) \overline{U}(\alpha, \tau)^{-2}$$
$$\times \exp i\{p^T\overline{V}p + p^T\overline{Y}(\overline{r} + \xi) - \tfrac{1}{4}[\overline{r}\overline{X}^\gamma\overline{r} + \sum_{\lambda\in\{\gamma\}\cup\mathcal{J}}$$
$$\times (2\overline{r}\overline{W}^\lambda\xi \prod_{\mu\supset\lambda} \tau_\mu^{-1} + \xi\overline{W}^\lambda\xi)]\}\Big|_{\substack{r_l=0 \\ \tau_\lambda=1}}. \tag{2.10}$$

Here t_τ^δ extracts the Taylor series in τ, centered at 0, to order δ (with $t_\tau^\delta = 0$ if $\delta < 0$), $\overline{r}_l = r_l \Pi_{\lambda\ni l}\tau_\lambda$, $p^T\overline{V}p = \sum V_{ij} p_i \cdot p_j$, etc., and $p_i \cdot p_j = p_{i0}p_{j0} - (1 - i\epsilon)\mathbf{p}_i \cdot \mathbf{p}_j$, $p_i \cdot r_l = p_{i0}r_{l0} - \mathbf{p}_i \cdot \mathbf{r}_l$, $r_l \cdot r_m = r_{l0}r_{m0} - (1 - i\epsilon)^{-1}\mathbf{r}_l \cdot \mathbf{r}_m$, and similarly with r_l, r_m replaced by ξ_l, ξ_m. (For us the condition $\mu \supset \lambda$ includes the possibility $\mu = \lambda$). We remark that (2.10), with $\xi_l = 0$ for all l, is the standard formula[3,4] for the $\mathcal{R}$ operation restricted to a single forest.

We want to justify (2.10) by showing its relation to (2.6)–(2.8). Suppose then that either γ is minimal in $\mathcal{J}$ or that we have justified (2.10) for all subgraphs of γ. If we write

$$Z_l(q) = Z_l\left(\frac{1}{i}\frac{\partial}{\partial r_l}\right) \exp(iq \cdot r_l)\big|_{r_l=0}$$

the integral of (2.6) becomes a Gaussian; evaluation of this integral yields (2.10) with no t_{τ_γ} operator with $\tau_\gamma = 1$. By (2.7) this completes the case $\gamma \in \mathcal{J}$. [We omit a detailed derivation of this result. The only difficulty is in evaluating certain combinations of matrix products which arise on completing the square in the exponential.

One observes, however, that these same expressions arise in a similar recursive evaluation of the $\mathcal{R}$ operation, and that in that case the amplitude may also be evaluated in one step by a rescaling of integration variables[4]; comparison of these formulas for the $\mathcal{R}$ operation then yields an evaluation of the needed expressions.]

Finally, if $\gamma \in \mathcal{J}$, we must apply the Taylor operator of (2.7), (2.8). It is quite complicated to apply (2.8) directly since factors $p_i \cdot p_j$ and $p_i \cdot \xi_l$ can both be generated by the spin terms; instead, to simplify our scheme, we scale with τ_γ the momentum variables in both the $p^T\overline{V}p$ and $p^T\overline{Y}\overline{r}$ (but not $p^T\overline{Y}\xi$) terms, leading immediately to (2.10). The effect of this choice is that for certain terms, involving overall $p \cdot \xi$ factors resulting from the r derivatives, we are undersubtracting in comparison with (2.8). However, the increased subtraction degrees we use suffice to give convergence despite this undersubtraction.

Definition 2.1: The *-renormalized amplitude for Γ is

$$\mathcal{R}_\Gamma^*(p; \xi) = \lim_{\epsilon\to 0} \int_0^\infty \cdots \int_0^\infty \Pi d\alpha_l \sum_{\mathcal{J}} \mathcal{Y}_{\mathcal{J}}^\Gamma(p_i; \xi; \alpha) \tag{2.11}$$

with $\mathcal{Y}_{\mathcal{J}}^\Gamma$ given by (2.10). Furthermore, (2.1) is defined by expanding in graphs and applying (2.11), with $\xi_l = \theta_i\xi$ if l is incident on $x + \theta_i\xi$ in $\hat{\Gamma}$; $\xi_l = 0$ otherwise.

E. Statement of results

We want to investigate Definition 2.1 for the case (2.2) arising in the N^* product of several currents as well as for general ξ. Thus let V be a fixed vertex of Γ and $\{\mathcal{L}_1, \ldots \mathcal{L}_k\}$ a partition of the lines incident on V; let $\mathcal{H}$ denote the set of 1PI $\gamma \subset \Gamma$ such that γ intersects at most one $\mathcal{L}_j$, and suppose for convenience that all lines incident on V are oriented into V. Then we consider the hypothesis: (H) $\xi_l = \zeta_j \in \mathbb{R}^4$ if $l \in \mathcal{L}_j (j = 1, \cdots, k)$; $\xi_l = 0$ otherwise.

The justification of Def. 2.1, proved in the next section, is:

Theorem 2.2: There exists a choice of subtraction degrees $\delta(\gamma)$ such that the integral in (2.11) is absolutely convergent for $\epsilon > 0$, and defines a tempered distribution in $p_1, \cdots, p_n$ which is continuous in ϵ, ξ for $\epsilon \geq 0$ and $\xi \in \mathbb{R}^{4N}$; if (H) holds we may take minimal subtraction degrees $(\delta(\gamma) = d(\gamma))$ for $\gamma \in \mathcal{H}$, and continuity then holds in $\epsilon > 0$ and $\zeta \in \mathbb{R}^{4k}$.

Remark 2.3: (a) In fact we give an explicit recursive formula for calculating $\delta(\gamma)$ but do not claim that the result is optimal.

(b) The subtractions for Γ itself are the terms of (2.11) with $\Gamma \in \mathcal{J}$. Now consider (2.2): from (2.10), a counterterm is a linear combination of integrals

$$\int d\alpha\, P(p_1, \cdots, p_m; \xi; \alpha) F(\xi^2) \exp i\sum_{j=1}^m p_j \cdot (x + c_j\xi), \tag{2.12}$$

where P is a polynomial in p. The Fourier transform in p of the integrand in (2.12) is supported at $x_j = x + c_j\xi$; it follows from the definition of Y_{is} that c_j is a convex combination of $\theta_1, \cdots, \theta_n$, and hence the counterterm is (formally) supported on the minimal line segment containing all points $x + \theta_i\xi$. Again, at least formally, this gives the same localization for $N^*(\Pi\phi_i(x + \theta_i\xi))$. Some further discussion is given in Sec. 4.

3. CONVERGENCE OF N^* PRODUCTS

We will divide our proof of Theorem 2.2 into several sections for clarity.

A. Decomposition of integration region

We follow Breitenlohner and Maison.[5] Consider a triple $(\mathcal{C}, \mathcal{B}, \sigma)$, where $\mathcal{C}$ is a maximal forest of 1PI subgraphs of Γ, $\mathcal{B} \subset \mathcal{C} \setminus \{\Gamma\}$, and σ is a map assigning to each $\gamma \in \mathcal{C}$ a line $\sigma(\gamma)$ in $\overline{\gamma} \equiv \gamma/\lambda_1 \cdots, \lambda_p$, with $\{\lambda_i\}$ the maximal subgraphs of γ in $\mathcal{C}$. The region $\mathcal{D}(\mathcal{C}, \mathcal{B}, \sigma) \subset \{\alpha \mid \alpha_l \geq 0\}$ is defined by

$$\alpha_l = \begin{cases} \prod_{\gamma\subset\lambda\in J} t_\lambda, & \text{if } l=\sigma(\gamma), \\ \beta_l \alpha_{\sigma(\gamma)}, & \text{if } l\in\overline{\gamma},\ l\neq\sigma(\gamma), \end{cases} \tag{3.1}$$

and the restrictions

$$t_\Gamma \geq 0;\quad 1\geq t_\gamma \geq 0,\ \gamma\notin\mathcal{B},\ \gamma\neq\Gamma;\ t_\gamma\geq 1,\ \gamma\in\mathcal{B};\ 1\geq\beta_l\geq 0. \tag{3.2}$$

We occasionally write $\beta_l \equiv 1$ if $l\in\sigma(\mathcal{C})$. Exactly as in Ref. 5, we show that (2.11) becomes

$$\lim_{\epsilon\to 0} \sum_{(\mathcal{C},\mathcal{B},\sigma)} \int_{D(\mathcal{C},\mathcal{B},\sigma)} \Pi d\alpha_l\, \mathcal{Y}^\Gamma_{\mathcal{C},\mathcal{B}}, \tag{3.3}$$

where $\mathcal{Y}^\Gamma_{\mathcal{C}}$ is given by formula (2.10) for $\mathcal{Y}^\Gamma_{\mathcal{C}}$ but with $-t_{\tau_\lambda}^{\delta(\lambda)}$ replaced by $(1-t_{\tau_\lambda}^{\delta(\lambda)})$ for $\lambda\notin\mathcal{B}$.

B. Evaluation of γ and τ derivatives

We first observe that the combinatoric functions $\overline{U}(\equiv\overline{U}^\Gamma)$, $\overline{V}$, etc. satisfy

(P1) $\overline{W}^\mu$ is independent of τ_λ unless $\lambda\subsetneq\mu$; $\overline{W}^\mu_{lm}=0$ unless lines l and m are in μ; if (H) holds, $\sum_m \overline{W}^\gamma_{lm}\xi_m = 0$ for $\gamma\in\mathcal{H}$.

(P2) for any $\lambda\in\mathcal{C}$, we have homogeneity in the variables τ_λ^{-2} and $\{\alpha_l \,|\, l\in\lambda\}$, with deg $\overline{U}=n(\lambda)$, deg $\overline{X}$ = deg $\overline{Y}$ = deg $\overline{V}$ = deg $\overline{W}^\mu = 0$ unless $\mu\subset\lambda$; deg $\overline{W}^\mu = -1$ if $\mu\subset\lambda$.

(P3) $\overline{Y}$, $\overline{V}$ and $(\prod_{\tau,s\in\lambda}\tau_\lambda^2)\overline{X}_{rs}$ have the form $P(\alpha,\tau^2)\overline{U}^{-1}$, and $\overline{W}^\lambda$ the form $P(\alpha,\tau^2)\,(\overline{U}^\lambda\Pi\,\overline{U}^{\lambda_i})^{-1}$, with P a polynomial and $\{\lambda_i\}$ the maximal subgraphs of λ in $\mathcal{C}$. [On first inspection it appears that $\overline{W}^\lambda$ might contain factors τ_μ^{-1} for $\mu\subset\lambda$, but this possibility may be eliminated using $\overline{U}^\lambda|_{\tau_\mu=0} = \overline{U}^\mu\overline{U}^{\lambda/\mu}$ (Ref. 6), etc.]

(P4) $\overline{W}^\lambda_{st}$ is a positive semidefinite matrix.

To evaluate the spin terms it is convenient to use $\partial/\partial r_l = \prod_{\lambda\ni l}\tau_\lambda\partial/\partial\overline{r}_l$. Differentiating and setting $\overline{r}=0$, we see that ΠZ_l is replaced by a sum of terms

$$Q_0(p)\prod_{\gamma\in\mathcal{C}}\tau_\gamma^{k(\gamma)}\mathcal{M}(\overline{Y})\prod_{\gamma\in\mathcal{C}\setminus\mathcal{H}}\mathcal{M}_\gamma(\overline{W}^\gamma\xi)\prod_{l,m}\Big(\overline{X}_{lm}\prod_{\lambda\ni l,m}\tau_\lambda^2\Big)^{a_{lm}}, \tag{3.4}$$

where Q_0 is a polynomial and $\mathcal{M}(\overline{Y})$, $\mathcal{M}_\gamma(\overline{W}^\gamma\xi)$ are monomials in $\{\overline{Y}_{is}\}$ and $\{(\overline{W}^\gamma\xi)_s = \sum_t\overline{W}^\gamma_{st}\xi_t\}$, respectively. If (3.4) arose from $\Pi(\partial/\partial\overline{r}_l)^{\rho'_l}$ (with $\rho'_l\leq\rho_l$) and if $c_\gamma = \deg\mathcal{M}_\gamma$, then

$$k(\gamma)\equiv\sum_{l\in\gamma}\rho'_l - 2\sum_{l,m\in\gamma}a_{lm} - \sum_{\substack{\mu\subset\gamma\\ \mu\notin\mathcal{H}}}c_\mu \geq 0; \tag{3.5}$$

note that we have used (P1) to restrict to $\gamma\notin\mathcal{H}$ (taking $\mathcal{H}=\emptyset$ by convention when (H) is not assumed), and have grouped the factors of τ as suggested by (P3). The only odd powers of τ now occur in $\tau_\gamma^{k(\gamma)}$, so we use

$$(-t_\tau^\delta)\tau^k F(\tau^2)|_{\tau=1} = \begin{cases} 0, & \text{if } \delta<k, \\ -t_\eta^{[(\delta-k)/2]}F(\eta)|_{\eta=1}, & \text{if } \delta\geq k; \end{cases}$$

here $[n]$ is $n/2$ [resp. $(n-1)/2$] if n is even (resp. odd). Thus, if we insert (3.4) into (3.3), the result vanishes unless $\delta(\lambda)\geq k(\lambda)$ for all $\lambda\in\mathcal{B}$, and otherwise is a sum of terms

$$\begin{aligned}&\int_{D(\mathcal{C},\mathcal{B},\sigma)}\Pi d\alpha\int_{\substack{\lambda\in\mathcal{C}\setminus\mathcal{B}\\ \delta(\lambda)\geq k(\lambda)}}\prod d\mu(\eta_\lambda)\prod_{\lambda\in\mathcal{C}}\Big(\frac{\partial}{\partial\eta_\lambda}\Big)^{e_\lambda}Q_0(p)\mathcal{M}(\hat{Y})\\ &\times\prod_{\gamma\notin\mathcal{H}}\mathcal{M}_\gamma(\hat{W}^\gamma\xi)\prod_{l,m}(\hat{X}_{lm}\prod_{\lambda\ni l,m}\eta_\lambda)^{a_{lm}}\hat{U}^{-2}\\ &\times\exp i\,[p^T\hat{V}p + p^T\hat{Y}\xi - \tfrac{1}{4}\sum_\lambda\xi\hat{W}^\lambda\xi - \sum\alpha_l m_l^2(1-i\epsilon)]\Big|_{\substack{\eta_\lambda=0,\ \lambda\in\mathcal{B}\\ \eta_\lambda=1,\ \lambda\notin\mathcal{B}\\ \delta(\lambda)<k(\lambda)}}\end{aligned} \tag{3.6}$$

with $\hat{U}(\alpha,\eta) = \overline{U}(\alpha,\tau)|_{\tau_\lambda^2=\eta_\lambda}$ etc.;

$$\int d\mu(\eta_\lambda) = \{[\delta(\lambda)-k(\lambda)]!\}^{-1}\int_0^1[1-\eta(\lambda)]^{\delta(\lambda)-k(\lambda)}d\eta_\lambda,$$

and

$$e_\lambda \begin{cases} = [(\delta(\lambda)-k(\lambda))/2]+1, & \text{if } \delta(\lambda)\geq k(\lambda),\ \lambda\notin\mathcal{B}, \\ =0,\ \delta(\lambda)<k(\lambda),\ \lambda\notin\mathcal{B}, \\ \leq[(\delta(\lambda)-k(\lambda))/2],\ \lambda\in\mathcal{B}. \end{cases} \tag{3.7}$$

We next carry out the η derivatives. Then, aside from the exponential, (3.6) becomes a sum of terms

$$\begin{aligned}&Q'_0(p)\mathcal{D}(\hat{U}^{-2})\mathcal{M}_1(\mathcal{D}\hat{Y})\mathcal{M}_2(\mathcal{D}\hat{V})\prod_{\gamma\notin\mathcal{H}}[\mathcal{M}''_\gamma(\mathcal{D}W^\gamma\xi)\mathcal{M}'_\gamma(\mathcal{D}\xi^T\hat{W}^\gamma\xi)]\\ &\times\prod_{l,m}(\mathcal{D}\hat{X}_{lm}\Pi\eta)^{a_{lm}}.\end{aligned} \tag{3.8}$$

Here (with some abuse of notation) $\mathcal{D}(\hat{U})^{-2}$ denotes some product of $(\partial/\partial\eta)$ operators on $\hat{U}^{-2}$, $\mathcal{M}_1(\mathcal{D}\hat{Y})$ a monomial in the η derivatives of $\{Y_{is}\}$, etc. We still have $c_\gamma = \deg\mathcal{M}''_\gamma$ and, if $\deg\mathcal{M}'_\gamma = c'_\gamma$, then from (P1)

$$\sum_{\gamma\subset\lambda}c'_\gamma \leq \sum_{\gamma\subsetneq\lambda}e_\gamma. \tag{3.9}$$

Further, (P3) implies that (3.8) has the form

$$\frac{Q(p,\xi)P(\alpha,\eta)}{\Pi(\hat{U}^\lambda)^{p(\lambda)}} \tag{3.10}$$

with P a polynomial, and (P2) shows that (3.10) is homogeneous in η_λ^{-1}, $\{\alpha_l \,|\, l\in\lambda\}$ of degree

$$-2n(\lambda) - \sum_{\mu\subset\lambda}(c_\mu + c'_\mu) - \sum_{l,m\in\lambda}a_{lm} + e_\lambda. \tag{3.11}$$

[Note that each $\partial/\partial\eta_\lambda$ contributes +1 to (3.11).]

Now set $\eta_\gamma = 0$, $\gamma\in\mathcal{B}$. For $\mu\in\mathcal{C}$ we let $\tilde{\mu}$ denote μ modulo its maximal proper subgraphs in in $\mathcal{B}$; then

$$\hat{U}^\lambda|_{\eta_\gamma=0,\,\gamma\in\mathcal{B}} = \hat{U}^{\tilde{\lambda}}\prod_{\substack{\gamma\in\mathcal{B}\\ \gamma\subsetneq\lambda}}\hat{U}^{\tilde{\gamma}},$$

so that (3.10) becomes

$$\frac{Q(p,\xi)R(\alpha,\eta)}{\Pi_\lambda(\hat{U}^{\tilde{\lambda}})^{q(\lambda)}}, \tag{3.12}$$

where $R = p|_{\eta_\gamma=0,\,\gamma\in\mathcal{B}}$. Moreover, (3.12) has degree (3.11) in

η_λ^{-1}, $\{\alpha_l | l \in \lambda\}$ for $\lambda \notin \mathcal{B}$ and in $\{\alpha_l | l \in \lambda\}$ for $\lambda \in \mathcal{B}$.

C. Change of integration variables

Introduce the variables $\{t_\lambda, \beta_l\}$ by (3.1). By standard arguments[7]

$$\hat{U}^{\tilde{\mu}} = \prod_{\nu\in\mathcal{C}} t_\nu^{j(\nu)} E^{\tilde{\mu}}(\alpha, \tau),$$

where $E^{\tilde{\mu}} \geq 1$, and $j(\nu) = n(\nu \cap \tilde{\mu})$ is precisely the degree of homogeneity of $\hat{U}^{\tilde{\mu}}$ in $\{\alpha_l | l \in \nu\}$, if $\nu \in \mathcal{B}$, or in η_ν^{-1}, $\{\alpha_l | l \in \nu\}$ if $\nu \notin \mathcal{B}$. Thus the homogeneity of (3.12) implies that when multiplied by the Jacobian $\Pi t_\lambda^{N(\lambda)-1}$ it becomes a sum of terms of the form

$$\prod_{\lambda\in\mathcal{C}} t_\lambda^{b_\lambda - 1} F(t, \beta, \eta) Q(p, \xi), \tag{3.13}$$

where $|F| \leq 1$ and

$$b_\lambda \geq N(\lambda) - 2n(\lambda) - \sum_{\mu\subset\lambda} (c_\mu + c'_\mu) - \sum_{l,m\in\lambda} a_{lm} + e_\lambda, \tag{3.14}$$

with equality in (3.14) if $\lambda \in \mathcal{B}$.

In order to be able to do the t_Γ integration explicitly we write

$$\sum_{\gamma\in\mathcal{C}} \xi \hat{W}^\gamma \xi \,|_{\alpha_l = \beta_l \Pi t_\lambda} \equiv -[t_\Gamma(1 - i\epsilon)]^{-1} A(t, \beta, \eta, \xi)^2.$$

By (P2), A is independent of t_Γ; recall that $\xi_s \cdot \xi_t = (1 - i\epsilon)^{-1} (\xi_s^0 \xi_t^0 (1 - i\epsilon) - \vec{\xi}_s \cdot \vec{\xi}_t)$, so by (P4) Im $A^2 \geq 0$ and we take Im $A > 0$. Then

$$\exp\left(-\frac{i}{4}\sum_\gamma \xi \hat{W}^\gamma \xi\right) = \int_{-\infty}^{\infty} d\omega [-it_\Gamma(1 - i\epsilon)/\pi]^{1/2} \times \exp\left(-i\{t_\Gamma(1 - i\epsilon)\omega^2 - \omega A\}\right). \tag{3.15}$$

Since (3.15) is absolutely convergent, it suffices to prove absolute convergence and ϵ, ξ continuity of (3.6) with the substitution (3.15), i.e., using (3.13), of

$$\int d\omega \,\Pi d\mu(\eta_\lambda) \,\Pi d\beta_l \Pi t_\lambda^{b_\lambda - 1} dt_\lambda t_\Gamma^{1/2} Q(p, \xi) F(t, \beta, \eta) \times \exp\{i(p^T \hat{Y} \xi + \omega A) \exp\{it_\Gamma[p^T \tilde{V} p - M(1 - i\epsilon)]\}. \tag{3.16}$$

Here $\tilde{V} = t_\Gamma^{-1} \hat{V}$ is independent of t_Γ,

$$M = \omega^2 + \sum_l m_l^2 \beta_l \prod_{l\in\lambda\neq\Gamma} t_\lambda,$$

and the integration region for β, t is (3.2).

D. Estimates for convergence and continuity

To verify absolute convergence of (3.16) we note that F and the first exponential are bounded by 1, the second exponential by $\exp(-\epsilon t_\Gamma M)$. When these bounds are inserted in (3.16), the resulting t_Γ integral can be done explicitly if $b_\Gamma > 0$ to give $\Gamma(b_\Gamma + \frac{1}{2})(\epsilon M)^{-(b_\Gamma + 1/2)}$; thus we need only the estimate (verified below)

$$\Pi t_\lambda^{b_\lambda} M^{-(b_\Gamma + 1/2)} \leq K(m_{\sigma(\Gamma)}^2 + \omega^2)^{-a} \prod_{\lambda\neq\Gamma} t_\lambda^{a_\lambda}, \tag{3.17}$$

where $a > \frac{1}{2}$, $a_\lambda < 0$ if $\lambda \in \mathcal{B}$ and $a_\lambda > 0$ if $\lambda \notin \mathcal{B}$. On the other hand, we may evaluate the t_Γ integral in (3.16) directly to give

$$\Gamma(b_\Gamma + 1/2) M^{-(b_\Gamma + 1/2)} [p^T \tilde{V} p / M - (1 - i\epsilon)]^{-(b_\Gamma + 1/2)}.$$

It is easy to verify that $\tilde{V}/M$ is uniformly bounded; hence[7] for any Schwartz test function $\psi(p)$,

$$\int \psi(p) (p^T \tilde{V} p / M - (1 - i\epsilon)]^{-(b_\Gamma + 1/2)} dp$$

is a continuous bounded function of t, β, η and ϵ for $\epsilon \geq 0$. Using Lebesgue dominated convergence and estimating as above we see that as a distribution (3.16) is continuous for $\xi \in \mathbb{R}^{4N}$ and $\epsilon \geq 0$ [or, under (H), for $\xi \in \mathbb{R}^{4k}$, $\epsilon \geq 0$].

It remains to verify that (3.17) and $b_\Gamma > 0$ can be ensured by suitable choice of $\delta(\gamma)$. Write $\delta(\gamma) = d(\gamma) + D(\gamma)$, and take $D(\gamma) = 0$ if (H) holds and $\gamma \in \mathcal{H}$; otherwise, choose $\delta(\gamma)$ recursively so that for any forest $\mathcal{J}$ of proper 1PI subgraphs of γ,

$$D(\gamma) \geq \sum_{l\in\gamma} \rho_l + \sum_{\lambda\in\mathcal{J}} [\delta(\lambda) + D(\lambda) + 1] \cdot \tag{3.18}$$

Then if $\gamma \in \mathcal{C}$, $\gamma \notin \mathcal{B}$ and $\mathcal{G} \subset \mathcal{B}$ is a family of disjoint subgraphs of γ, (3.14), (3.7), (3.5), and (3.18) imply

$$2(b_\gamma - \sum_{\mathcal{G}} b_\lambda) \geq D(\gamma) + 1 - \sum_{\gamma\in\mathcal{G}} D(\lambda) - \sum_{\mu\subset\gamma} (c_\mu + 2c'_\mu) \geq 1 \tag{3.19}$$

(or ≥ 2 since the b's are the integers): note $\gamma \in \mathcal{H}$ implies $\mu \in \mathcal{H}$ for $\mu \subset \gamma$, hence $c_\mu = c'_\mu = 0$.

In particular, with $\gamma = \Gamma$ and $\mathcal{G} = \emptyset$, (3.16) yields $b_\Gamma \geq 1$. For $\lambda \in \mathcal{B}$, define f_λ inductively by

$$f_\lambda = \max(0, b_\lambda - \sum_{\mu\subset\lambda} f_\mu),$$

and note that for any $\gamma \in \mathcal{C}$,

$$\sum_{\lambda\subset\lambda} f_\lambda \begin{cases} \geq b_\lambda, & \text{if } \gamma \in \mathcal{B}, \\ = \sum_{\mu\in\mathcal{G}} b_\mu, & \text{if } \gamma \notin \mathcal{B}, \end{cases} \tag{3.20}$$

where $\mathcal{G} \subset \mathcal{B}$ is some family as in (3.19). Choose θ with $0 < \theta < |\mathcal{B}|^{-1}$, then

$$\begin{aligned} M^{(b_\Gamma + 1/2)} &\geq (m_{\sigma(\Gamma)}^2 + \omega^2 + \sum_{\lambda\in\mathcal{B}} m_\sigma^2(\lambda) \prod_{\gamma\supset\lambda} t_\gamma)^{(b_\Gamma + 1/2)} \\ &\geq (m_{\sigma(\Gamma)}^2 + \omega^2)^{b_\Gamma + 1/2 - \Sigma_{\mathcal{B}}(d_\lambda + \theta)} \prod_{\lambda\in\mathcal{B}} (m_{\sigma(\lambda)}^2 \prod_{\gamma\supset\lambda} t_\gamma)^{d_\lambda + \theta} \\ &= \tilde{K}(m_{\sigma(\Gamma)}^2 + \omega^2)^{b_\Gamma + 1/2 - \Sigma_{\mathcal{B}}(d_\lambda + \theta)} \prod_{\substack{\gamma\in\mathcal{B} \\ \gamma\neq\Gamma}} t_\gamma^{\Sigma_{\lambda\subset\gamma}(d_\lambda + \theta)}. \end{aligned}$$

Now (3.14) follows immediately, using (3.19) and (3.20). This completes the proof of Theorem 2.6.

Remark: The basic requirement on $\delta(\gamma)$ is that

(3.19) be satisfied; (3.18) can be modified to give somewhat smaller $\delta(\gamma)$ while maintaining (3.19).

4. DISCUSSION OF LIGHT-CONE EXPANSIONS

Having specified a graph-by-graph subtraction procedure for Green functions of light-cone finite, multilocal products of fields, we now consider the question of whether such field products can be used to construct a light-cone expansion of the form (1.1). That this can be done is guaranteed (at least formally) by the recursive nature of the subtractions, and the broad outlines of such a derivation will be presented below. A detailed, rigorous treatment would require considerable additional effort and is probably premature. At this point, higher priority should be given to the task of improving the subtraction scheme so as to control the large-momentum behavior of vertex functions in a "renormalizable" way.

As a caveat to future investigators in this field, it should be pointed out that in searching for a suitable definition of light-cone finite normal products one must always make provisiion (if only at the level of a plausibility argument) for an eventual light-cone expansion. This is because the latter, by relating normal products to ordinary products of fields, allows one to establish, almost immediately, the legitimacy of the normal products as localized, covariant operator fields.

As mentioned above, the crucial property of our subtraction scheme which leads to an expansion (1.1) is its recursive nature, expressed in Eq. (2.6). That formula must be understood in the following sense: $\bar{\mathcal{Y}}_{\mathcal{J}}^{\gamma}$ is to be written in the standard α-parametric form, namely (2.10) with γ omitted from the final product over $\lambda \in \mathcal{J}$, and $t^{\delta(\gamma)}$ is to be understood as a Taylor series in τ_γ. We see that $\mathcal{Y}_{\mathcal{J}}^{\gamma}$ will then have the general form

$$\mathcal{Y}_{\mathcal{J}}^{\gamma} = \exp[i\tilde{p}^T(\bar{Y}^\gamma)_0\xi[\sum_a M_a^\gamma(\tilde{p}, \xi)G_a^\gamma(\alpha, \xi^2)], \qquad (4.1)$$

where $\{\tilde{p}_j\}$ is the set of external momenta of γ, $\xi_l = \theta_l\xi$ are assumed to be nonvanishing only for l incident on the normal-product vertex, M_a^γ is a monomial, and

$$(\bar{Y}^\gamma)_0 = Y^\gamma\big|_{\tau_\lambda=0,\, \lambda\in \mathcal{J}(\gamma)}.$$

Following Zimmermann[2] in his derivation of algebraic identities relating momentum-space integrands, we may iterate (2.6) to obtain

$$\mathcal{Y}_{\mathcal{J}}^{\Gamma} = \sum_a (\mathcal{Y}^{\Gamma/\gamma}_{\mathcal{J}\setminus\mathcal{J}(\gamma)})^a(\mathcal{Y}^{\gamma}_{\mathcal{J}(\gamma)})_a, \qquad (4.2)$$

where γ is the smallest element of $\mathcal{J}$ containing the normal product vertex and $\mathcal{J}(\gamma)$ is $\mathcal{J}$ restricted to γ. The index a runs through the terms of (4.1), with the factor $M_a^\gamma(\tilde{p}, \xi)\exp[i\tilde{p}^T(\bar{Y}^\gamma)_0\xi$ incorporated in $(\mathcal{Y}^{\Gamma/\gamma}_{\mathcal{J}\setminus\mathcal{J}(\gamma)})_a$ and $G_a^\gamma(\alpha, \xi^2)$ included in $(\mathcal{Y}^{\gamma}_{\mathcal{J}(\gamma)})_a$. Integrating over α and summing over all Γ and $\mathcal{J}$, one obtains an identity for Green functions which one can write in the shorthand form

$$N_\delta^*(\prod_{i=1}^n \phi_i(x+\theta_i\xi) = \prod_{i=1}^n \phi_i(x+\theta_i\xi) + \sum_r \int d\eta\, F_r(\eta, \xi^2) \times N_{\delta r}^*(\prod_j \mathcal{D}_{r_j}\psi_{r_j}\ (x+\eta_{r_j}\theta_j\xi)) \qquad (4.3)$$

where the derivative operator $\mathcal{D}_{r_j}$ may contain factors $\xi\cdot\partial/\partial x$ as well as $\partial/\partial x^\mu$, and the subtraction degrees, which may be greater than minimal.

Extracting powers of ξ^2 and $\ln\xi^2$ in $F_r(\eta, \xi^2)$, and applying the LSZ reduction formula to obtain an operator relation, one obtains a multilocal, light-cone expansion generalizing (1.1). Since, for any γ, $|\bar{Y}^\gamma| \leq 1$ [see Remark 2.3(b)], the support of $F_r(\eta, \xi^2)$ is contained in $|\eta| \leq 1$. The localization of $N^*(\prod_{i=1}^n \phi_i(x+\theta_i\xi))$ on the line segment connecting $x+\theta_{\min}\xi$ to $x+\theta_{\max}\xi$ then follows from the operator light-cone expansion by mathematical induction in the perturbative order.

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