# CHARACTERS OF INDUCTIVE LIMITS OF FINITE ALTERNATING GROUPS

SIMON THOMAS

ABSTRACT. If $G \not\cong \mathrm{Alt}(\mathbb{N})$ is an inductive limit of finite alternating groups, then the indecomposable characters of $G$ are precisely the associated characters of the ergodic invariant random subgroups of $G$.

## 1. INTRODUCTION

In [17], Vershik pointed out that the indecomposable characters of the group $\mathrm{Fin}(\mathbb{N})$ of finitary permutations of the natural numbers are closely connected with its ergodic invariant random subgroups; and in [16], he suggested that this should also be true of various other locally finite groups. In this paper, we will prove that if $G \not\cong \mathrm{Alt}(\mathbb{N})$ is an inductive limit of finite alternating groups, then the indecomposable characters of $G$ are precisely the associated characters of the ergodic invariant random subgroups of $G$.

Let $G$ be a countably infinite group and let $\mathrm{Sub}_G$ be the compact space of subgroups $H \leqslant G$. Then a Borel probability measure $\nu$ on $\mathrm{Sub}_G$ which is invariant under the conjugation action of $G$ on $\mathrm{Sub}_G$ is called an *invariant random subgroup* or *IRS*. For example, suppose that $G$ acts via measure-preserving maps on the Borel probability space $(Z, \mu)$ and let $f : Z \to \mathrm{Sub}_G$ be the $G$-equivariant map defined by

$$ z \mapsto G_z = \{\, g \in G \mid g \cdot z = z \,\}. $$

Then the corresponding *stabilizer distribution* $\nu = f_* \mu$ is an IRS of $G$. In fact, by a result of Abért-Glasner-Virag [1], every IRS of $G$ can be realized as the stabilizer distribution of a suitably chosen measure-preserving action. Moreover, by Creutz-Peterson [4], if $\nu$ is an ergodic IRS of $G$, then $\nu$ is the stabilizer distribution of an ergodic action $G \curvearrowright (Z, \mu)$.

If $G$ is a countable group, then a function $\chi : G \to \mathbb{C}$ is said to be a *character* if the following conditions are satisfied:

(i) $\chi(h\,g\,h^{-1}) = \chi(g)$ for all $g, \in G$.
(ii) $\sum_{i,j=1}^{n} \lambda_i \bar{\lambda}_j \chi(g_j^{-1} g_i) \geq 0$ for all $\lambda_1, \cdots, \lambda_n \in \mathbb{C}$ and $g_1, \cdots, g_n \in G$.
(iii) $\chi(1_G) = 1$.

For example, if $G \curvearrowright (Z, \mu)$ is any measure-preserving action on a Borel probability space, then we can define a character $\chi$ of $G$ by $\chi(g) = \mu(\mathrm{Fix}_Z(g))$. In particular, if $\nu$ is an IRS of $G$, then we can define a corresponding character $\chi$ by

$$ \chi(g) = \nu(\{\, H \in \mathrm{Sub}_G \mid gHg^{-1} = H \,\}) $$
$$ = \nu(\{\, H \in \mathrm{Sub}_G \mid g \in N_G(H) \,\}). $$

On the other hand, we can also define a second character $\chi'$ by

$$\chi'(g) = \nu(\{\, H \in \mathrm{Sub}_G \mid g \in H \,\}).$$

It is easily seen that $\chi' = \chi$ if and only if $N_G(H) = H$ for $\nu$-a.e. $H \in \mathrm{Sub}_G$. Fortunately, if $G \not\cong \mathrm{Alt}(\mathbb{N})$ is an inductive limit of finite alternating groups, then this is true of every ergodic IRS $\nu$ of $G$, except for the Dirac measure $\delta_1$ which concentrates on the identity subgroup $1$. (This result is proved during the proof of Thomas-Tucker-Drob [15, Theorem 3.21].) Since it turns out to be slightly more convenient to work with the character $\chi'$, we choose the following definition.

**Definition 1.1.** If $\nu$ is an IRS of the countable group $G$, then the *associated character* $\chi_\nu$ is defined to be $\chi_\nu(g) = \nu(\{\, H \in \mathrm{Sub}_G \mid g \in H \,\})$.

A character $\chi$ is said to be *indecomposable* or *extremal* if it is impossible to express $\chi = r\chi_1 + (1-r)\chi_2$, where $0 < r < 1$ and $\chi_1 \neq \chi_2$ are distinct characters. The set of characters of $G$ will be denoted by $\mathcal{F}(G)$ and the set of indecomposable characters will be denoted by $\mathcal{E}(G)$. The set $\mathcal{F}(G)$ always contains the two *trivial* characters $\chi_{\mathrm{con}}$ and $\chi_{\mathrm{reg}}$, where $\chi_{\mathrm{con}}(g) = 1$ for all $g \in G$ and $\chi_{\mathrm{reg}}(g) = 0$ for all $1 \neq g \in G$. It is well-known that $\chi_{\mathrm{con}}$ is indecomposable, and that $\chi_{\mathrm{reg}}$ is indecomposable if and only if $G$ is an i.c.c. group, i.e. the conjugacy class $g^G$ of every nonidentity element $g \in G$ is infinite. (For example, see Peterson-Thom [10].) Let $\delta_G$ and $\delta_1$ be the Dirac measures which concentrate on the normal subgroups $G$, $1$ respectively. Then $\delta_G$, $\delta_1$ are ergodic IRSs of $G$ and clearly $\chi_{\mathrm{con}} = \chi_{\delta_G}$ and $\chi_{\mathrm{reg}} = \chi_{\delta_1}$. Throughout this paper, we will refer to $\delta_G$, $\delta_1$ as the *trivial* ergodic IRSs of $G$.

**Definition 1.2.** A simple locally finite group $G$ is said to be an $L(\mathrm{Alt})$-group if we can express $G = \bigcup_{i \in \mathbb{N}} G_i$ as the union of a strictly increasing chain of finite alternating groups $G_i$. (Here we allow arbitrary embeddings $G_i \hookrightarrow G_{i+1}$.)

We are now in a position to state the main result of this paper.

**Theorem 1.3.** *If $G$ is an $L(\mathrm{Alt})$-group and $G \not\cong \mathrm{Alt}(\mathbb{N})$, then the indecomposable characters of $G$ are* precisely *the associated characters $\chi_\nu$ of the ergodic invariant random subgroups $\nu$ of $G$.*

Note that the statement of Theorem 1.3 makes two distinct assertions about the characters of the $L(\mathrm{Alt})$-group $G \not\cong \mathrm{Alt}(\mathbb{N})$. Firstly, if $\nu$ is any ergodic IRS of $G$, then the associated character $\chi_\nu$ is indecomposable; and, secondly, that every indecomposable character of $G$ is the associated character $\chi_\nu$ of some ergodic IRS $\nu$ of $G$. The former statement was proved in Thomas-Tucker-Drob [15], and so it will be enough for us to prove the latter statement in this paper. Also note that [15] contains a classification of the ergodic IRSs of the $L(\mathrm{Alt})$-group $G \not\cong \mathrm{Alt}(\mathbb{N})$. Thus, combining the results of this paper and [15], we obtain a classification of the indecomposable characters of the $L(\mathrm{Alt})$-group $G \not\cong \mathrm{Alt}(\mathbb{N})$. Of course, the indecomposable characters of $\mathrm{Alt}(\mathbb{N})$ have already been classified by Thoma [14]. (It is perhaps interesting to note that both of the assertions in Theorem 1.3 fail when $G = \mathrm{Alt}(\mathbb{N})$.)

The indecomposable characters of the diagonal limits $G = \bigcup_{i \in \mathbb{N}} G_i$ of finite alternating groups $G_i = \mathrm{Alt}(\Delta_i)$ such that $G \not\cong \mathrm{Alt}(\mathbb{N})$ were earlier classified by Leinen-Puglisi [7]. (Recall that $G = \bigcup_{i \in \mathbb{N}} G_i$ is a *diagonal limit* if for each $i \in \mathbb{N}$, every orbit of $G_i$ on $\Delta_{i+1}$ is either natural or trivial.) It should be stressed that

the proof of Theorem 1.3 makes essential use of the ideas and techniques of Leinen-Puglisi [7].

This paper is organized as follows. In Section 2, we will briefly discuss the ergodic IRSs of the $L(\mathrm{Alt})$-groups; and in Section 3, we will briefly discuss the irreducible characters of the finite alternating groups. In Sections 4 and 5, we will present the proof of Theorem 1.3. In Section 6, we will point out how both of the assertions in Theorem 1.3 fail when $G = \mathrm{Alt}(\mathbb{N})$.

Finally, we will explain our notation for the various kinds of limits that arise in this paper. Suppose that $(\, r_i \mid i \in \mathbb{N} \,)$ is a bounded sequence of real numbers. If $I \subseteq \mathbb{N}$ is an infinite subset which is enumerated in increasing order by the sequence $(\, i_k \mid k \in \mathbb{N} \,)$, then we will write $\lim_{i \in I} r_i$ instead of $\lim_{k \to \infty} r_{i_k}$. Also if $\mathcal{U}$ is a non-principal ultrafilter on $\mathbb{N}$, then $\lim_{\mathcal{U}} r_i$ will denote the unique real number $r$ such that $\{\, i \in \mathbb{N} : |r_i - r| < \varepsilon \,\} \in \mathcal{U}$ for all $\varepsilon > 0$.

## 2. The ergodic IRSs of the $L(\mathrm{Alt})$-groups

In this section, we will present a brief discussion of the ergodic IRSs of the $L(\mathrm{Alt})$-groups. First we need to introduce some notation. Suppose that $G = \bigcup_{i \in \mathbb{N}} G_i$ is the union of the strictly increasing chain of finite alternating groups $G_i = \mathrm{Alt}(\Delta_i)$. For each $i \in \mathbb{N}$, let

- $n_i = |\Delta_i|$;
- $s_{i+1}$ be the number of natural $G_i$-orbits on $\Delta_{i+1}$;
- $f_{i+1}$ be the number of trivial $G_i$-orbits on $\Delta_{i+1}$;
- $e_{i+1} = n_{i+1} - (s_{i+1}n_i + f_{i+1})$ is the number of points $x \in \Delta_{i+1}$ which lie in a nontrivial non-natural $G_i$-orbit.

Here an orbit $\Omega$ of $G_i = \mathrm{Alt}(\Delta_i)$ on $\Delta_{i+1}$ is said to be *natural* if $|\Omega| = |\Delta_i|$ and the action $G_i \curvearrowright \Omega$ is isomorphic to the natural action $G_i \curvearrowright \Delta_i$. Also for each $i < j$, let $s_{ij} = s_{i+1}s_{i+2}\cdots s_j$. Thus $s_{ij}$ is the number of "obvious" natural orbits of $G_i$ on $\Delta_j$.

The classification of the ergodic IRSs of the $L(\mathrm{Alt})$-groups involves a fundamental dichotomy which was introduced by Leinen-Puglisi [6, 7] in the more restrictive setting of diagonal limits of finite alternating groups, i.e. the linear vs sublinear natural orbit growth condition.

**Lemma 2.1** (Leinen-Puglisi [7]). *For each $i \in \mathbb{N}$, the limit $a_i = \lim_{j \to \infty} s_{ij}/n_j$ exists.*

**Definition 2.2.** An $L(\mathrm{Alt})$-groups $G = \bigcup_{i \in \mathbb{N}} G_i$ has *linear natural orbit growth* if $a_i > 0$ for some $i \in \mathbb{N}$. Otherwise, $G = \bigcup_{i \in \mathbb{N}} G_i$ has *sublinear natural orbit growth*.

**Remark 2.3.** Clearly if $G = \bigcup_{i \in \mathbb{N}} G_i$ has linear natural orbit growth, then there exists $i_0 \in \mathbb{N}$ such that $s_{i+1} > 0$ for all $i \geq i_0$. Also since $a_i = s_{i+1}a_{i+1}$, it follows that $a_i > 0$ for every $i \geq i_0$.

Since the proof of Theorem 1.3 makes use of the classification of the ergodic IRSs of the $L(\mathrm{Alt})$-groups of linear natural orbit growth, we will briefly describe this classification. So suppose that $G = \bigcup_{i \in \mathbb{N}} G_i$ has linear natural orbit growth. Then, after replacing the increasing union $G = \bigcup_{i \in \mathbb{N}} G_i$ by $G = \bigcup_{i_0 \leq i \in \mathbb{N}} G_i$ for some suitably chosen $i_0 \in \mathbb{N}$, we can suppose that $s_{i+1} > 0$ for all $i \in \mathbb{N}$. Let $t_0 = n_0$ and let $t_{i+1} = n_{i+1} - s_{i+1}n_i$. Then we can suppose that:

- $\Delta_0 = \{\, \alpha_\ell^0 \mid \ell < t_0 \,\}$; and

- $\Delta_{i+1} = \{\, \sigma\,\widehat{\phantom{x}}\,k \mid \sigma \in \Delta_i, 0 \le k < s_{i+1} \,\} \cup \{\, \alpha_\ell^{i+1} \mid 0 \le \ell < t_{i+1} \,\};$

and that the embedding $\varphi_i : \mathrm{Alt}(\Delta_i) \to \mathrm{Alt}(\Delta_{i+1})$ satisfies

$$\varphi_i(g)(\sigma\,\widehat{\phantom{x}}\,k) = g(\sigma)\,\widehat{\phantom{x}}\,k$$

for each $\sigma \in \Delta_i$ and $0 \le k < s_{i+1}$. Let $\Delta$ consist of all sequences of the form $(\,\alpha_\ell^i, k_{i+1}, k_{i+2}, k_{i+3}, \cdots)$, where $i \in \mathbb{N}$ and $k_j$ is an integer such that $0 \le k_j < s_j$. For each $i \in \mathbb{N}$ and $\sigma \in \Delta_i$, let $\Delta(\sigma) \subseteq \Delta$ be the subset of sequences of the form $\sigma\,\widehat{\phantom{x}}\,(\,k_{i+1}, k_{i+2}, k_{i+3}, \cdots)$. Then the sets $\Delta(\sigma)$ form a clopen basis for a locally compact topology on $\Delta$; and by Thomas-Tucker-Drob [15, Proposition 3.18], there exists a unique $G$-invariant ergodic probability measure $m$ on $\Delta$. By Thomas-Tucker-Drob [15, Corollary 2.5], since $G$ is a simple locally finite group, it follows that the product action $G \curvearrowright (\,\Delta^r, m^{\otimes r}\,)$ is also ergodic for all $r \ge 2$, and hence the corresponding stabilizer distribution $\nu_r$ is an ergodic IRS of $G$.

**Theorem 2.4** (Thomas-Tucker-Drob [15])**.** *If $G = \bigcup_{i \in \mathbb{N}} G_i$ has linear natural orbit growth, then the ergodic IRSs of $G$ are $\{\, \delta_1, \delta_G \,\} \cup \{\, \nu_r \mid r \in \mathbb{N}^+ \,\}$.*

From now on, whenever $G = \bigcup_{i \in \mathbb{N}} G_i$ has linear natural orbit growth, then we will refer to $G \curvearrowright (\,\Delta, m\,)$ as the *canonical ergodic action*. Since the proof of Theorem 1.3 does not require any knowledge of the ergodic IRSs of $L(\mathrm{Alt})$-group of sublinear natural orbit growth, we refer the interested reader to Thomas-Tucker-Drob [15] for the statements of the classification theorems. (The cases when $G \not\cong \mathrm{Alt}(\mathbb{N})$ and $G \cong \mathrm{Alt}(\mathbb{N})$ need to be handled separately.)

## 3. Irreducible characters of finite alternating groups

In this section, we will discuss some results of Leinen-Puglisi [7] concerning the asymptotic properties of the irreducible characters of $\mathrm{Alt}(n)$. But first, following Zalesskii [19], we will discuss the relationship between the irreducible characters of $\mathrm{Alt}(n)$ and $\mathrm{Sym}(n)$. It is well-known that the irreducible representations of the symmetric group $\mathrm{Sym}(n)$ are parametrized by the partitions $\lambda = (\,\ell_1, \ell_2, \cdots \ell_r\,)$ of $n$; i.e. sequences of integers such that $\ell_1 \ge \ell_2 \ge \cdots \ge \ell_r > 0$ and $\ell_1 + \ell_2 + \cdots + \ell_r = n$. For each such partition $\lambda$, let $\varphi_\lambda$ be the corresponding irreducible character of $\mathrm{Sym}(n)$ and let $D_\lambda$ be the corresponding Young diagram. Thus $D_\lambda$ is an array of cells with $\ell_k$ cells in the $k$th row for each $1 \le k \le r$. Also let $\lambda^*$ be the partition corresponding to the Young diagram obtained from $D_\lambda$ by reflection in the diagonal that runs rightwards and downwards from the upper left-hand corner of $D_\lambda$. For example, $(\,5, 2, 1\,)^* = (\,3, 2, 1, 1, 1\,)$. Finally, let $\trianglelefteq$ and $\le$ be the dominance and lexicographic orders on the set of partitions of $n$. (For example, see Sagan [12].)

If $\lambda$ is a partition of $n$ such that $\lambda \ne \lambda^*$, then $\varphi_\lambda \upharpoonright \mathrm{Alt}(n)$ is an irreducible character of $\mathrm{Alt}(n)$, which is equal to $\varphi_{\lambda^*} \upharpoonright \mathrm{Alt}(n)$. On the other hand, if $\lambda = \lambda^*$, then $\varphi_\lambda \upharpoonright \mathrm{Alt}(n)$ is the sum of two distinct irreducible representations of $\mathrm{Alt}(n)$. Furthermore, for every irreducible character $\theta$ of $\mathrm{Alt}(n)$, there exists a unique $\lambda$ such that $\lambda \ge \lambda^*$ and $\theta$ is an irreducible component of $\varphi_\lambda \upharpoonright \mathrm{Alt}(n)$. This allows us to associate a partition $\lambda$ such that $\lambda \ge \lambda^*$ with each irreducible character $\theta$ of $\mathrm{Alt}(n)$. If $\lambda > \lambda^*$, then $\lambda$ is associated with a unique irreducible character of $\mathrm{Alt}(n)$; while if $\lambda = \lambda^*$, then $\lambda$ is associated with a pair of irreducible characters of $\mathrm{Alt}(n)$. If $\lambda$ is associated with the irreducible character $\theta$ of $\mathrm{Alt}(n)$, then we write $D(\theta) = D_\lambda$ for the corresponding Young diagram. For later use, note that since $\lambda \ge \lambda^*$, it follows that the length of the first row of each Young diagram $D(\theta)$ is greater or equal to the length of the first column.

For each partition $\lambda = (\ell_1, \ell_2, \cdots \ell_r)$ of $n$ such that $\lambda \geq \lambda^*$, we define its *type* to be $\alpha_\lambda = (\ell_2, \cdots \ell_r)$ and its *depth* to be $d(\lambda) = \ell_2 + \cdots + \ell_r$. Similarly, we will refer to the types and depths of the corresponding Young diagrams and the corresponding irreducible characters of $\mathrm{Alt}(n)$; and if $\alpha = (\ell_2, \cdots \ell_r)$ is a type, then we will refer to $d(\alpha) = \ell_2 + \cdots + \ell_r$ as its depth. Of course, since $\ell_1 = n - d(\alpha)$, the corresponding partition $\lambda_\alpha$ of $n$ is uniquely determined by $\alpha$; and if $n \geq 2d(\alpha) + 1$, then $\lambda_\alpha > \lambda_\alpha^*$ and so there exists a unique irreducible character of $\mathrm{Alt}(n)$ of type $\alpha$, which we will denote by $\theta_\alpha$. Finally, for each integer $n \geq 2d(\alpha) + 1$, let $\Phi_\alpha$ be the set of partitions $(P_1, P_2, \cdots, P_r)$ of $n$ such that $|P_1| = n - d(\alpha)$ and $|P_k| = \ell_k$ for each $2 \leq k \leq r$, and let $\pi_\alpha$ be the permutation character of the action $\mathrm{Alt}(n) \curvearrowright \Phi_\alpha$. In the remainder of this section, we will present some results of of Leinen-Puglisi [7] concerning the asymptotic properties of $\theta_\alpha$ and $\pi_\alpha$ for some fixed type $\alpha$ as $n \to \infty$. We will be begin by stating two results concerning the growth rates of the degrees $\pi_\alpha(1), \theta_\alpha(1)$ of the representations. The first result is an easy exercise. For a proof of the second result, see Leinen-Puglisi [7, Lemma 3.1].

**Lemma 3.1.** *For each type $\alpha$, there exists a polynomial $p_\alpha \in \mathbb{Q}[x]$ of degree $d(\alpha)$ such that if $n \geq 2d(\alpha) + 1$, then $p_\alpha(n) = \pi_\alpha(1) = |\Phi_\alpha|$ is the degree of the permutation character $\pi_\alpha$ of the action $\mathrm{Alt}(n) \curvearrowright \Phi_\alpha$.*

**Lemma 3.2.** *For each type $\alpha$, there exists a polynomial $q_\alpha \in \mathbb{Q}[x]$ of degree $d(\alpha)$ such that if $n \geq 2d(\alpha) + 1$, then $q_\alpha(n) = \theta_\alpha(1)$ is the degree of the unique irreducible character $\theta_\alpha$ of $\mathrm{Alt}(n)$ of type $\alpha$.*

Before we can state the final result of this section, we first need to translate the dominance order on partitions to a corresponding partial order on types. So suppose that $\alpha$, $\beta$ are types. Let $n$ be an integer such that $n \geq \max\{2d(\alpha) + 1, 2d(\beta) + 1\}$ and let $\lambda_\alpha, \lambda_\beta$ be the corresponding partitions of $n$. Then we define

$$\alpha \trianglelefteq \beta \iff \lambda_\alpha \trianglelefteq \lambda_\beta.$$

It is easily checked that this definition is independent of the choice of the integer $n \geq \max\{2d(\alpha) + 1, 2d(\beta) + 1\}$. The following result, which is extracted from the proof of Leinen-Puglisi [7, Theorem 3.2], will play a key role in the next section. For the sake of completeness, we will sketch the main points of its proof.

**Lemma 3.3.** *Let $\alpha$ be a type of depth $d = d(\alpha)$, let $n$ be an integer such that $n \geq 2d + 1$, and let $\theta_\alpha$ be the irreducible character of $\mathrm{Alt}(n)$ of type $\alpha$. Then there exist integers $z_\beta \in \mathbb{Z}$, which are are independent of $n$, such that*

$$(3.3.a) \qquad\qquad \theta_\alpha = \sum_{\beta \trianglerighteq \alpha} z_\beta \pi_\beta.$$

*Furthermore, the integers $z_\beta$ satisfy:*

$$(3.3.b) \qquad\qquad \lim_{n \to \infty} \sum_{\substack{\beta \trianglerighteq \alpha \\ d(\beta) = d}} z_\beta \frac{\pi_\beta(1)}{\theta_\alpha(1)} = 1.$$

*Sketch proof.* Suppose that $\lambda$ is any partition of $n$ such that $n \geq 2d(\lambda) + 1$. If $\sigma$ is any partition of $n$ such that $\sigma \trianglerighteq \lambda$, then $d(\sigma) \leq d(\lambda)$ and so $n \geq 2d(\sigma) + 1$. In particular, letting $\varphi_\sigma$ be the corresponding irreducible character of $\mathrm{Sym}(n)$, we have that $\varphi_\sigma \restriction \mathrm{Alt}(n)$ is the unique irreducible character $\theta_\sigma$ associated with $\sigma$.

Thus Young's rule [12, Theorem 2.11.2] implies that

$$\text{(3.1)} \qquad\qquad \theta_\lambda = \pi_\lambda - \sum_{\sigma \rhd \lambda} \kappa_{\sigma\lambda}\theta_\sigma,$$

where $\kappa_{\sigma\lambda}$ is the corresponding Kostka number; i.e. the number of semi-standard tableaux of shape $\sigma$ and content $\lambda$. It is easily checked that, since

$$n \geq 2d(\sigma) + 1 \geq 2d(\lambda) + 1,$$

each of these Kostka numbers $\kappa_{\sigma\lambda}$ depends only on the types of $\sigma$ and $\lambda$. In particular, letting $\lambda$ be the partition of $n$ corresponding to the type $\alpha$, we can replace each partition in (3.1) by its corresponding type, and so obtain the following equality:

$$\theta_\alpha = \pi_\alpha - \sum_{\beta \rhd \alpha} \kappa_{\beta\alpha}\theta_\beta.$$

Proceeding inductively along the dominance order for types, we now easily obtain equation (3.3.a). In particular, we have that

$$\theta_\alpha(1) = \sum_{\beta \unrhd \alpha} z_\beta \pi_\beta(1)$$

and so

$$1 = \sum_{\beta \unrhd \alpha} z_\beta \frac{\pi_\beta(1)}{\theta_\alpha(1)}.$$

Using Lemmas 3.1 and 3.2, we easily obtain equation (3.3.b). $\qquad\qquad \square$

## 4. Full limits of finite alternating groups

In this section, we will prove Theorem 1.3 in the special case when $G = \bigcup_{i \in \mathbb{N}} G_i$ is a "full limit" of finite alternating groups. Our arguments in the first half of this section will follow those of Leinen-Puglisi [7, Section 3].

**Definition 4.1.** Suppose that $G = \bigcup_{i \in \mathbb{N}} G_i$ is the union of the strictly increasing chain of finite alternating groups $G_i = \text{Alt}(\Delta_i)$.
(i) The embedding $\text{Alt}(\Delta_i) \hookrightarrow \text{Alt}(\Delta_{i+1})$ is said to be *full* if $\text{Alt}(\Delta_i)$ has no trivial orbits on $\Delta_{i+1}$.
(ii) $G = \bigcup_{i \in \mathbb{N}} G_i$ is the *full limit* of the finite alternating groups $G_i = \text{Alt}(\Delta_i)$ if every embedding $\text{Alt}(\Delta_i) \hookrightarrow \text{Alt}(\Delta_{i+1})$ is full.

**Warning 4.2.** A composition of two full embeddings is not necessarily full. Consequently, if $G = \bigcup_{i \in \mathbb{N}} G_i$ is a full limit and $(k_i \mid i \in \mathbb{N})$ is a strictly increasing sequence of natural numbers, then $G = \bigcup_{i \in \mathbb{N}} G_{k_i}$ is not necessarily a full limit. The notion of a full limit is a purely technical one, first introduced in Thomas-Tucker-Drob [15], which is useful in the proofs of results about $L(\text{Alt})$-groups.

Most of this section will be devoted to the proof of the following result.

**Proposition 4.3.** *Suppose that $G = \bigcup_{i \in \mathbb{N}} G_i$ is the full limit of finite alternating groups $G_i = \text{Alt}(\Delta_i)$ and that $G$ has a nontrivial indecomposable character $\chi$. Then:*
    (a) *$\chi = \chi_\nu$ is the associated character of a nontrivial ergodic IRS $\nu$ of $G$; and*
    (a) *$G = \bigcup_{i \in \mathbb{N}} G_i$ has linear natural orbit growth.*

The proof of Proposition 4.3 will make use of the following result.

**Proposition 4.4** (Thomas-Tucker-Drob [15]). *If $G = \bigcup_{i \in \mathbb{N}} G_i$ is the full limit of finite alternating groups $G_i = \mathrm{Alt}(\Delta_i)$, then $G$ has a nontrivial ergodic IRS if and only if $G = \bigcup_{i \in \mathbb{N}} G_i$ has linear natural orbit growth.*

We will also make use of the following result, which is slight reformulation of Thomas-Tucker-Drob [15, Corollary 7.5].

**Lemma 4.5.** *If $G = \bigcup_{i \in \mathbb{N}} G_i$ is the full limit of the finite alternating groups $G_i = \mathrm{Alt}(\Delta_i)$, then $\liminf |\mathrm{supp}_{\Delta_i}(g)|/|\Delta_i| > 0$ for all $1 \neq g \in G$.*

From now on, suppose that $G = \bigcup_{i \in \mathbb{N}} G_i$ is the full limit of the finite alternating groups $G_i = \mathrm{Alt}(\Delta_i)$ and that $\chi$ is a nontrivial indecomposable character of $G$. Then, by Vershik-Kerov [18, Theorem 6], there exist irreducible characters $\theta_i$ of $G_i$ such that for all $g \in G$,

$$\chi(g) = \lim_{i \to \infty} \widehat{\theta}_i(g),$$

where $\widehat{\theta}_i = \theta_i/\theta_i(1)$ is the corresponding normalized irreducible character. For each $i \in \mathbb{N}$, let $d_i$ be the depth of the corresponding Young diagram $D(\theta_i)$. The proof of the next lemma is almost identical to that of Leinen-Puglisi [7, Proposition 3.5].

**Lemma 4.6.** $\limsup d_i < \infty$.

*Proof.* Since $\chi \neq \chi_{\mathrm{reg}}$, there exists a nonidentity element $1 \neq g \in G$ such that $\chi(g) \neq 0$. Applying Lemma 4.5, there exists $c > 0$ such that $|\mathrm{supp}_{\Delta_i}(g)| \geq cn_i$ for all sufficiently large $i$. Also, by Roichman [11, Theorem 5.4], since the length $n_i - d_i$ of the first row of the Young diagram $D(\theta_i)$ is greater or equal to the length of the first column, it follows that there exist constants $b > 0$ and $0 < q < 1$ such that if $i$ is sufficiently large, then

$$|\widehat{\theta}_i(g)| \leq \left( \max\left\{ q, \frac{n_i - d_i}{n_i} \right\} \right)^{b \cdot |\mathrm{supp}_{\Delta_i}(g)|}.$$

Since $\chi(g) = \lim_{i \to \infty} \widehat{\theta}_i(g) \neq 0$ and $\lim_{i \to \infty} |\mathrm{supp}_{\Delta_i}(g)| = \infty$, it follows that if $i$ is sufficiently large, then $\max\{ q, (n_i - d_i)/n_i \} = (n_i - d_i)/n_i$ and so

$$|\widehat{\theta}_i(g)| \leq \left( \frac{n_i - d_i}{n_i} \right)^{b \cdot |\mathrm{supp}_{\Delta_i}(g)|} = \left( 1 - \frac{d_i}{n_i} \right)^{b \cdot |\mathrm{supp}_{\Delta_i}(g)|}$$

It also now follows that $d_i/n_i \to 0$ as $i \to \infty$. Since $|\mathrm{supp}_{\Delta_i}(g)| \geq cn_i$ for all sufficiently large $i$, we have that

$$|\widehat{\theta}_i(g)| \leq \left( \left( 1 - \frac{d_i}{n_i} \right)^{\frac{n_i}{d_i}} \right)^{bcd_i}$$

Since $d_i/n_i \to 0$, it follows that

$$\left( 1 - \frac{d_i}{n_i} \right)^{\frac{n_i}{d_i}} \to \left( \frac{1}{e} \right)$$

and this implies that $\limsup d_i < \infty$. $\qquad\square$

Thus there exists an infinite subset $I \subseteq \mathbb{N}$ such that the irreducible character $\theta_i$ has the same type $\alpha$ for each $i \in I$. Let $d = d(\alpha)$ be the corresponding depth.

**Lemma 4.7.** $\chi(g) = \lim_{i \in I}(|\mathrm{Fix}_{\Delta_i}(g)|/|\Delta_i|)^d$ for all $g \in G$.

*Proof.* Suppose that $i \in I$ and that $n_i \gg d$. In order to simplify notation, we will write $n$, $\Delta$, $G$, $\theta$ instead of $n_i$, $\Delta_i$, $G_i$, $\theta_i$ and we will write limits as $\lim_{n \to \infty}$ instead of $\lim_{i \in I}$. For each type $\beta = (\ell_2, \cdots \ell_r) \trianglerighteq \alpha$, let $d(\beta)$ be the corresponding depth and let $\Phi_\beta$ be the corresponding set of partitions $(P_1, P_2, \cdots, P_r)$ of $\Delta$ such that $|P_1| = n - d(\beta)$ and $|P_k| = \ell_k$ for each $2 \leq k \leq r$. Let $\pi_\beta$ be the permutation character of the action $G \curvearrowright \Phi_\beta$ and let $\widehat{\pi}_\beta = \pi_\beta / \pi_\beta(1)$ be the corresponding normalized permutation character.

**Claim 4.8.** *For each type $\beta = (\ell_2, \cdots \ell_r) \trianglerighteq \alpha$ and element $g \in G$,*

$$\lim_{n \to \infty} \widehat{\pi}_\beta(g) = \lim_{n \to \infty} (|\operatorname{Fix}_\Delta(g)|/|\Delta|)^{d(\beta)}$$

*Proof of Claim 4.8.* Clearly we can suppose that $g \neq 1$. Let

$$F_0(g) = \{ (P_1, P_2, \cdots, P_r) \in \operatorname{Fix}_{\Phi_\beta}(g) \mid \operatorname{supp}_\Delta(g) \subseteq P_1 \}$$

and let $F_1(g) = \operatorname{Fix}_{\Phi_\beta}(g) \smallsetminus F_0(g)$. Let $c_\beta$ be the number of partitions of a $d(\beta)$-set into pieces of sizes $\ell_2, \cdots, \ell_r$. Then clearly

$$|\Phi_\beta| = c_\beta \binom{n}{d(\beta)} \quad \text{and} \quad |F_0(g)| = c_\beta \binom{|\operatorname{Fix}_\Delta(g)|}{d(\beta)}.$$

If $(P_1, P_2, \cdots, P_r) \in F_1(g)$, then $P_2 \sqcup \cdots \sqcup P_r$ is the union of $s$ nontrivial $g$-orbits $\sigma_1, \cdots, \sigma_s$ and $t = d(\beta) - \sum_{j=1}^{s} |\sigma_j|$ trivial $g$-orbits for some $1 \leq s \leq d(\beta)/2$. Clearly $0 \leq t \leq d(\beta) - 2s$. Since $g$ obviously has less than $n$ nontrivial orbits, it follows that

$$|F_1(g)| < c_\beta \sum_{s=1}^{d(\beta)/2} \binom{n}{s} \sum_{t=0}^{d(\beta)-2s} \binom{|\operatorname{Fix}_\Delta(g)|}{t}$$

$$< c_\beta \sum_{s=1}^{d(\beta)/2} \binom{n}{s} \sum_{t=0}^{d(\beta)-2s} \binom{n}{t}$$

and so there exists a polynomial $q(x) \in \mathbb{Z}[x]$ of degree at most $d(\beta) - 1$ such that $|F_1(g)| < q(n)$. Since $|\Phi_\beta|$ is a polynomial function of degree $d(\beta)$, it follows that $\lim_{n \to \infty} |F_1(g)|/|\Phi_\beta| = 0$. Hence

$$\lim_{n \to \infty} \widehat{\pi}_\beta(g) = \lim_{n \to \infty} |F_0(g)|/|\Phi_\beta|$$

$$= \lim_{n \to \infty} \binom{|\operatorname{Fix}_\Delta(g)|}{d(\beta)} / \binom{n}{d(\beta)}$$

$$= \lim_{n \to \infty} (|\operatorname{Fix}_\Delta(g)|/|\Delta|)^{d(\beta)}.$$

$\square$

Recall that $d(\alpha) = d$. Hence, applying Lemma 3.3, there exist integers $z_\beta \in \mathbb{Z}$, which are independent of $n$, such that

$$\theta = \theta_\alpha = \sum_{\beta \trianglerighteq \alpha} z_\beta \pi_\beta \quad \text{and} \quad \lim_{n \to \infty} \sum_{\substack{\beta \trianglerighteq \alpha \\ d(\beta) = d}} z_\beta \frac{\pi_\beta(1)}{\theta_\alpha(1)} = 1.$$

It follows that for each $g \in G$,

$$
\begin{aligned}
\chi(g) = \lim_{n\to\infty} \widehat{\theta}_\alpha(g) &= \lim_{n\to\infty} \sum_{\beta \trianglerighteq \alpha} z_\beta \frac{\pi_\beta(1)}{\theta_\alpha(1)} \widehat{\pi}_\beta(g) \\
&= \sum_{\beta \trianglerighteq \alpha} z_\beta \lim_{n\to\infty} \frac{\pi_\beta(1)}{\theta_\alpha(1)} \lim_{n\to\infty} \widehat{\pi}_\beta(g) \\
&= \sum_{\substack{\beta \trianglerighteq \alpha \\ d(\beta)=d}} z_\beta \lim_{n\to\infty} \frac{\pi_\beta(1)}{\theta_\alpha(1)} \lim_{n\to\infty} \widehat{\pi}_\beta(g) \\
&= \sum_{\substack{\beta \trianglerighteq \alpha \\ d(\beta)=d}} z_\beta \lim_{n\to\infty} \frac{\pi_\beta(1)}{\theta_\alpha(1)} \lim_{n\to\infty} (|\operatorname{Fix}_\Delta(g)|/|\Delta|)^d \\
&= \left( \lim_{n\to\infty} \sum_{\substack{\beta \trianglerighteq \alpha \\ d(\beta)=d}} z_\beta \frac{\pi_\beta(1)}{\theta_\alpha(1)} \right) \lim_{n\to\infty} (|\operatorname{Fix}_\Delta(g)|/|\Delta|)^d \\
&= \lim_{n\to\infty} (|\operatorname{Fix}_\Delta(g)|/|\Delta|)^d.
\end{aligned}
$$

This completes the proof of Lemma 4.7. $\qquad\square$

For each $i \in \mathbb{N}$, let $\Omega_i = \Delta_i^d$ and let $G_i \curvearrowright \Omega_i$ be the product action. Then the corresponding normalized permutation character of $G_i$ is

$$
|\operatorname{Fix}_{\Omega_i}(g)|/|\Omega_i| = (|\operatorname{Fix}_{\Delta_i}(g)|/|\Delta_i|)^d;
$$

and hence for each $g \in G$, we have that $\chi(g) = \lim_{i \in I} |\operatorname{Fix}_{\Omega_i}(g)|/|\Omega_i|$. We are now ready to complete the proof of Proposition 4.3. Our argument makes use of the Loeb measure construction [9]. Our exposition and notation follow that of Conley-Kechris-Tucker-Drob [3].

For each $i \in \mathbb{N}$, let $\mu_i$ be the uniform probability measure on $\Omega_i$ defined by $\mu_i(A) = |A|/|\Omega_i|$. Let $\mathcal{U}$ be a non-principal ultrafilter on $\mathbb{N}$ such that $I \in \mathcal{U}$ and let $\sim_{\mathcal{U}}$ be the equivalence relation on $X = \prod_{i\in\mathbb{N}} \Omega_i$ defined by

$$
(x_i) \sim_{\mathcal{U}} (y_i) \quad \Longleftrightarrow \quad \{\, i \in \mathbb{N} \mid x_i = y_i \,\} \in \mathcal{U}.
$$

For each $(x_i) \in X$, let $[(x_i)]_{\mathcal{U}}$ be the corresponding $\sim_{\mathcal{U}}$-equivalence class, and let

$$
X_{\mathcal{U}} = \{\, [(x_i)]_{\mathcal{U}} \mid (x_i) \in X \,\}.
$$

For each sequence $(A_i) \in \prod_{i\in\mathbb{N}} \mathcal{P}(\Omega_i)$, define the subset $[(A_i)]_{\mathcal{U}} \subseteq X_{\mathcal{U}}$ by

$$
[(x_i)]_{\mathcal{U}} \in [(A_i)]_{\mathcal{U}} \quad \Longleftrightarrow \quad \{\, i \in \mathbb{N} \mid x_i \in A_i \,\} \in \mathcal{U}.
$$

Then $\mathbf{B}_{\mathcal{U}}^0 = \{\, [(A_i)]_{\mathcal{U}} \mid (A_i) \in \prod_{i\in\mathbb{N}} \mathcal{P}(\Omega_i) \,\}$ is a Boolean algebra of subsets of $X_{\mathcal{U}}$, and we can define a finitely additive probability measure $\mu_{\mathcal{U}}$ on $\mathbf{B}_{\mathcal{U}}^0$ by

$$
\mu_{\mathcal{U}}([(A_i)]_{\mathcal{U}}) = \lim_{\mathcal{U}} \mu_i(|A_i|).
$$

Furthermore, there exists a $\sigma$-algebra $\mathbf{B}_{\mathcal{U}}$ of subsets of $X_{\mathcal{U}}$ such that $\mathbf{B}_{\mathcal{U}}^0 \subseteq \mathbf{B}_{\mathcal{U}}$ and such that $\mu_{\mathcal{U}}$ extends to a $\sigma$-additive probability measure on $\mathbf{B}_{\mathcal{U}}$, which we will also denote by $\mu_{\mathcal{U}}$. (A clear account of the construction of $\mathbf{B}_{\mathcal{U}}$ and $\mu_{\mathcal{U}}$ can be found in Conley-Kechris-Tucker-Drob [3, Section 2].) Thus $(X_{\mathcal{U}}, \mathbf{B}_{\mathcal{U}}, \mu_{\mathcal{U}})$ is a probability space. (Although this will not cause any difficulties in this proof, it is perhaps still worthwhile to note that this probability space is non-separable.)

Next for each $g \in G$ and $x \in \Omega_i$, we define

$$g \cdot x = \begin{cases} g(x), & \text{if } g \in G_i; \\ x, & \text{otherwise.} \end{cases}$$

Then we can define a measure-preserving action $G \curvearrowright (X_{\mathcal{U}}, \mathbf{B}_{\mathcal{U}}, \mu_{\mathcal{U}})$ by

$$g \cdot [(x_i)]_{\mathcal{U}} = [(g \cdot x_i)]_{\mathcal{U}}.$$

It is easily checked that $\operatorname{Fix}_{X_{\mathcal{U}}}(g) = [(\operatorname{Fix}_{\Omega_i}(g))]_{\mathcal{U}}$. Thus $\operatorname{Fix}_{X_{\mathcal{U}}}(g) \in \mathbf{B}_{\mathcal{U}}^0$ and

$$\mu_{\mathcal{U}}(\operatorname{Fix}_{X_{\mathcal{U}}}(g)) = \lim_{\mathcal{U}} |\operatorname{Fix}_{\Omega_i}(g)|/|\Omega_i| = \chi(g).$$

Let $f : X_{\mathcal{U}} \to \operatorname{Sub}_G$ be the $G$-equivariant map defined by $x \mapsto G_x$. Note that for each $g \in G$, we have that

$$f^{-1}(\{H \in \operatorname{Sub}_G \mid g \in H\}) = \operatorname{Fix}_{X_{\mathcal{U}}}(g) \in \mathbf{B}_{\mathcal{U}}^0.$$

It follows that $f$ is $\mathbf{B}_{\mathcal{U}}$-measurable and hence $\nu = f_* \mu_{\mathcal{U}}$ is an IRS of $G$. Furthermore, for each $g \in G$,

$$\chi(g) = \mu_{\mathcal{U}}(\operatorname{Fix}_{X_{\mathcal{U}}}(g)) = \nu(\{H \in Sub_G \mid g \in H\});$$

and so $\chi = \chi_\nu$ is the corresponding associated character. Finally, since $\chi$ is a nontrivial indecomposable character, it follows that $\nu$ is a nontrivial ergodic IRS; and thus Proposition 4.4 yields that $G = \bigcup_{i \in \mathbb{N}} G_i$ has linear natural orbit growth. This completes the proof of Proposition 4.3.

In the proof of Theorem 1.3, we will need to understand the decompositions of arbitary characters $\chi$ of full limits with linear natural orbit growth. So suppose that $G = \bigcup_{i \in \mathbb{N}} G_i$ is a full limit with linear natural orbit growth and let $G \curvearrowright (\Delta, m)$ be the canonical ergodic action. For each $r \geq 1$, let $\nu_r$ be the stabilizer distribution of the ergodic action $G \curvearrowright (\Delta^r, m^{\otimes r})$. Then, by Proposition 4.3 and Theorem 2.4, the set of indecomposable characters of $G$ is given by

$$\mathcal{E}(G) = \{\chi_{\text{reg}}, \chi_{\text{con}}\} \cup \{\chi_{\nu_r} \mid r \in \mathbb{N}^+\}.$$

**Proposition 4.9.** *With the above hypotheses, for every character $\chi \in \mathcal{F}(G)$, there exist uniquely determined non-negative real coefficients $\alpha$, $\beta$, and $\gamma_r$ for $r \geq 1$ such that:*

(i) $\alpha + \beta + \sum_{r \geq 1} \gamma_r = 1$; *and*

(ii) $\chi = \alpha \chi_{reg} + \beta \chi_{con} + \sum_{r \geq 1} \gamma_r \chi_{\nu_r}$.

*Consequently, $\mu = \alpha \delta_1 + \beta \delta_G + \sum_{r \geq 1} \gamma_r \nu_r$ is the* unique *IRS of $G$ such that $\chi_\mu = \chi$.*

*Proof.* As in the proof of Leinen-Puglisi [7, Theorem 3.6], every convergent sequence of elements of $\mathcal{E}(G)$, which does not tend to one of the functions in

$$\{\chi_{\text{con}}\} \cup \{\chi_{\nu_r} \mid r \in \mathbb{N}^+\},$$

must converge to

$$\lim_{r \to \infty} \chi_{\nu_r} = \lim_{r \to \infty} (\chi_{\nu_1})^r = \chi_{\text{reg}},$$

since $\chi_{\nu_1}(g) = m(\operatorname{Fix}_\Delta(g)) < 1$ for all $1 \neq g \in G$. Thus $\mathcal{E}(G)$ is a closed subset of $\mathcal{F}(G)$. By Thoma [13], $\mathcal{F}(G)$ is a Choquet simplex; and, applying Choquet's theorem, we obtain that if $\chi \in \mathcal{F}(G)$, then there exist uniquely determined non-negative real coefficients $\alpha$, $\beta$, and $\gamma_r$ for $r \geq 1$ such that:

(i) $\alpha + \beta + \sum_{r \geq 1} \gamma_r = 1$; and

(ii) $\chi = \alpha \chi_{\text{reg}} + \beta \chi_{\text{con}} + \sum_{r \geq 1} \gamma_r \chi_{\nu_r}$.

In particular, the IRS $\mu = \alpha\,\delta_1 + \beta\,\delta_G + \sum_{r\geq 1}\gamma_r\,\nu_r$ satisfies $\chi_\mu = \chi$. By considering an element $1 \neq g \in G$ such that $0 < m(\mathrm{Fix}_\Delta(g)) < 1$, we see that if $\nu \neq \nu'$ are two distinct ergodic IRSs of $G$, then $\chi_\nu \neq \chi_{\nu'}$; and it follows that $\mu$ is the unique IRS of $G$ such that $\chi_\mu = \chi$. $\qquad\square$

**Remark 4.10.** It is not true in general that if $G$ is a simple locally finite group and $\nu \neq \nu'$ are two distinct ergodic IRSs of $G$, then $\chi_\nu \neq \chi_{\nu'}$. For example, let $\mathbb{F} = GF(q)$ be the finite field with $q$ elements and let $V$ be a vector space over $\mathbb{F}$ having a countably infinite basis $\mathcal{B} = \{\,v_1, v_2, \cdots, v_n, \cdots\,\}$. For each $n \geq 1$, let $G_n$ be the group of linear transformations of $V$ that leave the subspace $V_n = \langle\,v_1, \cdots, v_n\,\rangle$ invariant, induce an element of $SL(V_n)$ on $V_n$ and fix each of the basis vectors in $\mathcal{B} \smallsetminus \{\,v_1, v_2, \cdots, v_n\,\}$. Then the *stable special linear group* $G = \bigcup_{n\geq 1} G_n$ is a simple locally finite group.

Let $V^*$ be the dual space of linear functionals $\varphi : V \to \mathbb{F}_p$ and let $\lambda$ be the Haar measure on $V^*$. Then $G$ acts ergodically on $(V^*, \lambda)$; and, letting $\nu$ be the corresponding stabilizer distribution, the associated character is $\chi_\nu(g) = 1/q^{\mathrm{rank}(g-1)}$.

Next let $X = \mathbb{F}^{\mathbb{N}^+}$ and let $\mu$ be the uniform product probability measure on $X$. Then, identifying $\mathbb{F}^n$ with $V_n$, we can define an ergodic action of $G$ on $(X, \mu)$ by letting each subgroup $G_n$ act via

$$g \cdot (\alpha_1, \cdots, \alpha_n, \alpha_{n+1}, \cdots, \alpha_m, \cdots) = (g(\alpha_1, \cdots, \alpha_n), \alpha_{n+1}, \cdots, \alpha_m, \cdots).$$

Let $\nu'$ be the corresponding stabilizer distribution. Then it is easily checked that the associated character is $\chi_{\nu'}(g) = 1/q^{\mathrm{rank}(g-1)}$ and so $\nu \neq \nu'$ are two distinct ergodic IRSs of $G$ such that $\chi_\nu = \chi_{\nu'}$.

## 5. The proof of Theorem 1.3

In this section, we will present the proof of Theorem 1.3. Suppose that $G$ is an $L(\mathrm{Alt})$-group and that $G \not\cong \mathrm{Alt}(\mathbb{N})$. Then, as explained in Section 1, it is enough to show that every indecomposable character of $G$ is the associated character $\chi_\nu$ of some ergodic IRS $\nu$ of $G$. First suppose that $G$ has no nontrivial indecomposable characters. Then, since $\chi_{\mathrm{con}} = \chi_{\delta_G}$ and $\chi_{\mathrm{reg}} = \chi_{\delta_1}$, the desired conclusion holds. Hence we can suppose that $G$ has a nontrivial indecomposable character $\chi$. Let $G = \bigcup_{i\in\mathbb{N}} G_i$ be the (not necessarily full) union of the increasing chain of finite alternating groups $G_i = \mathrm{Alt}(\Delta_i)$. We will begin by expressing $G = \bigcup_{\ell\in\mathbb{N}} G(\ell)$ as a (not necessarily strictly) increasing union of subgroups $G(\ell)$, each of which can be expressed as a full limit of finite alternating groups.

Applying Hall [5, Theorem 5.1], since $G \not\cong \mathrm{Alt}(\mathbb{N})$, it follows that for each $i \in \mathbb{N}$, the number $c_{ij}$ of nontrivial $G_i$-orbits on $\Delta_j$ is unbounded as $j \to \infty$. Hence, after passing to a suitable subsequence, we can suppose that each $G_i$ has at least 2 nontrivial orbits on $\Delta_{i+1}$. Since $G_i$ is simple, this implies that if $1 \neq G_i' \leqslant G_i$, then $G_i'$ also has at least 2 nontrivial orbits on $\Delta_{i+1}$. For each $\ell \in \mathbb{N}$, we define sequences of subsets $\Delta_j^\ell \subseteq \Delta_j$ and subgroups $G(\ell)_j = \mathrm{Alt}(\Delta_j^\ell)$ for $j \geq \ell$ inductively as follows:

- $\Delta_\ell^\ell = \Delta_\ell$;
- $\Delta_{j+1}^\ell = \Delta_{j+1} \smallsetminus \mathrm{Fix}_{\Delta_{j+1}}(G(\ell)_j)$.

Clearly each $G(\ell)_j$ is strictly contained in $G(\ell)_{j+1}$ and $G(\ell) = \bigcup_{\ell \leq j \in \mathbb{N}} G(\ell)_j$ is the full limit of the alternating groups $G(\ell)_j = \mathrm{Alt}(\Delta_j^\ell)$. It is also easily checked that

if $\ell < m$ and $i < j$, then

$$G_\ell \leqslant G(\ell)_i \leqslant G(m)_i < G(m)_j.$$

It follows that if $\ell < m$, then $G(\ell) \leqslant G(m)$ and that $G = \bigcup_{\ell \in \mathbb{N}} G(\ell)$. For each $\ell \in \mathbb{N}$, let $\chi_\ell = \chi \upharpoonright G(\ell)$.

**Lemma 5.1.** *The subgroup $G(\ell) = \bigcup_{\ell \leq j \in \mathbb{N}} G(\ell)_j$ has linear natural orbit growth for all but finitely many $\ell \in \mathbb{N}$.*

*Proof.* For the sake of contradiction, suppose that there exists an infinite subset $I \subseteq \mathbb{N}$ such that for all $\ell \in I$, the subgroup $G(\ell) = \bigcup_{\ell \leq j \in \mathbb{N}} G(\ell)_j$ does *not* have linear natural orbit growth. Then, by Proposition 4.3, for each $\ell \in I$, the only indecomposable characters of $G(\ell)$ are $\chi_{\mathrm{con}}$ and $\chi_{\mathrm{reg}}$. Hence there exists a real number $0 \leq r_\ell \leq 1$ such that $\chi_\ell = r_\ell \chi_{\mathrm{con}} + (1 - r_\ell) \chi_{\mathrm{reg}}$. If $\ell < m$ are distinct elements of $I$, then $G(\ell) \leqslant G(m)$ and it follows that $r_\ell = r_m$. But then there exists a fixed $r$ such that $r_\ell = r$ for all $\ell \in I$ and this implies that $\chi = r \chi_{\mathrm{con}} + (1 - r) \chi_{\mathrm{reg}}$, which is a contradiction. $\qquad\square$

Hence we can suppose that $G(\ell) = \bigcup_{\ell \leq j \in \mathbb{N}} G(\ell)_j$ has linear natural orbit growth for all $\ell \in \mathbb{N}$. Let $G(\ell) \curvearrowright (\Delta_\ell, m_\ell)$ be the canonical ergodic action and for each $r \in \mathbb{N}^+$, let $\nu(\ell)_r$ be the stabilizer distribution of $G(\ell) \curvearrowright (\Delta_\ell^r, m_\ell^{\otimes r})$. Then for each $\ell \in \mathbb{N}$, there exist $\alpha(\ell), \beta(\ell), \gamma(\ell)_r \in [0, 1]$ with $\alpha(\ell) + \beta(\ell) + \sum_{r \in \mathbb{N}^+} \gamma(\ell)_r = 1$ such that

$$(5.1) \qquad \chi_\ell = \alpha(\ell) \chi_{\mathrm{reg}} + \beta(\ell) \chi_{\mathrm{con}} + \sum_{r \in \mathbb{N}^+} \gamma(\ell)_r \chi_{\nu(\ell)_r}.$$

Thus $\chi_\ell$ is the associated character $\chi_{\nu_\ell}$ of the IRS $\nu_\ell$ of $G(\ell)$ defined by

$$(5.2) \qquad \nu_\ell = \alpha(\ell) \delta_1 + \beta(\ell) \delta_{G(\ell)} + \sum_{r \in \mathbb{N}^+} \gamma(\ell)_r \nu(\ell)_r.$$

For each $\ell \in \mathbb{N}$, let $f_\ell : \mathrm{Sub}_{G(\ell+1)} \to \mathrm{Sub}_{G(\ell)}$ be the continuous map defined by $H \mapsto H \cap G(\ell)$.

**Lemma 5.2.** $(f_\ell)_* \nu_{\ell+1} = \nu_\ell.$

*Proof.* Let $\theta_\ell$ be the character associated with the IRS $(f_\ell)_* \nu_{\ell+1}$ of $G(\ell)$. Then for each element $g \in G(\ell)$,

$$\begin{aligned}
\theta_\ell(g) &= (f_\ell)_* \nu_{\ell+1}(\{ K \in \mathrm{Sub}_{G(\ell)} \mid g \in K \}) \\
&= \nu_{\ell+1}(\{ H \in \mathrm{Sub}_{G(\ell+1)} \mid g \in H \}) \\
&= \chi_{\ell+1}(g) \\
&= \chi_\ell(g).
\end{aligned}$$

Hence the result follows from Proposition 4.9. $\qquad\square$

Thus $\{ (\mathrm{Sub}_{G(\ell)}, \nu_\ell) \mid \ell \in \mathbb{N} \}$ is an inverse family of topological probability spaces in the sense of Choksi [2]; and clearly we can naturally identify the inverse limit $\varprojlim \mathrm{Sub}_{G(\ell)}$ with $\mathrm{Sub}_G$. For each $\ell \in \mathbb{N}$, let $f_{\infty \ell} : \mathrm{Sub}_G \to \mathrm{Sub}_{G(\ell)}$ be the continuous map defined by $H \mapsto H \cap G(\ell)$. Applying Choksi [2, Theorem 2.2], since each $\mathrm{Sub}_{G(\ell)}$ is a compact Hausdorff space, it follows that there exists a measure

$\nu$ on $\mathrm{Sub}_G$ such that $(f_{\infty\ell})_*\nu = \nu_\ell$ for each $\ell \in \mathbb{N}$. Note that for each $\ell \in \mathbb{N}$ and element $g \in G(\ell)$, we have that

$$\begin{aligned}
\chi(g) &= \nu_\ell(\{ K \in \mathrm{Sub}_{G(\ell)} \mid g \in K \}) \\
&= (f_{\infty\ell})_*\nu(\{ K \in \mathrm{Sub}_{G(\ell)} \mid g \in K \}) \\
&= \nu(\{ H \in \mathrm{Sub}_G \mid g \in H \}).
\end{aligned}$$

Thus $\chi$ is the character associated with the IRS $\nu$ of $G$; and since $\chi$ is a nontrivial indecomposable character, it follows that $\nu$ is a nontrivial ergodic IRS. This completes the proof of Theorem 1.3.

## 6. The indecomposable characters of $\mathrm{Alt}(\mathbb{N})$

In this final section, we will point out the two ways in which Theorem 1.3 fails when $G = \mathrm{Alt}(\mathbb{N})$. Firstly, it follows from Thomas-Tucker-Drob [15, Theorem 9.2] that there exist ergodic IRSs $\nu$ of $\mathrm{Alt}(\mathbb{N})$ such that the associated character

$$\chi_\nu(g) = \nu(\{ H \in \mathrm{Sub}_G \mid g \in H \})$$

is *not* indecomposable. Secondly, as we will explain in the remainder of this section, there exist indecomposable characters $\chi$ of $\mathrm{Alt}(\mathbb{N})$ for which there does *not* exist an ergodic IRS $\nu$ such that $\chi = \chi_\nu$.

We will begin by recalling Thoma's classification [14] of the indecomposable characters of $\mathrm{Alt}(\mathbb{N})$. For each $g \in \mathrm{Alt}(\mathbb{N})$ and $n \geq 2$, let $c_n(g)$ be the number of cycles of length $n$ in the cyclic decomposition of the permutation $g$. Then the indecomposable characters of $\mathrm{Alt}(\mathbb{N})$ are precisely the functions $\chi : \mathrm{Alt}(\mathbb{N}) \to \mathbb{C}$ such that there exist two sequences $(\alpha_i \mid i \in \mathbb{N}^+)$ and $(\beta_i \mid i \in \mathbb{N}^+)$ of non-negative real numbers satisfying

- $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_i \geq \cdots \geq 0$;
- $\beta_1 \geq \beta_2 \geq \cdots \geq \beta_i \geq \cdots \geq 0$;
- $\sum_{i=1}^\infty \alpha_i + \sum_{i=1}^\infty \beta_i \leq 1$;

and such that for all $g \in \mathrm{Alt}(\mathbb{N})$,

$$\chi(g) = \prod_{n=2}^\infty s_n^{c_n(g)}, \qquad \text{where } s_n = \sum_{i=1}^\infty \alpha_i^n + (-1)^{n+1}\sum_{i=1}^\infty \beta_i^n.$$

(In these products, $s_n^0$ is always taken to be 1, including the case when $s_n = 0$.)

**Proposition 6.1.** *If $\chi$ is the indecomposable character for which $\alpha_1 = \beta_1 = 1/2$ and $\alpha_i = \beta_i = 0$ for all $i > 1$, then there does not exist an ergodic IRS $\nu$ of $\mathrm{Alt}(\mathbb{N})$ such that $\chi = \chi_\nu$.*

*Proof.* Suppose that $\nu$ is an ergodic IRS of $\mathrm{Alt}(\mathbb{N})$ such that $\chi = \chi_\nu$; i.e. such that

$$\chi(g) = \nu(\{ H \in \mathrm{Sub}_{\mathrm{Alt}(\mathbb{N})} \mid g \in H \}).$$

Since $\chi((a\,b\,c)) = 1/4$, it follows that there exist $n \neq m$ such that

$$\nu(\{ H \in \mathrm{Sub}_{\mathrm{Alt}(\mathbb{N})} \mid (1\,2\,n), (1\,2\,m) \in H \}) > 0;$$

and, since $(1\,2\,n)(1\,2\,m) = (1\,n)(2\,m)$, it follows that

$$\nu(\{ H \in \mathrm{Sub}_{\mathrm{Alt}(\mathbb{N})} \mid (1\,n)(2\,m) \in H \}) > 0.$$

But this contradicts that fact that $\chi((1\,n)(2\,m)) = 0$. $\qquad\square$

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Mathematics Department, Rutgers University, 110 Frelinghuysen Road, Piscataway, New Jersey 08854-8019, USA

*E-mail address*: simon.rhys.thomas@gmail.com