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MATROIDS WITH WEIGHTED BASES

AND FEYNMAN INTEGRALS

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by

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Matroids with weighted bases and Feynman integrals

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We ask when it is possible to weigh the bases B of a matroid E by non-negative real numbers w_B, so that the sum of w_B over all bases is 1 and, for each element $e \in E$, the sum of w_B over all bases B containing e is bounded below and above by specified real numbers λ_e, μ_e. In other words, given a matroid E with the rank function r and the set of bases $\mathcal{B}$, and given real vectors $(\lambda_e)_{e\in E}$, $(\mu_e)_{e\in E}$, we wish to determine if the linear constraints

$$\sum_{B\in\mathcal{B}} w_B = 1,$$

$$\lambda_e \le \sum_{B\ni e} w_B \le \mu_e \qquad \text{(for each } e \in E\text{)}, \tag{1}$$

$$w_B \ge 0 \qquad \text{(for each } B \in \mathcal{B}\text{)},$$

are feasible. We shall find the following necessary and sufficient conditions for the feasibility of (1):

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$$\lambda_e \leq \mu_e \qquad \text{(for each } e \in E),$$

$$\sum_{e \in A} \lambda_e \leq r(A) \qquad \text{(for each } A \subseteq E), \tag{2}$$

$$\sum_{e \notin A} \mu_e \geq r(E) - r(A) \qquad \text{(for each } A \subseteq E).$$

We found it interesting to compare two different proofs of this fact: One uses only the duality theorem of linear programming and the greedy algorithm for finding an optimal base; the other depends directly on some deeper results of matroid theory. The question was motivated by a problem of uniform estimation of certain Feynman integrals, [11]; we shall describe briefly this application of our theorem. We shall also consider the special case $\lambda_e = \mu_e = \lambda$ (for each $e \in E$), especially in the context of graphic matroids, and make an observation relating edge-connectivity and edge-density.

<u>Theorem</u>. The constraints (1) are feasible if and only if (2) holds.

It is easy to see that the conditions (2) are necessary for the feasibility of (1). Indeed, if $(w_B)_{B \in \mathcal{B}}$ is a solution of (1), then clearly $\lambda_e \leq \mu_e$ for each $e \in E$, and for each $A \subseteq E$,

$$\sum_{e \in A} \lambda_e \leq \sum_{e \in A} \sum_{B \ni e} w_B = \sum_{B \in \mathcal{B}} |A \cap B| w_B \leq r(A) \sum_{B \in \mathcal{B}} w_B = r(A),$$

$$\sum_{e \notin A} \mu_e \geq \sum_{e \notin A} \sum_{B \ni e} w_B = \sum_{B \in \mathcal{B}} |B \setminus A| w_B = \sum_{B \in \mathcal{B}} (|B| - |A \cap B|) w_B \geq r(E) - r(A).$$

Lemma 1. The constraints (1) are feasible if and only if for every pair of vectors $(x_e)_{e\in E}$, $(y_e)_{e\in E}$ with non-negative coordinates,

$$\sum_{e\in E} (\lambda_e x_e - \mu_e y_e) \le \max_{B\in\mathcal{B}} \sum_{e\in B} (x_e - y_e). \tag{3}$$

Proof. This can be easily seen using the Farkas Lemma [3] or its equivalents [7], and amounts to an application of the duality theorem of linear programming. For example, according to [4, Theorem 2.8], the constraints

$$\sum_{B\in\mathcal{B}} w_B \le 1, \qquad \sum_{B\in\mathcal{B}} - w_B \le -1,$$

$$\sum_{B\ni e} w_B \le \mu_e, \qquad \sum_{B\ni e} - w_B \le -\lambda_e \qquad \text{(for each } e \in E\text{)},$$

$$w_B \ge 0 \qquad \text{(for each } B \in \mathcal{B}\text{)},$$

(equivalent to (1)) are feasible if and only if the constraints

$$\sum_{e\in B} (y_e - x_e) + v - t \ge 0 \qquad \text{(for each } B \in \mathcal{B}\text{)},$$

$$\sum_{e\in E} (\mu_e y_e - \lambda_e x_e) + v - t < 0,$$

$$v \ge 0, \quad t \ge 0,$$

$$x_e \ge 0, \quad y_e \ge 0 \qquad \text{(for each } e \in E\text{)},$$

are not. Setting $u = v - t$, we obtain the condition that for any vectors $(x_e)_{e\in E}$, $(y_e)_{e\in E}$ with non-negative coordinates,

$$\sum_{e\in B} (x_e - y_e) \le u \quad \text{for all } B \in \mathcal{B} \text{ implies } \sum_{e\in E} (\lambda_e x_e - \mu_e y_e) \le u,$$

which is equivalent to (3).

Given vectors $(x_e)_{e\in E}$, $(y_e)_{e\in E}$, we write $z_e = x_e - y_e$. The quantity $\max_{B\in\mathcal{B}} \Sigma_{e\in B} z_e$, which occurs in (3), can be found by applying the greedy algorithm, [8, p. 277], which for our purposes may be described as follows: Let $E = \{e_1, e_2, \dots, e_m\}$ be an enumeration of the elements of E such that $z_i = z_{e_i}$ satisfy $z_1 \ge z_2 \ge \dots \ge z_m$. Let $A_0 = \emptyset$, $A_k = \{e_1, e_2, \dots, e_k\}$ for $1 \le k \le m$, and $B^* = \{e_k : 1 \le k \le m, r(A_k) = r(A_{k-1}) + 1\}$. Then B^* is a base of E and $\Sigma_{e\in B^*} z_e = \max_{B\in\mathcal{B}} \Sigma_{e\in B} z_e$. (In fact, we really need only the fact that B^* is a base.)

To complete the first proof of the theorem, it remains to show that (2) implies that for any vectors $(x_e)_{e\in E}$, $(y_e)_{e\in E}$ of non-negative numbers,

$$\sum_{e\in E} (\lambda_e x_e - \mu_e y_e) \le \sum_{e\in B^*} z_e .$$

Assume that $z_1 \ge \dots \ge z_n \ge 0 \ge z_{n+1} \ge \dots \ge z_m$ and write $\lambda_i = \lambda_{e_i}$, $\mu_i = \mu_{e_i}$. Then

$$\sum_{k=1}^{n} [(z_k - z_{k+1}) \sum_{i=1}^{k} \lambda_i] \le \sum_{k=1}^{n} (z_k - z_{k+1}) r(A_k)$$

(where in this inequality only we set $z_{n+1} = 0$), whence

$$\sum_{k=1}^{n} \lambda_k z_k \le \sum_{k=1}^{n} z_k (r(A_k) - r(A_{k-1})) = \sum_{e\in B^*\cap A_n} z_e .$$

Similarly,

$$\sum_{k=n+1}^{m} [(z_{k-1} - z_k) \sum_{i=k}^{m} \mu_i] \geq \sum_{k=n+1}^{m} (z_{k-1} - z_k)(r(E) - r(A_{k-1}))$$

(where again, in this inequality only, we set $z_n = 0$), and so

$$\sum_{k=n+1}^{m} \mu_k z_k \leq \sum_{k=n+1}^{m} z_k(r(A_k) - r(A_{k-1})) = \sum_{e\in B^*\setminus A_n} z_e .$$

Thus

$$\sum_{e\in E} (\lambda_e x_e - \mu_e y_e) = \sum_{k=1}^{n} [\lambda_k z_k + (\lambda_k - \mu_k) y_k] + \sum_{k=n+1}^{m} [\mu_k z_k + (\lambda_k - \mu_k) x_k] \leq$$

$$\leq \sum_{k=1}^{n} \lambda_k z_k + \sum_{k=n+1}^{m} \mu_k z_k \leq \sum_{e\in B^*} z_e ,$$

as required.

For the second proof of the theorem, we observe that, without loss of generality, the vectors $(\lambda_e)_{e\in E}$ and $(\mu_e)_{e\in E}$ may be assumed to have rational coordinates. Indeed, if real vectors $(\lambda_e)_{e\in E}$, $(\mu_e)_{e\in E}$ satisfy (2), then so do any rational vectors $(\lambda_e^i)_{e\in E}$, $(\mu_e^i)_{e\in E}$, where $\lambda_e^i \leq \lambda_e$ and $\mu_e \leq \mu_e^i$ for each $e \in E$; hence for any rational sequences $\{\lambda_e^i\}_{i=1}^{\infty}$, $\{\mu_e^i\}_{i=1}^{\infty}$ (for each $e \in E$) with $\lambda_e^i \leq \lambda_e$, $\mu_e \leq \mu_e^i$ and $\lim_{i\to\infty} \lambda_e^i = \lambda_e$, $\lim_{i\to\infty} \mu_e^i = \mu_e$ (for each $e \in E$), the sets

$$S_i = \{(w_B)_{B\in\mathcal{B}} : \sum_{B\in\mathcal{B}} w_B = 1,\ \lambda_e^i \leq \sum_{B\ni e} w_B \leq \mu_e^i\ (e \in E),\ w_B \geq 0\ (B \in \mathcal{B})\}$$

are nonempty compact subsets of $\mathbb{R}^{\mathcal{B}}$. Therefore $\bigcap_{i=1}^{\infty} S_i \neq \emptyset$, and (1) is feasible.

Let k, ℓ_e $(e \in E)$ and m_e $(e \in E)$ be integers such that, for all $e \in E$, $\lambda_e = \ell_e/k$ and $\mu_e = m_e/k$. It remains to show that

$$\ell_e \le m_e \qquad \text{(for each } e \in E),$$

$$\sum_{e\in A} \ell_e \le k\,r(A) \qquad \text{(for each } A \subseteq E), \tag{4}$$

$$\sum_{e\notin A} m_e \ge k(r(E) - r(A)) \qquad \text{(for each } A \subseteq E),$$

imply that

$$\sum_{B\in\mathcal{B}} w_B = k,$$

$$\ell_e \le \sum_{B\ni e} w_B \le m_e \qquad \text{(for each } e \in E), \tag{5}$$

$$w_B \ge 0 \qquad \text{(for each } B \in \mathcal{B}),$$

is feasible.

Consider the sets $\tilde{E} = \{(e,i) : e \in E,\ 1 \le i \le m_e\}$ and $\tilde{\mathcal{B}}$ defined to consist of all subsets X of $\tilde{E}$ satisfying

(a) for each $e \in E$ there is at most one i such that $(e,i) \in X$

(b) $\{e : (e,i) \in X$ for some $i\}$ is a base of E.

It is easy to verify that $\tilde{\mathcal{B}}$ is the set of bases of a matroid on $\tilde{E}$; we shall refer to that matroid as $\tilde{E}$ and its rank function shall be denoted by $\tilde{r}$. Finally, let $T = \{(e,i) : e \in E,\ 1 \le i \le \ell_e\}$.

<u>Lemma 2</u>. (5) has an integer solution if and only if $\tilde{E}$ admits k disjoint bases whose union contains T.

<u>Proof</u>. Let $X_1, X_2, \ldots, X_k$ be disjoint bases of $\tilde{E}$ whose union contains T. Let $B_j = \{e : (e,i) \in X_j$ for some $i\}$, $1 \le j \le k$; by (b) each B_j is a base of $\tilde{E}$, but the B_j's are not necessarily disjoint,

or even distinct. Let, for $B \in \mathcal{B}$, w_B be the number of j's such that $B_j = B$. Then evidently $\Sigma_{B \in \mathcal{B}} w_B = k$; moreover, for any $e \in E$, $\Sigma_{B \ni e} w_B \leq m_e$ because the X_j's are disjoint; finally, for each $e \in E$, $\Sigma_{B \ni e} w_B \geq \ell_e$ because $\bigcup_{j=1}^{k} X_j$ contains T. Hence w_B, $B \in \mathcal{B}$, are integers satisfying (5). Conversely, given integers w_B, $B \in \mathcal{B}$, satisfying (5), it is easy to construct k disjoint bases of $\tilde{E}$ whose union contains T.

<u>Lemma 3</u>. There exist in $\tilde{E}$ k disjoint bases whose union contains T if and only if there exist in $\tilde{E}$ k disjoint bases and there exist in $\tilde{E}$ k bases whose union contains T.

<u>Proof</u>. Lemma 3 holds for an arbitrary matroid $\tilde{E}$ and any subset T. Accordingly, in this proof no use shall be made of the specific form of $\tilde{E}$ and T. We first construct a new matroid $\tilde{E}^{(k)}$ on $\tilde{E}$, whose independent sets are precisely the sets that can be written as unions of k sets, independent in $\tilde{E}$. (In other words, $X \subseteq \tilde{E}$ is independent in $\tilde{E}^{(k)}$ if and only if $X = I_1 \cup I_2 \cup \ldots \cup I_k$ with each I_j independent in $\tilde{E}$.) It follows from [9] that $\tilde{E}^{(k)}$ is indeed a matroid. Assume that $\tilde{E}$ has k disjoint bases $X_1, \ldots, X_k$. Then $\bigcup_{i=1}^{k} X_i$ is an independent set in $\tilde{E}^{(k)}$ and hence the rank of $\tilde{E}^{(k)}$ is $k\tilde{r}(\tilde{E})$. In other words, every base of $\tilde{E}^{(k)}$ is the union of k disjoint bases of $\tilde{E}$. If at the same time $\tilde{E}$ admits k bases $Y_1, Y_2, \ldots, Y_k$ such that $T \subseteq \bigcup_{i=1}^{k} Y_i$, then T is also an independent set in $\tilde{E}^{(k)}$. Therefore T can be extended to a base of $\tilde{E}^{(k)}$; i.e., some k disjoint bases of $\tilde{E}$ contain T in their union. The converse is obvious.

Our second proof is now completed by an application of the covering and packing theorems of Edmonds, [2]. They imply that $\tilde{E}$ has k disjoint bases if and only if $|\tilde{E}\setminus A| \geq k((\tilde{r}(\tilde{E}) - \tilde{r}(A))$ for all $A \subseteq \tilde{E}$; this is easily seen equivalent to $\Sigma_{e \notin A} m_e \geq k((r(E) - r(A))$ for all $A \subseteq E$. They also imply that T can be covered by k bases (or, equivalently, independent sets) of $\tilde{E}$ if and only if $|A| \leq k\,\tilde{r}(A)$ for all $A \subseteq \tilde{E}$; this is easily seen equivalent to $\Sigma_{e \in A} \ell_e \leq k\; r(A)$ for all $A \subseteq E$. Hence conditions (4) are, by Lemmas 2 and 3, sufficient to assure that (5) admits an integer solution.

Corollary 1. There is an integer solution to (5) if and only if (4) holds.

Proof. The necessity of (4) follows from the Theorem: Letting $\lambda_e = \ell_e/k$, $\mu_e = m_e/k$, feasibility of (5) implies feasibility of (1) and thus (2) holds, which is equivalent to (4).

There is an obvious interpretation of an integer solution to (5) as a set of k bases (with possible repetition) with the property that each $e \in E$ belongs to at least ℓ_e and at most m_e of these bases.

Corollary 2. (a) The constraints

$$\sum_{B \in \mathcal{B}} w_B = 1,$$

$$\sum_{B \ni e} w_B \geq \lambda_e \qquad \text{(for each } e \in E),$$

$$w_B \geq 0 \qquad \text{(for each } e \in E),$$

are feasible if and only if

$$\sum_{e\in A} \lambda_e \le r(A) \qquad \text{(for each } A \subseteq E).$$

(b) The constraints

$$\sum_{B\in \mathcal{B}} w_B = 1,$$

$$\sum_{B\ni e} w_B \le \mu_e \qquad \text{(for each } e \in E),$$

$$w_B \ge 0 \qquad \text{(for each } B \in \mathcal{B}),$$

are feasible if and only if

$$\sum_{e\notin A} \mu_e \ge r(E) - r(A) \qquad \text{(for each } A \subseteq E).$$

Proof. These results follow from our theorem by choosing, in (a), $\mu_e = 1$ for all $e \in E$, and in (b), $\lambda_e = 0$ for all $e \in E$. We remark that (a) and (b) are related by duality, that is, (a) for E is equivalent to (b) for the matroid dual to E and vice versa. We also remark that similar one-sided integral versions of (5) may be deduced from Corollary 1 by taking $m_e = k$, respectively $\ell_e = 0$, for all $e \in E$.

Corollary 2(b) for graphic matroids has been applied in [10] to estimate the magnitude of the Feynman amplitudes which are associated with (Feynman) graphs in perturbative quantum field theory. The essential idea may be described in the context of an arbitrary matroid as follows. To the matroid E and positive real numbers a and $(b_e)_{e\in E}$ we associate the amplitude

$$I_E(a,b) = \int_0^\infty \cdots \int_0^\infty U(\alpha)^{-a} \prod_{e\in E} \{\Gamma(b_e)^{-1} \alpha_e^{b_e - 1} \exp(-\alpha_e)\, d\alpha_e\}, \tag{6}$$

where

$$U(\alpha) = \sum_{B\in\mathcal{B}} \prod_{e\in B} \alpha_e .$$

Note that $I_E(0,b) \equiv 1$; on the other hand, if (b_e) is fixed and a is sufficiently large, the integral diverges. In fact, the condition for convergence is known to be

$$\sum_{e\in A} b_e - a[r(E) - r(E\setminus A)] > 0 \tag{7}$$

for each $A \subseteq E$.

The sufficiency of (7) for convergence follows from the argument below. We briefly sketch a standard argument (see, e.g., [11]) showing necessity also. Consider the integral (6) taken over a sector $\alpha_{e_1} \le \dots \le \alpha_{e_m}$ and make the variable change $\alpha_{e_k} = \Pi_{j=k}^{m} t_j$, $k = 1, \dots, m$, with the Jacobian $\Pi_{j=1}^{m} t_j^{j-1}$. Let $A_j = \{e_1, \dots, e_j\}$ and

$$B^* = \{e_k : 1 \le k \le m,\ r(A_k) = r(A_{k-1}) + 1\},$$

as in the first proof of the theorem. Now

$$U(\alpha(t)) = \sum_{B\in\mathcal{B}} \prod_{j=1}^{m} t_j^{|A_j\cap B|} = \prod_{j=1}^{m} t_j^{[r(E)-r(E\setminus A_j)]}\, V(t),$$

where, since $|A_j \cap B| \ge r(E) - r(E\setminus A_j)$, $V(t)$ is a polynomial. Moreover,

$$U(\alpha(t)) \ge \prod_{e\in B^*} \alpha_e(t) = \prod_{j=1}^{m} t_j^{[r(E)-r(E\setminus A_j)]},$$

so $V(t) \ge 1$, and, since $V(t)$ is independent of t_m, it is uniformly bounded above on the compact set $\{0 \le t_j \le 1 : j = 1, \dots, m-1\}$. Thus the integral over this sector becomes

$$\prod_{e\in E} \Gamma(b_e)^{-1}\int_0^\infty dt_m \int_0^1 dt_{m-1} \cdots \int_0^1 dt_1 \prod_{j=1}^m t_j^{c(A_j)-1} V(t)^{-a} \exp[-\sum_{e\in E} \alpha_e(t)], \quad (8)$$

and the upper and lower bounds on V imply that (8) converges if and only if $c(A_j) > 0$, $j = 1, \ldots, m$.

The problem in [10] is to give an estimate for I_E of the special form

$$I_E(a,b) \leq K^{|E|} \quad (9)$$

where the constant K does not depend on E. Such a uniform estimate will not of course hold in the entire convergence region specified by (7), but only on compact subregions; here we specify such a subregion by a parameter $\varepsilon > 0$ and by the condition

$$\sum_{e\in A} b_e - a[r(E) - r(E\setminus A)] \geq \varepsilon|A| \quad (10)$$

(the specification in [10] is slightly different). Then we have

<u>Corollary 3</u>: For all matroids E and non-negative parameters a, $(b_e)_{e\in E}$ satisfying (10), there is a constant K depending only on ε such that (9) holds.

<u>Proof</u>: By Corollary 2(b) there are non-negative weights $(w_B)_{B\in\mathcal{B}}$ with $\Sigma_{\mathcal{B}} w_B = 1$ and

$$\sum_{B\ni e} w_B \leq a^{-1}(b_e - \varepsilon).$$

From the standard inequality between arithmetic and geometric means,

$$U(\alpha) \geq \sum_{B\in\mathcal{B}} w_B \prod_{e\in B} \alpha_e$$

$$\geq \prod_{B\in\mathcal{B}} \left(\prod_{e\in B} \alpha_e\right)^{w_B} = \prod_{e\in E} \alpha_e^{(\Sigma_{B\ni e} w_B)} .$$

Thus

$$I_\varepsilon(a,b) \leq \prod_{e\in E} \Gamma(b_e)^{-1} \int_0^\infty \alpha_e^{d_e - 1} e^{-\alpha_e} d\alpha_e = \prod_{e\in E} \Gamma(d_e)/\Gamma(b_e)$$

where $d_e = b_e - a\,\Sigma_{B\ni e}\, w_B$ satisfies $b_e \geq d_e \geq \varepsilon$. Since for $t > 0$, $\Gamma(t)$ is convex with a minimum at $t = 1$, we have

$$\Gamma(d_e)/\Gamma(b_e) \leq \begin{cases} \Gamma(\varepsilon)/\Gamma(1) = \Gamma(\varepsilon), & \text{if } d_e \leq 1, \\ 1 \leq \Gamma(\varepsilon), & \text{if } d_e \geq 1. \end{cases}$$

Thus (9) holds with $K = \Gamma(\varepsilon)$.

Next we explore the theorem in the special case $\lambda_e = \mu_e$, $e \in E$. It implies that the system of linear equations

$$\sum_{B\in\mathcal{B}} w_B = 1,$$

$$\sum_{B\ni e} w_B = \lambda_e \qquad \text{(for each } e \in E\text{)},$$

admits a non-negative solution w_B, $B \in \mathcal{B}$, if and only if

$$r(E) - r(E\setminus A) \leq \sum_{e\in A} \lambda_e \leq r(A) \qquad \text{(for each } A \subseteq E\text{)}. \tag{11}$$

Note that for $A = E$, (11) implies that $\Sigma_{e\in E}\, \lambda_e = r(E)$. In particular, where each $\lambda_e = \mu_e = \lambda$ $(e \in E)$, (11) implies that $\lambda = r(E)/|E|$. In fact, if $\Sigma_{B \in \mathcal{B}} w_B = x$ and $\Sigma_{B \ni e} w_B = y$ (for each $e \in E$), then either one of x, y determines the other, since

$$y|E| = \sum_{e \in E} \sum_{B \ni e} w_B = \sum_{B \in \mathcal{B}} \sum_{e \in B} w_B = r(E) \sum_{B \in \mathcal{B}} w_B = x\,r(E).$$

Hence if $y = r(E)/|E|$ then $x = 1$, and the feasibility of (1) with all $\lambda_e = \mu_e = r(E)/|E|$ $(e \in E)$, is equivalent to the feasibility of

$$\sum_{B \ni e} w_B = \frac{r(E)}{|E|} \qquad \text{(for each } e \in E\text{)},$$

$$w_B \geq 0 \qquad \text{(for each } B \in \mathcal{B}\text{)},$$

or of

$$\sum_{B \ni e} w_B = 1 \qquad \text{(for each } e \in E\text{)} \tag{12}$$

$$w_B \geq 0 \qquad \text{(for each } B \in \mathcal{B}\text{)}.$$

The system (12) is particularly interesting. We are trying to weigh the bases of E by non-negative reals so that the sum of weights of all bases containing each fixed element e is 1. Matroids in which this is possible, i.e., in which (12) is feasible, shall be called <u>baseable</u>.

<u>Corollary 4</u>. A matroid E is baseable if and only if any of the following conditions holds:

(a) $\frac{|A|}{r(A)} \leq \frac{|E|}{r(E)}$ (for each $A \subseteq E$),

(b) $\frac{|K|}{r(K)} \geq \frac{|E|}{r(E)}$ (for each contraction K of E),

(c) There exist bases $B_1, B_2, \ldots, B_{|E|}$ of E (not necessarily distinct), such that each $e \in E$ belongs to precisely $r(E)$ bases among $B_1, B_2, \ldots, B_{|E|}$.

(d) (12) is solvable by integer multiples of $1/r(E)$.

Proof. The feasibility of (12) is equivalent to (11) with each $\lambda_e = r(E)/|E|$, i.e., to

$$r(E) - r(E \setminus A) \leq \frac{r(E)}{|E|} |A| \leq r(A) \qquad \text{(for each } A \subseteq E).$$

Since a contraction K of E has $r(K) = r(E) - r(E \setminus K)$, [13, p. 63], E is baseable if and only if (a) and (b) hold. On the other hand, (a) and (b) are algebraically equivalent, so either is equivalent to the baseability of E. Condition (c) is equivalent to the integer feasibility of

$$\sum_{B \ni \mathcal{B}} w_B = |E|,$$

$$\sum_{B \ni e} w_B = r(E) \qquad \text{(for each } e \in E), \tag{13}$$

$$w_B \geq 0 \qquad \text{(for each } B \in \mathcal{B}).$$

(Given $B_1, B_2, \ldots, B_{|E|}$, let w_B equal the number of times B occurs among $B_1, B_2, \ldots, B_{|E|}$; given w_B, take each B w_B times to form $B_1, B_2, \ldots, B_{|E|}$.) Since we have already observed that the first equation in (13) is redundant, (c) and (d) are equivalent. Moreover, according to Corollary 1, (13) has an integer solution if and only if (a) and (b) hold.

We define a graph to be arboreal if its cycle matroid is baseable. The density of a connected graph $G = (V,E)$ is the ratio $|E|/(|V|-1)$. Using condition (a) of Corollary 4 and the fact that $(|E_1| + |E_2|)/((|V_1|-c_1) + (|V_2|-c_2))$ lies between $|E_1|/(|V_1|-c_1)$ and

$|E_2|/(|V_2|-c_2)$, we can see that G is arboreal if and only if all components of G are arboreal and have the same density. Therefore we shall confine our remarks to connected arboreal graphs. There is the obvious interpretation of Corollary 4, where "base" is replaced by "spanning tree," "matroid E" by "connected graph $G = (V,E)$," and "baseable" by "arboreal." It is easy to see that (a) and (b) can be weakened to

(a′) No connected subgraph of G has greater density than G,

(b′) No contraction of G has smaller density than G.

Corollary 4(a) has led us to wonder when can (12) be solved by a constant vector $(w_B)_{B\in\mathcal{B}}$ (i.e., when can all w_B be chosen the same). In [12] we asked for connected graphs having the property that each edge lies in the same number of spanning trees. E. Mendelsohn suggested that such graphs be called <u>equiarboreal</u>. Equiarboreal graphs are necessarily arboreal. Obviously trees and edge-transitive graphs are equiarboreal. Also, equiarboreal graphs with the same density can be attached together at a vertex to form another equiarboreal graph. C. Godsil, [5], has shown that any colour class in an association scheme is equiarboreal. This means that any distance regular graph, [1], and in particular any strongly regular graph is equiarboreal. Even more generally, he defines a graph to be <u>1-homogeneous</u> if, for every ℓ, the number of closed walks of length ℓ through a vertex v is the same for every v, and the number of closed walks of length ℓ through an edge vw is the same for every vw. Godsil gave an elegant proof showing that 1-homogeneous graphs are equiarboreal, and observed that colour classes of association schemes are 1-homogeneous, and that 1-homogeneous graphs are closed under conjunction. Thus there is a wide supply of equiarboreal graphs.

Godsil also noticed that the edge-connectivity of a connected equiarboreal graph is at least as large as its density, and was led to a number of interesting results and conjectures about the edge-connectivity of highly regular graphs, [5,6]. He later extended his observation on edge-connectivity to all regular arboreal graphs (private communication). In our last corollary, we show how Corollary 4(c) can be used to extend Godsil's proof to all arboreal graphs:

Corollary 5. The edge-connectivity of a connected arboreal graph is at least as large as its density.

Proof. By Corollary 4(c), a connected arboreal graph $G = (V,E)$ admits spanning trees $T_1, T_2, \ldots, T_{|E|}$ with the property that each edge of G belongs to precisely $|V| - 1$ of them. Consider a set of c edges whose removal disconnects G. Each spanning tree of G must contain one of the c edges, and hence

$$|E| \leq c\,(|V| - 1).$$

Therefore the edge-connectivity of G is at least $|E|/(|V|-1)$.

It may also be fruitful to study "equibaseable" matroids, i.e., matroids in which each element belongs to the same number of bases.

References

[1] N. Biggs, Algebraic Graph Theory, Cambridge University Press, 1974.

[2] J. Edmonds, "Lehman's switching game and a theorem of Tutte and Nash-Williams," J. Res. Nat. Bur. Stand. 69B (1965), 73-77.

[3] J. Farkas, "Über die Theorie der einfachen Ungleichungen," J. Reine Angew. Math. 124 (1902), 1-24.

[4] D. Gale, The Theory of Linear Economic Models, McGraw-Hill, N.Y., 1958.

[5] C. D. Godsil, "Equiarboreal graphs," Combinatorica, to appear.

[6] C. D. Godsil and B. D. McKay, "Edge-connectivity of regular graphs with diameter 2," to appear.

[7] H. W. Kuhn, "Solvability and consistency for linear equations and inequalities," Amer. Math. Monthly 63 (1956), 217-232.

[8] E. L. Lawler, Combinatorial Optimization, Holt, Rinehart and Winston 1976.

[9] C. St. J. A. Nash-Williams, "An application of matroids to graph theory," in Theory of Graphs, International Symposium Rome, Dunod (Paris) 1966, pp. 263-265.

[10] V. Rivasseau and E. Speer, "The Borel transform in Euclidean φ_{ν}^{4}. Local existence for Re $\nu < 4$," Commun. Math. Phys. 72 (1980), 293-302.

[11] E. R. Speer, Generalized Feynman Amplitudes, Princeton University Press, 1969.

[12] Vancouver Problem Book, N.S.E.R.C. Summer Research Workshop in Combinatorics, Simon Fraser University, 1979.

[13] D. J. A. Welsh, Matroid Theory, Academic Press, 1976.