JOURNAL OF MATHEMATICAL PHYSICS VOLUME 9, NUMBER 9 SEPTEMBER 1968

Analytic Renormalization

EUGENE R. SPEER*
Department of Mathematics, Princeton University, Princeton, New Jersey

(Received 1 December 1967)

Renormalized Feynman amplitudes are defined by a method of analytic continuation in subsidiary parameters. The results are shown to belong to the class of renormalized amplitudes defined by Bogoliubov, Parasiuk, and Hepp.

1. INTRODUCTION

In the perturbation-series expansion of the S matrix or the time-ordered vacuum expectation values in a Lagrangian field theory, there occur formal expressions of the form

$$\prod_{l\in\mathfrak{L}} \Delta_F^l(x_{i_l} - x_{f_l}), \tag{1.1}$$

where $\mathfrak{L}$ is the collection of lines of a certain Feynman graph $G(V_1, \cdots, V_n; \mathfrak{L})$, with vertices $\{V_i\}$, and V_{i_l} and V_{f_l} are the initial and final vertices of the lth line. Δ_F^l is given in p space by

$$\tilde{\Delta}_F^l(p) = iP_l(p)(p^2 - m_l^2 + i0)^{-1}, \tag{1.2}$$

with $P_l(p)$ a polynomial of degree r_l. In general, however, (1.1) is not well defined (even as a distribution) because the convolutions in p space diverge. In the theory of renormalization, (1.1) is given a well-defined meaning by a variety of methods, among which that of Hepp[1] is distinguished by its mathematical coherence.

In this paper we apply to (1.1) a method of defining divergent quantities which was originated by Riesz[2] and has been used in various contexts by many authors.[3] To define a formally divergent quantity I, these authors introduce a function $I(\lambda)$, analytic in some region Ω of the complex plane, and defined by an expression which is formally equal to I for $\lambda = \lambda_0$. I is then defined as the analytic continuation of $I(\lambda)$ from Ω to $\lambda = \lambda_0$. In some cases $I(\lambda)$ has a pole at λ_0; an acceptable definition of I may then be obtained as the constant term of the Laurent series of $I(\lambda)$ about λ_0.

To apply these techniques to (1.1) we find it necessary to consider functions of several complex variables $\lambda_1, \cdots, \lambda_L$, one associated with each line of the Feynman graph. The main difficulty is the extension of the above treatment of poles to the more complicated singularities which occur in several complex variables. Such an extension is given and a renormalized value of (1.1) is defined. It is shown that this definition is one of the class of renormalized values of (1.1) defined by Boguliubov, Parasiuk, and Hepp.[1]

We remark that we are interested only in defining (1.1) as a tempered distribution in $\mathcal{S}'(R^{4n})$. We restrict attention to the case of $m_l > 0$, and without loss of generality assume that $G(V_1, \cdots, V_n; \mathfrak{L})$ is connected.

2. ANALYTIC PROPERTIES

We generalize (1.2) by defining, for any complex λ_l,[4]

$$\tilde{\Delta}_{\lambda_l}^l(p) = P_l(p)e^{\frac{1}{2}i\pi\lambda_l}(p^2 - m_l^2 + i0)^{-\lambda_l}, \tag{2.1}$$

and use Hepp's regularization to write, for Re $\lambda_l > 0$,

$$\Delta_{\lambda_l}^l = \lim_{\epsilon\to 0+}\lim_{r\to 0+} \Delta_{\lambda_l,\epsilon,r}^l,$$

where

$$\tilde{\Delta}_{\lambda_l,\epsilon,r}^l(p) = P_l(p)\Gamma(\lambda_l)^{-1} \times \int_r^\infty d\alpha_l \alpha_l^{\lambda_l-1} \exp i\alpha_l(p^2 - m_l^2 + i\epsilon). \tag{2.2}$$

The distributions $\Delta_{\lambda_l}^l$ and $\Delta_{\lambda_l,\epsilon,r}^l$ are entire functions of λ_l. Moreover, when $\epsilon > 0$ and $r > 0$, $\tilde{\Delta}_{\lambda_l,\epsilon,r}^l$ is in is in $\mathcal{O}_C'(R^4)$ (the space of rapidly decreasing distributions), and its Fourier transform $\Delta_{\lambda_l,\epsilon,r}^l$ is in $\mathcal{O}_M(R^4)$ (the space of polynomially bounded infinitely differentiable functions).[5] Thus we may define unambiguously

$$\mathcal{T}_{\lambda_1,\cdots,\lambda_L,\epsilon,r}(V_1, \cdots, V_n; \mathfrak{L}) = \prod_{l\in\mathfrak{L}} \Delta_{\lambda_l,\epsilon,r}^l(x_{i_l} - x_{f_l}). \tag{2.3}$$

* Supported by a National Science Foundation Graduate Fellowship.

[1] K. Hepp, Commun. Math. Phys. **2**, 301 (1966). See also N. N. Bogoliubov and O. S. Parasiuk, Acta Math. **97**, 227 (1957); O. S. Parasiuk, Ukr. Math. J. **12**, 287 (1960).

[2] M. Riesz, Acta Math. **81**, 1, 1949.

[3] See, e.g., N. E. Fremberg, Proc. Roy. Soc. (London) **A188**, 18 (1946); T. Gustafson, Arkiv Mat. Astron. Fysik **34A** No. 2 (1947); S. B. Nilsson, Arkiv Fysik **1**, 369 (1950); G. Källen, Arkiv Fysik **5**, 130 (1951); E. Karlson, Arkiv Fysik **7**, 221 (1954); I. M. Gel'fand and G. E. Shilov, *Generalized Functions, Vol. 1* (Academic Press Inc., New York, 1964), Chap. 3; and C. G. Bollini, J. J. Ciambiagi, and A. Gonzalez Dominguez, Nuovo Cimento **31**, 550 (1964).

[4] See I. M. Gel'fand and G. E. Shilov, Ref. 3, Chap. 3, Sec. 2.4. This is a good basic reference for the properties of distributions depending analytically on a parameter.

[5] These spaces are discussed in L. Schwarts, *Théorie des distributions* (Hermann & Cie., Paris, 1966), pp. 243–244.

In this section we investigate the analytic properties of (2.3) after the limit $r \to 0+$. For convenience we write $(\lambda_1, \cdots, \lambda_L) = \boldsymbol{\lambda}$.

We remark that our results in this section would not be changed if, in (2.1), we also generalized $P_l(p)$ to $P_l(\lambda_l, p)$. Here $P_l(\lambda_l, p)$ is a covariant polynomial in p of degree r_l, whose coefficients are entire functions of λ_l which satisfy $P_l(1, p) = P_l(p)$. Consistent renormalization of a theory would require in addition that $P_l(\lambda_l, p)$ depend only on the particle associated with the lth line. Such a change in P_l would result in a finite change in the renormalization constants.

Theorem 1: Let $G(V_1, \cdots, V_n; \mathfrak{L})$ be a connected Feynman graph, as above. Define $N = L - n + 1$ to be the number of loops of G, and $\Omega = \{\boldsymbol{\lambda} \in \mathbb{C}^L \mid \operatorname{Re} \lambda_l > M, l = 1, \cdots, L\}$, where $M = N(2 + \sum_1^L r_l)$. For $\boldsymbol{\lambda} \in \Omega$, define

$$\mathcal{G}_{\boldsymbol{\lambda},\epsilon}(V_1, \cdots, V_n; \mathfrak{L}) = \lim_{r \to 0+} \mathcal{G}_{\boldsymbol{\lambda},\epsilon,r}(V_1, \cdots, V_n; \mathfrak{L}). \quad (2.4)$$

Then: (a) The limit (2.4) exists [in $\mathcal{S}'(R^{4n})$] and $\mathcal{G}_{\boldsymbol{\lambda},\epsilon}(V_1, \cdots, V_n; \mathfrak{L})$ is holomorphic in Ω.

(b) $\mathcal{G}_{\boldsymbol{\lambda},\epsilon}(V_1, \cdots, V_n; \mathfrak{L})$ may be analytically continued to a function meromorphic in $\mathbb{C}^L$. If we use the same notation for the continued function, then

$$\mathcal{G}_{\boldsymbol{\lambda},\epsilon}(V_1, \cdots, V_n; \mathfrak{L}) \prod_A \Gamma\left[\sum_{l \in A} (\lambda_l - M)\right]^{-1} \quad (2.5)$$

is holomorphic in $\mathbb{C}^L$. Here $\prod_A$ is taken over all subsets A of $\{1, \cdots, L\}$.

We remark that a more detailed discussion of the singularities of $\mathcal{G}_{\boldsymbol{\lambda},\epsilon}$ is possible but is not needed in this paper.

Proof: Let p_j be the momentum dual to x_j. We may evaluate (2.3) in p space by attaching to each vertex V_j an external line directed into the diagram and carrying momentum p_j, and then applying the integration methods of Chisholm.[6] That is, we assign paths through the diagram for the external momenta and choose loops and loop momenta $k_1, \cdots, k_N$, so that the lth line is assigned momentum

$$q_l = \sum_{i=1}^N a_{li} k_i + \sum_{j=1}^n b_{lj} p_j. \quad (2.6)$$

Then (2.3) becomes

$$\mathcal{G}_{\boldsymbol{\lambda},\epsilon,r}(V_1, \cdots, V_n; \mathfrak{L}) = \delta\left(\sum_1^n p_j\right) \int dk_1 \cdots dk_N \prod_{l=1}^L \tilde{\Delta}^l_{\lambda_l,\epsilon,r}(q_l). \quad (2.7)$$

[6] J. S. R. Chisholm, Proc. Cambridge Phil. Soc. **48**, 300 (1952). See, e.g., R. J. Eden, P. V. Landshoff, D. I. Olive, and J. C. Polkinghorne, *The Analytic S-Matrix* (Cambridge University Press, Cambridge, England, 1966), pp. 31–34.

If we interchange the k and α integrations and use $k_i = -i(\partial/\partial S_i) e^{ik_i S_i}|_{S_i=0}$ in the factors $P_l(q_l)$, we may write (2.7) as a sum of terms of the form

$$\begin{aligned}(\text{const})\delta(\textstyle\sum p_j) A(p) \displaystyle\int_r^\infty \cdots \int_r^\infty \prod_1^L [d\alpha_l \alpha_l^{\lambda_l - 1} \Gamma(\lambda_l)^{-1}] \\ \times \left\{\int dk_1 \cdots dk_N A'(-i\nabla_S) \exp i\left[\sum_{i,j=1}^N \theta_{ij} k_i k_j \right.\right. \\ \left.\left. + \sum_1^N (2\phi_i + S_i) k_i + \psi + i\epsilon \sum_1^L \alpha_l\right]\right\}\Big|_{S=0}. \quad (2.8)\end{aligned}$$

Here A and A' are monomials of degree $\le \rho = \sum_1^L r_l$, and

$$\theta_{ij} = \sum_{l=1}^L \alpha_l a_{li} a_{lj}, \quad (2.9a)$$

$$\phi_i = \sum_{l=1}^L \sum_{j=1}^n \alpha_l a_{li} b_{lj} p_j, \quad (2.9b)$$

$$\psi = \sum_{l=1}^L \sum_{j,k=1}^n \alpha_l b_{lj} b_{lk} p_j p_k - \sum_{l=1}^L \alpha_l m_l^2. \quad (2.9c)$$

When all α_l are positive, θ_{ij} is positive-definite. Thus, if we now do the k integrations, the bracket in (2.8) becomes, up to a constant factor,

$$\begin{aligned}(\det \theta)^{-2} A'(-i\nabla_S) \exp i\left[\psi - \frac{1}{4} \sum_{i,j=1}^N (2\phi_i + S_i)\right. \\ \left. \times (\theta^{-1})_{ij} (2\phi_j + S_j) + i\epsilon \sum_1^L \alpha_l\right].\end{aligned}$$

Using $\theta^{-1} = \theta^{\mathrm{Ad}}/\det \theta$, where θ^{Ad} is the transpose of the matrix of cofactors, performing the S derivatives, and setting $S = 0$, we may finally write

$$\begin{aligned}\mathcal{G}_{\boldsymbol{\lambda},\epsilon,r}(V_1, \cdots, V_n; \mathfrak{L}) \\ = \sum_{m=0}^\rho \delta(\textstyle\sum p_j) \displaystyle\int_r^\infty \cdots \int_r^\infty \prod_1^L [d\alpha_l \alpha_l^{\lambda_l - 1} \Gamma(\lambda_l)^{-1}] \\ \times B_m(p, \alpha) C(\alpha)^{-(m+2)} \exp i[D(\alpha, p)/C(\alpha) + i\epsilon \textstyle\sum \alpha_l], \\ (2.10)\end{aligned}$$

where B_m is a polynomial, $C(\alpha) = \det \theta$, and $D(\alpha, p) = \det \chi$, with

$$\chi = \begin{vmatrix} \theta_{11} & \cdots & \theta_{1N} & \phi_1 \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \theta_{N1} & & & \phi_N \\ \phi_1 & \cdots & \phi_N & \psi \end{vmatrix}. \quad (2.11)$$

The "ultraviolet divergences" occur in the limit $r \to 0+$ because $C(\alpha)$ vanishes when certain $\alpha_l \to 0$. We now investigate this behavior in a region $0 \le \alpha_{l_1} \le \cdots \le \alpha_{l_L}$; for simplicity, we consider

$$0 \le \alpha_1 \le \alpha_2 \cdots \le \alpha_L. \quad (2.12)$$

Within this region we introduce new variables $t_1, \cdots, t_L$, defined by $\alpha_l = t_L t_{L-1} \cdots t_l$, so that (2.12) becomes

$$0 \le t_L \le \infty,$$
$$0 \le t_l \le 1 \quad \text{if} \quad l = 1, \cdots, L-1. \qquad (2.13)$$

Let G_l be the graph consisting of lines 1 through l with their vertices, and let N_l be the number of loops of G_l.

Lemma 1: For α in (2.12),

$$C(\alpha) = \prod_1^L t_l^{N_l} E(t_1, \cdots, t_{L-1}), \qquad (2.14a)$$

$$D(\alpha, p) = t_L \prod_1^L t_l^{N_l} F(t_1, \cdots, t_{L-1}, p), \qquad (2.14b)$$

where E and F polynomials, and E does not vanish in (2.13).

Proof: Since $N_1 = 0$, $N_L = N$, and $(N_{l+1} - N_l)$ is always 0 or 1, there exist integers $1 < l_1 < \cdots < l_N \le L$ such that $N_{l_i} = N_{(l_i - 1)} + 1$. Thus we may choose loop variables so that the ith loop is contained in G_{l_i}, that is, so that $a_{l_i} = 0$ unless $l \le l_i$ [see (2.6)]. From (2.9) and (2.11) we see that the ith row and column $(1 \le i \le N)$ of θ and χ contain a factor $t_L \cdots t_{l_i}$, and the $(N+1)$th row and column of D contain a factor t_L. We remove these factors from the rows to produce new matrices θ' and χ'; this gives (2.14) with $E = \det \theta'$, $F = \det \chi'$.

To show that E does not vanish, we consider instead of θ' the matrix θ'', obtained from θ by removing a factor $(t_L \cdots t_{l_i})^{\frac{1}{2}}$ from the ith row *and* column of θ. θ'' is symmetric, and $E = \det \theta''$. Suppose $E(t) = 0$ at some point $t = \tau$ in (2.13). Then there exist numbers $\delta_1, \cdots, \delta_N$ such that $\sum \delta_i \theta''_{ij}(\tau) \delta_j = 0$, or

$$\sum_{l=1}^{L} \left[\sum_{i=1}^{N} \delta_i a_{li} \prod_{l \le l' < l_i} \tau_{l'}^{\frac{1}{2}} \right]^2 = 0. \qquad (2.15)$$

Each term in the sum over l must vanish. Let $I = \max \{i \mid \delta_i \ne 0\}$ and consider the term with $l = l_I$. $\delta_i = 0$ for $i > I$, while $a_{l_I i} = 0$ for $i < I$. Thus we must have $\delta_I a_{l_I I} = 0$. But $\delta_I \ne 0$, and the Ith loop must go through the l_Ith line, so $a_{l_I I} \ne 0$. This contradiction proves the lemma.

Now consider an integrand of (2.10) in the region (2.12) and change variables to $t_1, \cdots, t_L$. The Jacobian of this change is $\prod_1^L t_l^{l-1}$, so that (2.10) becomes a sum of terms of the form

$$\delta(\sum p_j) \int_r^\infty dt_L \int^1 dt_{L-1} \cdots \int^1 dt_1$$
$$\times \prod_1^L [\Gamma(\lambda_l)^{-1} t_l^{(\mu_l - (m+2)N_l - 1)}] B'_m(p, t) E(t)^{-(m+2)}$$
$$\times \exp i t_L [F/E + i\epsilon(1 + t_{L-1} + \cdots)], \qquad (2.16)$$

where $\mu_l = \sum_{l'=1}^{l} \lambda_{l'}$, and B'_m is a polynomial. The lower limits of the $t_{L-1}, \cdots, t_1$ integrations in (2.16) are complicated functions whose only relevant property is that they approach 0 when $r \to 0+$. For $\lambda \in \Omega$, $\operatorname{Re} \mu_l > (m+2)N$, so the integrand of (2.16) is absolutely integrable in all of the region (2.13). This justifies the limit $r \to 0+$ in Ω; the analyticity is clear. Thus $\mathcal{C}_{\lambda,\epsilon}(V_1, \cdots, V_n; \mathfrak{L})$ for $\boldsymbol{\lambda} \in \Omega$ is a sum of terms of the form (2.16) with 0 as the lower limit on all integrals, and, in general, with $\mu_l = \sum_{l' \in A} \lambda_{l'}$ for some $A \subset \{1, \cdots, L\}$.

We now prove part (b) of the theorem. Given a positive integer M', we may construct a continuation of (2.16) into the region

$$\Omega_{M'} = \{\boldsymbol{\lambda} \in \mathbb{C}^L \mid \operatorname{Re} \lambda_l > X_{M'}, \quad l = 1, \cdots, L\},$$

where

$$X_{M'} = \begin{cases} M - M', & \text{if } M - M' \ge 0, \\ (M - M')/L, & \text{if } M - M' < 0, \end{cases}$$

as follows. We do M' integrations by parts with respect to each of $t_1, \cdots, t_{L-1}$, integrating the factor $t_l^{[\mu_l - (m+2)N_l - 1]}$ (or the higher powers of t arising from this) and differentiating the rest. This is permissible for $\boldsymbol{\lambda} \in \Omega$; in each partial integration the integrated terms vanish as the lower limit. Finally, the t_L integration may be done explicitly with the use of the formula

$$\int_0^\infty dt\, t^{\mu-1} e^{it\kappa} = e^{\frac{1}{2}i\pi\mu} \Gamma(\mu) \kappa^{-\mu}, \qquad (2.17)$$

valid for $\operatorname{Re} \mu > 0$, $\operatorname{Im} \kappa > 0$. Thus $\mathcal{C}_{\lambda,\epsilon}(V_1, \cdots, V_n; \mathfrak{L})$ may be written as a sum of terms of the form

$$H(\boldsymbol{\lambda}) \int_0^1 \cdots \int_0^1 \prod_{l=1}^{L'} \{dt'_l t_l'^{[\mu_l - (m+2)N_l + M' - 1]}\} G(t', p, \epsilon) E(t')_i$$
$$\times [F/E + i\epsilon(1 + t_{L-1} + \cdots)]^{(j - \sum_1^L \lambda_l)}. \qquad (2.18)$$

Here $\{t'_1, \cdots, t'_L\}$ is a subset of $\{t_1, \cdots, t_{L-1}\}$ (the rest having been set equal to 1 during some partial integration), G is a polynomial, i and j are integers, and $H(\boldsymbol{\lambda})$ contains factors from (2.17) as well as factors $(\mu_l - k)^{-1}$ arising from the partial integrations. Since $[\operatorname{Re} \mu_l - (m+2)N_l + M'] > 0$ for $\boldsymbol{\lambda} \in \Omega_{M'}$, (2.18) provides a continuation of $\mathcal{C}_{\lambda,\epsilon}(V_1, \cdots, V_n; \mathfrak{L})$ to the region $\Omega_{M'}$; moreover,

$$H(\boldsymbol{\lambda}) \prod_{A \subset \{1 \cdots L\}} \Gamma\left[\sum_{l \in A} (\lambda_l - M) \right]^{-1}$$

is an entire function of $\boldsymbol{\lambda}$. Since $\Omega_{M'}$ increases to $\mathbb{C}^L$ as M' approaches infinity, part (b) of the theorem is proved.

3. RENORMALIZATION

It would now be natural to define (1.1) as

$$\lim_{\epsilon\to 0} \mathcal{G}_{1,\cdots 1,\epsilon}(V_1, \cdots, V_n; \mathfrak{L});$$

however, Theorem 1 implies that $\mathcal{G}_{\lambda,\epsilon}$ may have a complicated singularity at $\boldsymbol{\lambda} = (1, \cdots, 1)$. In one complex variable we could discard the singular part by using the constant term of the Laurent series. In this section we generalize this procedure to several variables.

Definition: Let $U \subset \mathbb{C}^L$ $(L \geq 1)$ be an open neighborhood of $(1, \cdots, 1)$. Let $\mathcal{A}_L(U) = \{f(\boldsymbol{\lambda}) \mid f(\boldsymbol{\lambda}) \prod_{A\subset\{1,\cdots,L\}}[\sum_{l\in A}(\lambda_l - 1)]^m$ is analytic in U for some integer $m \geq 0\}$, and let $\mathcal{A}_L = \cup \mathcal{A}_L(U)$, the union taken over all neighborhoods U. Then a family of maps $\mathcal{F} = \{\mathcal{F}_L\}_{L=1}^{\infty}$, $\mathcal{F}_L: \mathcal{A}_L \to \mathbb{C}$, is a *generalized evaluator* [at $(1, \cdots, 1)$] if the following conditions are satisfied for each L:

(1) $\mathcal{F}_L$ is linear;

(2) if $f \in \mathcal{A}_L$ is analytic at $(1, \cdots, 1)$, then $\mathcal{F}_L f = f(1, \cdots, 1)$;

(3) if $f_n \in \mathcal{A}_L(U)$, for $n = 0, 1, \cdots, g_n(\boldsymbol{\lambda}) = f_n(\boldsymbol{\lambda}) \prod_A [\sum_{l\in A} (\lambda_l - 1)]^m$, is analytic in U, and $g_n \to g_0$ uniformly on U, then $\mathcal{F}_L f_n \to \mathcal{F}_L f_0$;

(4) if σ is a permutation of $\{1, \cdots, L\}$, $f \in \mathcal{A}_L$, and $f_\sigma \in \mathcal{A}_L$ is defined by

$$f_\sigma(\lambda_1, \cdots, \lambda_L) = f(\lambda_{\sigma(1)}, \cdots, \lambda_{\sigma(L)}),$$

then $\mathcal{F}_L f_\sigma = \mathcal{F}_L f$;

(5) if $f \in \mathcal{A}_L$ depends only on $\lambda_1, \cdots, \lambda_{L'}$, where $L' < L$, then $\mathcal{F}_{L'} f = \mathcal{F}_L f$;

(6) if $f_1, f_2 \in \mathcal{A}_L$, and f_1 depends only on $\lambda_1, \cdots, \lambda_{L'}$, f_2 only on $\lambda_{L'+1}, \cdots, \lambda_L$, then $\mathcal{F}_L(f_1 f_2) = (\mathcal{F}_L f_1) \times (\mathcal{F}_L f_2)$.

If $f \in \mathcal{A}_L$, we use Conditions (4) and (5) to write without ambiguity $\mathcal{F}f = \mathcal{F}_L f = \mathcal{F}_{L'} f$ for any $L' \geq L$. Conditions (1)–(5) are rather natural; the utility of (6) will be shown in Sec. 5. It is this condition which would be violated by setting $\lambda_1 = \cdots = \lambda_L = \lambda$ and defining $\mathcal{F}f$ as the constant term of the Laurent series of $f(\lambda, \lambda, \cdots, \lambda)$ at $\lambda = 1$.

Example: Suppose $f \in \mathcal{A}_L(U)$, and let U contain the poly disc $|\lambda_l - 1| < R$. Choose $0 < R_1 < \cdots < R_L < R$, in such a way that $R_i > \sum_{j=1}^{i-1} R_j$, and let C_i be the contour $|z - 1| = R_i$ oriented counterclockwise. Define

$$\mathcal{F}_L f = \frac{1}{L!} \sum_\sigma \frac{1}{(2\pi i)^L} \int_{C_{\sigma(1)}} d\lambda_1 \cdots \times \int_{C_{\sigma(L)}} d\lambda_L f(\lambda) \prod_1^L (\lambda_l - 1)^{-1}, \quad (3.1)$$

where $\sum_\sigma$ runs over all permutations σ of $\{1, \cdots, L\}$. One easily checks that $\mathcal{F}$ is well defined, independent of the choice of $\{R_i\}$, and satisfies (1)–(6).

We want to be able to apply a generalized evaluator to meromorphic distributions. Consider such a distribution:

$$S(\boldsymbol{\lambda}) = S'(\boldsymbol{\lambda}) \prod_{A\subset\{1\cdots L\}} \left[\sum_{l\in A} (\lambda_l - 1)\right]^{-m},$$

where $S'(\boldsymbol{\lambda})$ is an analytic function of $(\lambda_1, \cdots, \lambda_L)$ in some neighborhood U of $(1, \cdots, 1)$, taking values in $\mathcal{S}'(R^n)$. Then the formula $(\mathcal{F}_L S)(\psi) = \mathcal{F}_L(S(\psi))$ defines a linear functional $\mathcal{F}_L S$ on $\mathcal{S}(R^n)$. Now $S': U \to \mathcal{S}'(R^n)$ is continuous [when $\mathcal{S}'(R^n)$ is given the usual weak topology], so that if $K \to U$ is compact, $S'(K)$ is (weakly) compact in $\mathcal{S}'(R^n)$, and hence is strongly bounded.[7] That is, there is a constant C_K and a norm $\| \ \|$ on $\mathcal{S}(R^n)$ (one of the norms defining the topology) such that $|S'(\boldsymbol{\lambda})(\psi)| \leq C_K \|\psi\|$ for any $\boldsymbol{\lambda} \in K$ and any $\psi \in \mathcal{S}(R^n)$. So for any sequence $\{\psi_i\}$ of elements of $\mathcal{S}(R^n)$, converging to an element ψ_0, the sequence $\{S'(\boldsymbol{\lambda})(\psi_i)\}$ converges uniformly for $\boldsymbol{\lambda} \in K$ to $S'(\boldsymbol{\lambda})(\psi_0)$. Then property (3) of $\mathcal{F}$ implies that $\mathcal{F}S$ as defined above is continuous.

Definition: The renormalized value of (1.1) is defined to be

$$\mathcal{G}(V_1, \cdots, V_n; \mathfrak{L}) = \lim_{\epsilon\to 0+} \mathcal{F}\mathcal{G}_{\lambda,\epsilon}(V_1, \cdots, V_n; \mathfrak{L}). \quad (3.2)$$

The existence of the $\epsilon \to 0+$ limit follows from the theorem we prove in Sec. 5: the agreement of this definition with that of Boguliubov, Parasiuk, and Hepp. It may also be proved directly that:

(a) $\lim_{\epsilon\to 0} \mathcal{G}_{\lambda,\epsilon} = \mathcal{G}_\lambda$ exists and is a meromorphic function of $\boldsymbol{\lambda}$ with the same singularities as $\mathcal{G}_{\lambda,\epsilon}$;

(b) $\mathcal{G} = \mathcal{F}\mathcal{G}_\lambda$.

We remark that a change in the generalized evaluator used in (3.2) is reflected in a finite change in the renormalization constants.

4. BOGULIUBOV–PARASIUK–HEPP RENORMALIZATION

We now review the renormalization methods of Boguliubov, Parasiuk, and Hepp,[1] and extend their results slightly. We follow the notation of Hepp.

Definition: A graph $G(V_1, \cdots, V_n; \mathfrak{L})$ is *one-particle irreducible* (OPI) if, for any $l \in \mathfrak{L}$ and $\mathfrak{L}' = \mathfrak{L} - \{l\}$, $G(V_1, \cdots, V_n; \mathfrak{L}')$ is connected. Otherwise G is *one-particle reducible* (OPR). A *generalized vertex* of G is a nonempty subset $U = \{V_1' \cdots V_m'\}$ of $\{V_1 \cdots V_n\}$.

[7] I. M. Gel'fand and G. E. Schilow, *Verallgemeinerte Funktionen II* (VEB Deutscher Verlag der Wissenschaften, Berlin, 1962), Chap. 1, Sec. 5.

If $U_1, \cdots, U_m$ are pairwise-disjoint generalized vertices, with $\bigcup_{i=1}^m U_i = \{V_1', \cdots, V_s'\}$, the graph $G(U_1, \cdots, U_m; \mathfrak{L})$ is obtained from $G(V_1', \cdots, V_s'; \mathfrak{L})$ by collapsing each generalized vertex U_i, and any lines which join two vertices in U_i, to a single point. The *superficial divergence* of $U = \{V_1', \cdots, V_m'\}$ is defined by

$$\nu(V_1', \cdots, V_m') = \sum_{\text{conn}} (r_l + 2) - 4(m-1), \quad (4.1)$$

where $\sum_{\text{conn}}$ runs over all lines of $\mathfrak{L}$ connecting different vertices of $\{V_1', \cdots, V_m'\}$. We do not distinguish between the vertex V_i and the generalized vertex $\{V_i\}$.

Definition: A *finite renormalization* is a map assigning to each generalized vertex $U = \{V_1', \cdots, V_m'\}$ a distribution $\hat{\mathfrak{X}}_\epsilon(V_1', \cdots, V_m'; \mathfrak{L})$ [also written $\hat{\mathfrak{X}}_\epsilon(U; \mathfrak{L})$] in $\mathcal{S}'(R^{4m})$ such that

$$\hat{\mathfrak{X}}_\epsilon(V_1', \cdots, V_m'; \mathfrak{L}) = \begin{cases} 1, & \text{for } m = 1, \\ 0, & \text{for IPR } G(V_1', \cdots, V_m'; \mathfrak{L}), \\ \delta\left(\sum_1^m p_j'\right) P_\epsilon(p_1', \cdots, p_m'), & \text{otherwise.} \end{cases} \quad (4.2)$$

Here P_ϵ is a covariant polynomial of degree $\leq \nu(V_1' \cdots V_m')$, whose coefficients approach finite limits as $\epsilon \to 0$, and which depends only on the structure of the graph $G(V_1', \cdots, V_m'; \mathfrak{L})$.

Definition: Given a finite renormalization $\hat{\mathfrak{X}}_\epsilon$, $U_1, \cdots, U_r$ pairwise-disjoint generalized vertices, define recursively for $\{U_1', \cdots, U_m'\} \subset \{U_1, \cdots, U_r\}$:

$$\mathfrak{X}_{\lambda,\epsilon,r}(U_1', \cdots U_m'; \mathfrak{L}) = \begin{cases} \hat{\mathfrak{X}}_\epsilon(U_1'; \mathfrak{L}), & \text{if } m = 1, \quad (4.3a) \\ 0, & \text{for OPR } G(U_1', \cdots, U_m'; \mathfrak{L}), \quad (4.3b) \\ -\mathcal{M}\bar{\mathfrak{R}}_{\lambda,\epsilon,r}(U_1', \cdots, U_m'; \mathfrak{L}), & \text{otherwise,} \quad (4.3c) \end{cases}$$

$$\bar{\mathfrak{R}}_{\lambda,\epsilon,r}(U_1', \cdots, U_m'; \mathfrak{L}) = \sum_P' \prod_{j=1}^{k(P)} \mathfrak{X}_{\lambda,\epsilon,r}(U_{j1}^P, \cdots, U_{jr(j)}^P; \mathfrak{L}) \prod_{\text{conn}} \Delta_{\lambda_l,\epsilon,r}^l, \quad (4.4)$$

$$\mathfrak{R}_{\lambda,\epsilon,r}(U_1', \cdots, U_m'; \mathfrak{L}) = \bar{\mathfrak{R}}_{\lambda,\epsilon,r}(U_1', \cdots, U_m'; \mathfrak{L}) + \mathfrak{X}_{\lambda,\epsilon,r}(U_1', \cdots, U_m'; \mathfrak{L}). \quad (4.5)$$

Here $\sum_P'$ in (4.4) runs over all partitions of $\{U_1', \cdots, U_m'\}$ into $k(P) \geq 2$ disjoint subsets

$$\{U_{j1}^P, \cdots, U_{jr(j)}^P\}$$

and $\prod_{\text{conn}}$ runs over those $l \in \mathfrak{L}$ which connect different subsets of the partition. When

$$G(U_1', \cdots, U_m'; \mathfrak{L})$$

is OPI, and

$$\bigcup_{i=1}^m U_i' = \{V_1', \cdots, V_s'\},$$

then $\bar{\mathfrak{R}}$ has in p space the form $\delta(\sum_{i=1}^s p_i')F(p_1', \cdots, p_s')$, and $\mathcal{M}$ is the operation of truncating the Taylor series of F about the origin at order $\nu(V_1', \cdots, V_s')$ [$\mathcal{M} = 0$ if $\nu(V_1', \cdots, V_s') < 0$].

In the case where each U_i is a single vertex V_i, we also define

$$\mathfrak{X}_{\lambda,\epsilon,r}'(V_1', \cdots, V_m'; \mathfrak{L}) = \begin{cases} 1, & \text{if } m = 1, \quad (4.3a') \\ 0, & \text{for OPR } G(V_1', \cdots, V_m'; \mathfrak{L}), \quad (4.3b') \\ -\mathcal{M}\bar{\mathfrak{R}}_{\lambda,\epsilon,r}'(V_1', \cdots, V_m'; \mathfrak{L}) + \hat{\mathfrak{X}}_\epsilon(V_1', \cdots, V_m'; \mathfrak{L}), & \text{otherwise,} \quad (4.3c') \end{cases}$$

$$\bar{\mathfrak{R}}_{\lambda,\epsilon,r}'(V_1', \cdots, V_m'; \mathfrak{L}) = \sum_P' \prod_{j=1}^{k(P)} \mathfrak{X}_{\lambda,\epsilon,r}'(V_{j1}^P, \cdots, V_{jr(j)}^{(P)}; \mathfrak{L}) \prod_{\text{conn}} \Delta_{\lambda_l,\epsilon,r}^l, \quad (4.4')$$

$$\mathfrak{R}_{\lambda,\epsilon,r}'(V_1', \cdots, V_m'; \mathfrak{L}) = \bar{\mathfrak{R}}_{\lambda,\epsilon,r}'(V_1', \cdots, V_m'; \mathfrak{L}) + \mathfrak{X}_{\lambda,\epsilon,r}'(V_1', \cdots, V_m'; \mathfrak{L}), \quad (4.5')$$

with $\sum_P'$, $\prod_{\text{conn}}$, and $\mathcal{M}$ as above. The following lemma may be proved by straightforward manipulation of these definitions.

Lemma 2: With the above definitions, we have

$$\bar{\mathfrak{R}}_{\lambda,\epsilon,r}'(V_1', \cdots, V_m'; \mathfrak{L}) = \sum_P \bar{\mathfrak{R}}_{\lambda,\epsilon,r}(U_1^P, \cdots, U_{m(P)}^P; \mathfrak{L}),$$

$$\mathfrak{X}_{\lambda,\epsilon,r}'(V_1', \cdots, V_m'; \mathfrak{L}) = \sum_P \mathfrak{X}_{\lambda,\epsilon,r}(U_1^P, \cdots, U_{m(P)}^P; \mathfrak{L}),$$

and hence

$$\mathfrak{R}_{\lambda,\epsilon,r}'(V_1', \cdots, V_m'; \mathfrak{L}) = \sum_P \mathfrak{R}_{\lambda,\epsilon,r}(U_1^P, \cdots, U_{m(P)}^P; \mathfrak{L}),$$

where $\sum_P$ runs over all partitions of $\{V_1', \cdots, V_m'\}$ into $m(P)$ generalized vertices $\{U_j^P\}$.

Now Boguliubov, Parasiuk, and Hepp define the renormalized value of (1.1) to be

$$\lim_{\epsilon \to 0+} \lim_{r \to 0+} \mathfrak{R}_{1,\cdots,1,\epsilon,r}'(V_1, \cdots, V_n; \mathfrak{L}); \quad (4.6)$$

that is, they define a class of values of (1.1) which depend on the finite renormalization used. The main result of Hepp is the existence of the $r \to 0+$ limit in (4.6); it may be generalized as follows.

Theorem 2: Let

$$\Omega' = \{\boldsymbol{\lambda} \in \mathbb{C}^L \mid \operatorname{Re} \lambda_l > 1 - 1/2L,\ l = 1, \cdots, L\}.$$

Then

$$\mathfrak{R}_{\lambda,\epsilon}'(V_1, \cdots, V_n; \mathfrak{L}) = \lim_{r \to 0+} \mathfrak{R}_{\lambda,\epsilon,r}'(V_1, \cdots, V_n; \mathfrak{L}) \quad (4.7)$$

exists in $\mathcal{S}'(R^{4n})$ and is analytic for $\boldsymbol{\lambda} \in \Omega'$.

Proof: Hepp actually proves the existence of

$$\lim_{r\to 0+} \mathcal{R}_{1,\cdots,1,\epsilon,r}(V_1,\cdots,V_n;\mathfrak{L}),$$

that is, the existence of (4.7) for $\boldsymbol{\lambda} = (1,\cdots,1)$ when $\mathcal{R}'$ is defined using zero finite renormalization. However, it is a trivial modification of his proof to show the existence and analyticity in Ω' of

$$\lim_{r\to 0+} \mathcal{R}_{\boldsymbol{\lambda},\epsilon,r}(U_1,\cdots,U_r;\mathfrak{L}),$$

for any pairwise-disjoint generalized vertices $U_1,\cdots,U_r$. The theorem then follows from Lemma 2.

5. EQUIVALENCE OF THE DEFINITIONS

In this section we show that our definition (3.2) of the renormalized amplitude agrees with the Bogu-liubov definition (4.6), calculated using a certain finite renormalization.

Definition: We write

$$J_L(\boldsymbol{\lambda}) = \prod_{A\subset\{1,\ldots,L\}} \Gamma\Big[\sum_{l\in A}(\lambda_l - M)\Big] \qquad \Big[\text{recall } M = N\Big(2+\sum_1^L r_l\Big)\Big].$$

Let $\mathcal{B}(L,m)$ be the set of mappings $\phi:\mathbb{C}^L\to\mathcal{S}'(R^{4m})$ with the form

$$\phi(\boldsymbol{\lambda})(p_1,\cdots,p_m) = \delta\Big(\sum_{i=1}^m p_i\Big)J_L(\boldsymbol{\lambda})f(\boldsymbol{\lambda},p_1,\cdots,p_m), \tag{5.1}$$

where

(a) $f\in C^\infty(R^{2L+4m})$;

(b) f is analytic in $\boldsymbol{\lambda}$ for fixed p;

(c) if D is a monomial in the p derivatives and $K\subset\mathbb{C}^L$ a compact set, there are positive constants C_1 and C_2 such that

$$|Df(\boldsymbol{\lambda},p_1,\cdots,p_m)| \le C_1(1+\|p\|^2)^{C_2}$$

uniformly for $\boldsymbol{\lambda}\in K$.

For any integer ν, define $\mathcal{M}_\nu:\mathcal{B}(L,m)\to\mathcal{B}(L,m)$ by

$$[\mathcal{M}_\nu(\phi)](\boldsymbol{\lambda})(p_1,\cdots,p_m) = \delta\Big(\sum_1^m p_i\Big)J_L(\boldsymbol{\lambda})F_\nu(\boldsymbol{\lambda},p_1,\cdots,p_m),$$

where ϕ is given by (5.1) and F_ν is the Taylor series of f in p about the origin up to order ν ($\mathcal{M}_\nu = 0$ if $\nu < 0$).

Lemma 3: Let $\mathcal{F}$ be a generalized evaluator. Then $\mathcal{F}:\mathcal{B}(L,m)\to\mathcal{B}(L,m)$, and $\mathcal{F}$ commutes with $\mathcal{M}_\nu$ on $\mathcal{B}(L,m)$.

Proof: $\mathcal{F}$ is defined on an element $\phi\in\mathcal{B}(L,m)$ by $(\mathcal{F}\phi)(\psi) = \mathcal{F}[\phi(\psi)]$, for any $\psi\in\mathcal{S}(R^{4m})$. We claim that, if ϕ has the form (5.1),

$$\mathcal{F}\phi(p) = \delta\Big(\sum_1^m p_i\Big)\mathcal{F}[J(\boldsymbol{\lambda})f(\boldsymbol{\lambda},p)]. \tag{5.2}$$

Note first that the difference quotient defining a p derivative of f converges uniformly in $\boldsymbol{\lambda}$ (on compact sets), so that property (3) of $\mathcal{F}$ implies that

$$\mathcal{F}[J(\boldsymbol{\lambda})f(\boldsymbol{\lambda},p)]\in C^\infty(R^{4m}).$$

Moreover, for $\boldsymbol{\lambda}\in K$, $f(\boldsymbol{\lambda},p)\times(1+\|p\|^2)^{-(C_2+1)}\to 0$ as $\|p\|\to\infty$, so that (3) implies $\mathcal{F}[J(\boldsymbol{\lambda})f(\boldsymbol{\lambda},p)]\in\mathcal{O}_M(R^{4m})$, that is, (5.2) is indeed in $\mathcal{B}(L,m)$ (as a constant function of $\boldsymbol{\lambda}$). Now

$$\phi(\boldsymbol{\lambda})(\psi) = \int_{\Sigma p_i=0}\psi(p)J(\boldsymbol{\lambda})f(\boldsymbol{\lambda},p)\,dp,$$

and this integral may be approximated uniformly in compact subsets of $\mathbb{C}^L$ by Riemann sums. The linearity and continuity of $\mathcal{F}$ then imply (5.2). The fact that $\mathcal{M}_\nu$ and $\mathcal{F}$ commute follows again from the uniformity of the limit defining a p derivative.

The results of Sec. 2 imply that

$$\mathcal{T}_{\boldsymbol{\lambda},\epsilon}(V_1',\cdots,V_m';\mathfrak{L})\in\mathcal{B}(L,m)$$

for any $\{V_1',\cdots,V_m'\}$. Thus we may define

$$\hat{\mathfrak{X}}_\epsilon(V_1',\cdots,V_m';\mathfrak{L}) = \begin{cases}1, & \text{for } m=1,\\ 0, & \text{for OPR } G(V_1',\cdots,V_m';\mathfrak{L}),\\ \mathcal{F}\mathcal{M}\mathcal{T}_{\boldsymbol{\lambda},\epsilon}(V_1',\cdots,V_m';\mathfrak{L}), & \text{otherwise.}\end{cases} \tag{5.3}$$

Here $\mathcal{M} = \mathcal{M}_{\nu(V_1',\cdots V_m')}$.

Lemma 4: $\hat{\mathfrak{X}}_\epsilon(V_1',\cdots,V_m';\mathfrak{L})$ as given by (5.3) is a finite renormalization.

Proof: $\mathfrak{X}_\epsilon$ clearly has the correct form (4.2); property (4) guarantees that $\hat{\mathfrak{X}}_\epsilon$ depends only on the structure of the graph $G(V_1',\cdots,V_m';\mathfrak{L})$. The existence of the $\epsilon\to 0+$ limit follows from the explicit form of $\mathcal{T}_{\boldsymbol{\lambda},\epsilon}$ given in (2.18).

Now we may define $\mathfrak{X}'_{\boldsymbol{\lambda},\epsilon,r}(V_1',\cdots,V_m';\mathfrak{L})$,

$$\overline{\mathcal{R}}'_{\boldsymbol{\lambda},\epsilon,r}(V_1';\cdots,V_m';\mathfrak{L}),$$

and $\mathcal{R}'_{\boldsymbol{\lambda},\epsilon,r}(V_1',\cdots,V_m';\mathfrak{L})$ by formulas (4.3′)–(4.5′), using (5.3) as finite renormalization. We have already discussed the behavior of $\lim_{r\to 0+}\mathcal{R}'_{\boldsymbol{\lambda},\epsilon,r}$.

Lemma 5: Let Ω be as in Theorem 1. Then

$$\mathfrak{X}'_{\lambda,\epsilon}(V'_1,\cdots,V'_m;\mathfrak{L}) = \lim_{r\to 0+} \mathfrak{X}'_{\lambda,\epsilon,r}(V'_1,\cdots,V'_m;\mathfrak{L}),$$

$$\bar{\mathcal{R}}'_{\lambda,\epsilon}(V'_1,\cdots,V'_m;\mathfrak{L}) = \lim_{r\to 0+} \bar{\mathcal{R}}'_{\lambda,\epsilon,r}(V'_1,\cdots,V'_m;\mathfrak{L})$$

exist for $\boldsymbol{\lambda}\in\Omega$ and may be analytically continued to $\mathbb{C}^L$; they are in $\mathcal{B}(L, m)$.

Proof: Similar to Theorem 1. We note in particular that $\mathfrak{X}'_{\lambda,\epsilon}(V'_1,\cdots,V'_m;\mathfrak{L})$ has the form

$$\delta\left(\sum_1^m p'_j\right)\sum_{|i|<\nu(V_1'\cdots V_m')} f_{(i)}(\boldsymbol{\lambda},\epsilon)p'^{(i)}, \tag{5.4}$$

where (i) is a multi-index,

$$p'^{(i)} = \prod_{j=1}^{m}\prod_{\mu=0}^{4} p'^{i_{j\mu}}_{j\mu},$$

and $f_{(i)}(\boldsymbol{\lambda},\epsilon)\in\mathcal{A}_L$.

Theorem 3: Let $\mathcal{R}'_{\lambda,\epsilon,r}(V_1,\cdots,V_n;\mathfrak{L})$ be defined using (5.3) as finite renormalization. Then

$$\mathcal{F}\mathcal{T}_{\lambda,\epsilon}(V_1,\cdots,V_n;\mathfrak{L}) = \lim_{r\to 0+}\mathcal{R}'_{1,\cdots,1,\epsilon,r}(V_1,\cdots,V_n;\mathfrak{L}). \tag{5.5}$$

We remark that Hepp has shown that the $\epsilon\to 0$ limit of the right-hand side of (5.5) exists. This justifies our definition (3.2) of $\mathcal{T}(V_1,\cdots,V_n;\mathfrak{L})$, and the $\epsilon\to 0$ limit of (5.5) is just the equality of the two definitions of the renormalized amplitudes.

Proof: We first show that, for $m'>1$,

$$\mathcal{F}\mathfrak{X}'_{\lambda,\epsilon}(V'_1,\cdots,V'_{m'};\mathfrak{L}) = 0. \tag{5.6}$$

The statement is, of course, true (vacuously) for $m'=1$; we assume it for all $1\le m'<m$, and consider an OPI graph $G(V'_1,\cdots,V'_m;\mathfrak{L})$.

From (4.3C′),

$$\mathfrak{X}'_{\lambda,\epsilon,r}(V'_1,\cdots,V'_m;\mathfrak{L}) = -\left\{\sum_P{}'\prod_{j=1}^{k(P)}\mathfrak{X}'_{\lambda,\epsilon,r}(V^P_{j1},\cdots,V^P_{jr(j)};\mathfrak{L})\prod_{\text{conn}}\Delta^l\right\} + \hat{\mathfrak{X}}_\epsilon(V'_1,\cdots,V'_m;\mathfrak{L}). \tag{5.7}$$

Consider a term from $\sum'_P$ in (5.7) in which $r(j)>1$ for some j, say $j=1$ [note $k(P)\ge 2$, so we must have $r(j)<m$]. From (5.4) this has the form in p space

$$W_P(\boldsymbol{\lambda},\epsilon,r) = \sum_{(i)} f_{(i)}(\boldsymbol{\lambda},\epsilon,r)\{(\delta(\textstyle\sum p)p^{(i)}) * V\}, \tag{5.8}$$

where V is the Fourier transform of

$$\prod_{j=2}^{k(P)}\mathfrak{X}'_{\lambda,\epsilon,r}(V^P_{j1},\cdots)\prod_{\text{conn}}\Delta^l.$$

For $\boldsymbol{\lambda}\in\Omega$, we can let $r\to 0+$ in (5.8). The bracketed term converges to an element in $\mathcal{B}(L,m)$, and $f_{(i)}(\boldsymbol{\lambda},\epsilon,r)$ converges to $f_{(i)}(\boldsymbol{\lambda},\epsilon)\in\mathcal{A}_L$. Actually, however, $f_{(i)}(\boldsymbol{\lambda},\epsilon)$ depends only on those λ_l such that lth line joins two vertices of $\{V^P_{11},\cdots,V^P_{1r(1)}\}$, while the bracket in (5.8) depends on those λ_l such that the lth line has at least one end point outside this set. Thus property (6) of $\mathcal{F}$ implies

$$\mathcal{F}\left[\lim_{r\to 0+}W_P\right] = \sum_{(i)}[\mathcal{F}f_{(i)}(\boldsymbol{\lambda},\epsilon)]\left[\mathcal{F}\lim_{r\to 0}\{\quad\}\right].$$

But by the induction assumption

$$\mathcal{F}\mathfrak{X}'_{\lambda,\epsilon}(V^P_{11},\cdots,V^P_{1r(1)};\mathfrak{L}) = 0,$$

so that $\mathcal{F}f_{(i)}(\boldsymbol{\lambda},\epsilon)=0$ and hence

$$\mathcal{F}\left[\lim_{r\to 0+}W_P(\boldsymbol{\lambda},\epsilon,r)\right] = 0. \tag{5.9}$$

Now, using Lemma 3,

$$\mathcal{F}\mathfrak{X}'_{\lambda,\epsilon}(V_1,\cdots,V_m;\mathfrak{L}) = -\mathcal{M}\mathcal{F}\left(\sum_P{}'\left[\lim_{r\to 0+}W_P\right]\right) + \hat{\mathfrak{X}}_\epsilon(V_1,\cdots,V_m;\mathfrak{L}), \tag{5.10}$$

since property (2) of $\mathcal{F}$ implies $\mathcal{F}^2=\mathcal{F}$. But by (5.9), all terms of $\sum'_P$ in (5.10) vanish except for that partition in which $r(j)=1$ for all j. However, this term is exactly cancelled by $\hat{\mathfrak{X}}_\epsilon(V_1,\cdots,V_m;\mathfrak{L})$; this proves (5.6).

Equation (4.5′), defining $\mathcal{R}'$, may be written

$$\mathcal{R}'_{\lambda,\epsilon,r}(V_1,\cdots,V_n;\mathfrak{L}) = \prod_{\mathfrak{L}}\Delta^l_{\lambda_l,\epsilon,r} + \sum_P{}''\prod_{j=1}^{k(P)}\mathfrak{X}'_{\lambda,\epsilon,r}(V^P_{j1},\cdots,V^P_{jr(j)};\mathfrak{L})\prod_{\text{conn}}\Delta^l, \tag{5.11}$$

where $\sum''_P$ is over all partitions of $\{V_1,\cdots,V_n\}$ with $1\le k(P)<n$. For $\boldsymbol{\lambda}\in\Omega$, we let $r\to 0+$ in (5.11) and then apply $\mathcal{F}$ to both sides. Equation (5.6) and another use of property (6) show that $\mathcal{F}$ annihilates the second term on the right-hand side. But the first term on this side is just $\mathcal{T}_{\lambda,\epsilon}(V_1,\cdots,V_n;\mathfrak{L})$, so that (5.11) becomes

$$\mathcal{F}\mathcal{R}'_{\lambda,\epsilon}(V_1,\cdots,V_n;\mathfrak{L}) = \mathcal{F}\mathcal{T}_{\lambda,\epsilon}(V_1,\cdots,V_n;\mathfrak{L}).$$

Theorem 2 and property (2) of $\mathcal{F}$ show that

$$\mathcal{F}\mathcal{R}'_{\lambda,\epsilon}(V_1,\cdots,V_n;\mathfrak{L}) = \mathcal{R}'_{1,\cdots,1,\epsilon}(V_1,\cdots,V_n;\mathfrak{L});$$

this completes the proof of the theorem.

ACKNOWLEDGMENT

I would like to thank Professor Arthur S. Wightman for suggesting this problem and for encouraging and guiding its development.