ON THE STEINHAUS AND BERGMAN PROPERTIES FOR INFINITE PRODUCTS OF FINITE GROUPS

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ABSTRACT. We study the relationship between the existence of nonprincipal ultrafilters over ω and the failure of the Steinhaus and Bergman properties for infinite products of finite groups.

1. INTRODUCTION

In this paper, we will investigate the status of the Steinhaus and Bergman properties for infinite products of finite groups in various axiomatic frameworks. Our interest in these properties is partially motivated by the automatic continuity problem for Polish groups. More specifically, we will be interested in the question of which infinite products $G = \prod G_n$ of nontrivial finite groups have the *automatic continuity property*; i.e. have the property that *every* homomorphism $\varphi : G \to H$ from G into a Polish group H is necessarily continuous. In set theory with the Axiom of Choice, infinite products of finite groups typically fail to have this property; and, in fact, no examples of infinite products of finite groups with this property are currently known. The basic example of a non-continuous homomorphism involves a nonprincipal ultrafilter $\mathcal{U}$ over the set ω of natural numbers.

Example 1.1. Suppose that there exists a fixed nontrivial finite group F such that $G_n \cong F$ for all $n \in \omega$. Then the corresponding ultraproduct $\prod_{\mathcal{U}} G_n$ is isomorphic to F and it is clear that the associated homomorphism $\varphi : \prod G_n \to F$ is not continuous.

The automatic continuity property for some more interesting infinite products of finite groups can be shown to fail for more complicated reasons.

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Example 1.2. Let $d \geq 2$ and suppose that $G_n = SL(d, p_n)$, where $(p_n \mid n \in \omega)$ is an increasing sequence of primes. If $K = \prod_{\mathcal{U}} \mathbb{F}_{p_n}$ is the corresponding ultraproduct of the fields $\mathbb{F}_{p_n}$ of order p_n, then

$$\prod_{\mathcal{U}} SL(d, p_n) \cong SL(d, \prod_{\mathcal{U}} \mathbb{F}_{p_n}) = SL(d, K)$$

and thus $SL(d, K)$ is a homomorphic image of $\prod SL(d, p_n)$. Since K is a field of characteristic 0 and cardinality $2^{\aleph_0}$, it follows that K embeds into $\mathbb{C}$ and hence $SL(d, K)$ embeds into $SL(d, \mathbb{C})$. Once again, it is clear that the associated homomorphism $\varphi : \prod SL(d, p_n) \to SL(d, \mathbb{C})$ is not continuous.

Remark 1.3. In Section 3, we will present a more sophisticated construction involving an embedding of K into the field of Puiseux series over the field $\overline{\mathbb{Q}}$ of algebraic numbers, which yields a non-continuous homomorphism of $\prod SL(d, p_n)$ into the infinite symmetric group $\mathrm{Sym}(\omega)$.

It is natural to ask whether the existence of a nonprincipal ultrafilter $\mathcal{U}$ over ω is either necessary or sufficient in the above constructions of non-continuous homomorphisms. (The existence of a nonprincipal ultrafilter $\mathcal{U}$ is clearly sufficient in Example 1.1. However, the construction in Example 1.2 also makes use of the existence of an embedding of the field $K = \prod_{\mathcal{U}} \mathbb{F}_{p_n}$ into $\mathbb{C}$ and the usual proofs of this result rely on the existence of transcendence bases for both K and $\mathbb{C}$.) Of course, when considering this kind of question, we cannot work with the usual ZFC axioms of set theory since these already imply the existence of nonprincipal ultrafilters over arbitrary infinite sets. Instead we will work with the axiom system $ZF + DC$, where DC is the following weak form of the Axiom of Choice.

Axiom of Dependent Choice (DC)**.** Suppose that X is a nonempty set and that R is a binary relation on X such that for all $x \in X$, there exists $y \in X$ with $x \, R \, y$. Then there exists a function $f : \omega \to X$ such that $f(n) \, R \, f(n+1)$ for all $n \in \omega$.

The axiom system $ZF + DC$ is sufficient to develop most of real analysis and descriptive set theory, but is insufficient to prove the existence of pathologies such as nonmeasurable sets. (For example, see Moschovakis [22].) In particular, since nonprincipal ultrafilters over ω are nonmeasurable when regarded as subsets of the

Cantor space $2^{\mathbb{N}}$, it follows that $ZF + DC$ does not prove the existence of such ultrafilters.

We will provide a structured answer to the above question. Firstly, the following result is well-known. (For example, see Rosendal [24, Section 2].)

Theorem 1.4. *It is consistent with $ZF + DC$ that if G, H are* any *Polish groups, then every homomorphism $\varphi : G \to H$ is continuous.*

In fact, assuming the existence of suitable large cardinals, this is true in $L(\mathbb{R})$, the canonical minimal model of ZF which contains all of the ordinals and all of the real numbers. Of course, this implies the well-known result that $L(\mathbb{R})$ does not contain any nonprincipal ultrafilters over ω. While it seems almost certain that the existence of a nonprincipal ultrafilter over ω is necessary to prove the failure of the automatic continuity property for suitably chosen infinite products $G = \prod G_n$ of finite groups, we have not completely settled this question. However, in Section 4, we will prove a number of partial results in this direction, including the following theorem.

Theorem 1.5 ($ZF + DC$)**.** *Suppose that $d \geq 2$ and that $(p_n \mid n \in \omega)$ is an increasing sequence of primes. If there exists a non-continuous homomorphism $\varphi : \prod SL(d, p_n) \to \mathrm{Sym}(\omega)$, then there exists a nonprincipal ultrafilter over ω.*

On the other hand, we will show that the existence of a nonprincipal ultrafilter over ω is not sufficient to prove the failure of the automatic continuity property for suitably chosen infinite products of finite groups. In order to explain this result, it will be necessary in the remainder of this section to assume the existence of suitable large cardinals. We will not specify the precise large cardinal hypothesis that we need until it becomes necessary to do so in Section 7. (This paper has been written so that the first six sections can be read by mathematicians with no knowledge of advanced set theory, such as forcing, large cardinals, etc. It is only in the final section that some knowledge of advanced set theory is needed and this section can be omitted by mathematicians without the necessary background.) Following the usual convention [31], we will indicate the use of a large cardinal hypothesis by writing (LC) before the statement of the relevant theorem. The following result is a special case of a more general result that we will present in Section 5.

Theorem 1.6 (LC)**.** *It is consistent with $ZF + DC$ that*

(i) *there exists a nonprincipal ultrafilter $\mathcal{U}$ over ω; and*

(ii) *for each $d \geq 2$, if $(p_n \mid n \in \omega)$ is a sufficiently fast growing sequence of primes, then $\prod SL(d, p_n)$ has the automatic continuity property.*

In fact, assuming the existence of suitable large cardinals, this is true in $L(\mathbb{R})[\mathcal{U}]$, the minimal model of ZF containing all of the ordinals and real numbers, together with a Ramsey ultrafilter $\mathcal{U}$ over ω. Under a suitable large cardinal hypothesis, $L(\mathbb{R})[\mathcal{U}]$ has canonicity features parallel to those of $L(\mathbb{R})$; and, in particular, its theory does not depend on the choice of the Ramsey ultrafilter $\mathcal{U}$. Di Prisco-Todorcevic [7] have shown that many of the regularity properties of $L(\mathbb{R})$ continue to hold in $L(\mathbb{R})[\mathcal{U}]$. For example, in $L(\mathbb{R})[\mathcal{U}]$, every uncountable set of reals has a perfect subset. Thus it seems natural to regard $L(\mathbb{R})[\mathcal{U}]$ as a canonical model of $ZF + DC$ in which a minimal number of the pathological consequences of the Axiom of Choice hold, modulo the existence of a nonprincipal ultrafilter $\mathcal{U}$ over ω. The results of this paper provide yet more evidence for this point of view.

Up until this point, we have considered two examples of infinite products of finite groups; namely, infinite products of a fixed finite group F and infinite products of the form $\prod SL(d, p_n)$ for various increasing sequences $(p_n \mid n \in \omega)$ of primes. In the first example, we have seen that the existence of a nonprincipal ultrafilter $\mathcal{U}$ over ω is sufficent to prove the failure of the automatic continuity property; while in the second example, this is not sufficent. Now we should also consider a third example; namely, the infinite product $\prod \mathrm{Alt}(n)$ of the finite alternating groups. In this case, as we will explain in Section 6, it is natural to conjecture that the automatic continuity property holds. So what is the essential difference between these three examples? Perhaps surprisingly, the key to our analysis of the infinite product $\prod G_n$ of finite groups turns out to be the "asymptotic representation theory" of the sequence $(G_n \mid n \in \omega)$. In order to state this more precisely, it is necessary to introduce the following definitions.

Definition 1.7. Let H be a nontrivial finite group.

(i) If K is a field, then $d_K(H)$ denotes the minimal dimension of a nontrivial K-representation of H; i.e. the least d such that there exists a nontrivial homomorphism $\theta : H \to GL(d, K)$.

(ii) $d(H) = \min\{ d_K(H) \mid K \text{ is a field } \}$.

Example 1.8. Suppose that $p \geq 5$ is a prime and that $H = SL(d,p)$.

(i) If $d = 2$, then $d_{\mathbb{C}}(H) = (p-1)/2$ and $d(H) = 2$.

(ii) If $d > 2$, then $d_{\mathbb{C}}(H) = (p^d - p)/(p-1)$ and $d(H) = d$.

(For example, see Humphreys [12] and Tiep-Zalesskii [30].)

Let $(G_n \mid n \in \omega)$ be a sequence of nontrivial finite groups. In this paper, we will prove the following results.

(a) If $\liminf d_{\mathbb{C}}(G_n) < \infty$, then the existence of a nonprincipal ultrafilter $\mathcal{U}$ over ω is enough to prove that $\prod G_n$ does not have the automatic continuity property.

(b) Assuming (LC), if $(d_{\mathbb{C}}(G_n) \mid n \in \omega)$ grows sufficiently fast, then $\prod G_n$ has the automatic continuity property in $L(\mathbb{R})[\mathcal{U}]$.

(c) If $\liminf d(G_n) < \infty$, then $\prod G_n$ does not have the automatic continuity property in the actual set-theoretic universe V.

Furthermore, we conjecture that the converse of (c) also holds.

This paper is organized as follows. In Section 2, we will discuss the Steinhaus and Bergman properties for infinite products of finite groups. In Section 3, working with the usual ZFC axioms of set theory, we will prove that the Steinhaus and Bergman properties fail for various infinite products of finite groups. In Section 4, working with the axiom system $ZF + DC$, we will prove that the failure of the Bergman property for suitably chosen infinite products of finite groups implies the existence of a nonprincipal ultrafilter over ω; and we will show that the failure of a weak form of the Steinhaus property also implies the existence of such an ultrafilter. In Section 5, we will present a partition property PP for products of finite sets with measures; and we will show that $ZF + DC + PP$ implies that various infinite products of finite groups have both the Bergman property and the Steinhaus property. In Section 6, we will briefly discuss the questions of which infinite products of nonabelian finite simple groups have either the Bergman property or the Steinhaus property in the actual set-theoretic universe V. Finally, in Section 7, assuming the existence of suitable large cardinals, we will prove that $L(\mathbb{R})[\mathcal{U}]$ satisfies PP.

Notation 1.9. Let $(\, H_n \mid n \in \omega \,)$ be a sequence of finite groups and let $H = \prod H_n$. Suppose that $A \subseteq \omega$.

(i) $\prod_{n \in A} H_n$ denotes the subgroup of H consisting of those elements $(\, h_n \,) \in H$ such that $h_n = 1$ for all $n \in \omega \smallsetminus A$.

(ii) If $h = (\, h_n \,) \in H$, then $h \upharpoonright A$ denotes the element $(g_n) \in \prod_{n \in A} H_n$ such that $g_n = h_n$ for all $n \in A$.

Recall that $H = \prod H_n$ is a Polish topological group with neighborhood basis of the identity given by $\{\, \prod_{n \in A} H_n \mid A \text{ is a cofinal subset of } \omega \,\}$.

Suppose that U is a subset of the group G. Then for each $t \geq 1$, U^t denotes the set of elements $g \in G$ which can be expressed as a product $g = u_1 \cdots u_t$, where each $u_i \in U$. The subset U is said to be *symmetric* if $U = U^{-1}$ is closed under taking inverses.

2. The Steinhaus and Bergman Properties

In this section, we will discuss the Steinhaus and Bergman properties for infinite products of finite groups. The Steinhaus property was introduced by Rosendal-Solecki [23] in the context of the automatic continuity problem for homomorphisms between topological groups. In the following definition, a subset W of a group G is said to be *countably syndetic* if there exist elements $g_n \in G$ for $n \in \omega$ such that $G = \bigcup_{n \in \omega} g_n W$.

Definition 2.1. Let G be a topological group. Then G has the *Steinhaus property* if there exists a *fixed* integer $k \geq 1$ such that for every symmetric countably syndetic subset $W \subseteq G$, the k-fold product W^k contains an open neighborhood of the identity element 1_G.

Proposition 2.2 (Rosendal-Solecki [23])**.** *If G is a topological group with the Steinhaus property and $\varphi : G \to H$ is a homomorphism into a separable group H, then φ is necessarily continuous.*

The class of groups with the Steinhaus property includes Polish groups with ample generics [15], $\mathrm{Aut}(\mathbb{Q}, <)$, $\mathrm{Homeo}(\mathbb{R})$ [23] and full groups of ergodic countable Borel equivalence relations [16]. However, no infinite product of finite groups is currently known to have the Steinhaus property. Of course, by Example 1.2 and

Proposition 2.2, it follows that if $(p_n \mid n \in \omega)$ is an increasing sequence of primes and $d \geq 2$, then $\prod SL(d, p_n)$ does not have the Steinhaus property. We will prove the following more general result in Section 3.

Theorem 2.3. *Suppose that $(G_n \mid n \in \omega)$ is a sequence of nontrivial finite groups. If $\liminf d(G_n) < \infty$, then $\prod G_n$ does not have the automatic continuity property and hence does not have the Steinhaus property.*

As the reader has probably guessed, the proof of Theorem 2.3 involves the use of a suitable ultraproduct $\prod_{\mathcal{U}} G_n$. However, the following strengthening of Theorem 1.6, which we will prove in Section 5, shows that the existence of a nonprincipal ultrafilter over ω is not always enough to prove that such a product $\prod G_n$ does not have the Steinhaus property.

Theorem 2.4 (LC)**.** *It is consistent with $ZF + DC$ that*

(i) *there exists a nonprincipal ultrafilter $\mathcal{U}$ over ω; and*
(ii) *for each $d \geq 2$, if $(p_n \mid n \in \omega)$ is a sufficiently fast growing sequence of primes, then $\prod SL(d, p_n)$ has the Steinhaus property.*

Once again, assuming the existence of suitable large cardinals, this is true in $L(\mathbb{R})[\mathcal{U}]$, the minimal model of ZF containing all of the ordinals and real numbers, together with a Ramsey ultrafilter $\mathcal{U}$ over ω.

The Bergman property was introduced by Bergman [2] as a strengthening of the notion of uncountable cofinality which was introduced earlier by Macpherson-Neumann [20].

Definition 2.5. Suppose that G is a non-finitely generated group.

(a) G has *countable cofinality* if $G = \bigcup_{n \in \omega} G_n$ can be expressed as the union of a countable increasing chain of proper subgroups. Otherwise, G has *uncountable cofinality*.
(b) G is *Cayley bounded* if for every symmetric generating set S, there exists an integer $n \geq 1$ such that every element $g \in G$ can be expressed as a product $g = s_1 \cdots s_n$, where each $s_i \in S \cup \{ 1 \}$.
(c) G has the *Bergman property* if G has uncountable cofinality and is Cayley bounded.

By de Cornulier [6], a group G has the Bergman property if and only if whenever G acts isometrically on a metric space, every G-orbit has a finite diameter. For this reason, groups with the Bergman property are often said to be "strongly bounded". The class of groups with the Bergman property includes the symmetric groups over infinite sets [2], automorphism groups of various infinite structures [9, 13] and oligomorphic groups with ample generics [15] The following easy observation is essentially contained in Bergman [2, Lemma 10].

Lemma 2.6. *If G is a non-finitely generated group, then the following conditions are equivalent.*

(a) *G has the Bergman property.*
(b) *If $G = \bigcup_{n\in\omega} U_n$ is the union of an increasing chain of symmetric subsets such that $U_n U_n \subseteq U_{n+1}$ for all $n \in \omega$, then there exists an $n \in \omega$ such that $U_n = G$.*

In [6], improving an earlier result of Koppelberg-Tits [17], de Cornulier proved that if G is a product of infinitely many copies of a fixed finite perfect group, then G has the Bergman property; and Zalan Gyenis has recently checked that the arguments of Saxl-Shelah-Thomas [25] can be modified to prove that an infinite product $\prod S_n$ of finite simple groups has the Bergman property if and only if $\prod S_n$ has uncountable cofinality. This yields an explicit classification of the infinite products $\prod S_n$ of finite simple groups satisfying the Bergman property, which we will discuss in Section 6. On the other hand, there are many infinite products of finite groups which are known not to have the Bergman property. In particular, the following result holds.

Theorem 2.7. *If $d \geq 2$ and $(\, p_n \mid n \in \omega \,)$ is an increasing sequence of primes, then:*

(a) *$\prod SL(d, p_n)$ has countable cofinality; and*
(b) *$\prod SL(d, p_n)$ is not Cayley bounded.*

Theorem 2.7(a) is essentially contained in Saxl-Shelah-Thomas [25]. However, for the sake of completeness, we will quickly sketch the very easy proof. (We will present the proof of Theorem 2.7(b) in Section 3.) Let $\mathcal{U}$ be a nonprincipal ultrafilter over ω and let $K = \prod_{\mathcal{U}} \mathbb{F}_{p_n}$ be the corresponding ultraproduct of the fields $\mathbb{F}_{p_n}$ of order p_n. Then K is an uncountable field and

$$\prod_{\mathcal{U}} SL(d, p_n) \cong SL(d, \prod_{\mathcal{U}} \mathbb{F}_{p_n}) = SL(d, K).$$

It follows that $SL(d, K)$ is a homomorphic image of $\prod SL(d, p_n)$ and hence Theorem 2.7(a) is an immediate consequence of the following observation.

Proposition 2.8. *If F is an uncountable field, then $SL(d, F)$ has countable cofinality.*

Proof. Let B be a transcendence basis of F over its prime subfield. Then B is uncountable and hence we can express $B = \bigcup_{n \in \omega} B_n$ as the union of a countable strictly increasing chain of proper subsets. For each $n \in \omega$, let F_n be the algebraic closure of B_n in F. Then the strictly increasing chain of proper subgroups

$$SL(d, F) = \bigcup_{n \in \omega} SL(d, F_n)$$

witnesses that $SL(d, F)$ has countable cofinality. □

The following result, which will be proved in Section 4, shows that the existence of a nonprincipal ultrafilter over ω is necessary in order to prove either Theorem 2.7(a) or Theorem 2.7(b).

Theorem 2.9 ($ZF + DC$)**.** *Let $d \geq 2$ and let $(\, p_n \mid n \in \omega \,)$ be an increasing sequence of primes. If $\prod SL(d, p_n)$ does not have the Bergman property, then there exists a nonprincipal ultrafilter over ω.*

On the other hand, we will also show that the existence of a nonprincipal ultrafilter over ω is not sufficient to prove either of the parts of Theorem 2.7.

Theorem 2.10 (LC)**.** *If $d \geq 2$ and $(\, p_n \mid n \in \omega \,)$ is an increasing sequence of primes, then $\prod SL(d, p_n)$ has the Bergman property in $L(\mathbb{R})[\mathcal{U}]$.*

Examining the above proof of Theorem 2.7(a), we see that it relies upon the following three consequences of the Axiom of Choice:

(i) the existence of a nonprincipal ultrafilter $\mathcal{U}$ over ω;

(ii) the existence of a transcendence basis B of the field $\prod_{\mathcal{U}} \mathbb{F}_{p_n}$; and

(iii) the existence of an expression of B as the union of a countable strictly increasing chain of proper subsets.

Clearly $L(\mathbb{R})[\mathcal{U}]$ satisfies (i); and since DC implies that every infinite set has a denumerably infinite subset, it follows easily that every infinite set can be expressed as the union of a countable strictly increasing chain of proper subsets in $L(\mathbb{R})[\mathcal{U}]$. Consequently, assuming LC, if $(\, p_n \mid n \in \omega\,)$ is an increasing sequence of primes, then (ii) must fail in $L(\mathbb{R})[\mathcal{U}]$.

Corollary 2.11 (LC)**.** *If $(\, p_n \mid n \in \omega\,)$ is an increasing sequence of primes, then the field $\prod_{\mathcal{U}} \mathbb{F}_{p_n}$ does not have a transcendence basis in $L(\mathbb{R})[\mathcal{U}]$.*

3. On the failure of the Bergman and Steinhaus properties

In this section, we will first that if $(\, p_n \mid n \in \omega\,)$ is an increasing sequence of primes and $d \geq 2$, then:

- There exists a non-continuous homomorphism of $\prod SL(d, p_n)$ into $\mathrm{Sym}(\omega)$.
- $\prod SL(d, p_n)$ is not Cayley bounded.

Then we will prove prove that if $(\, G_n \mid n \in \omega\,)$ is a sequence of nontrivial finite groups such that $\liminf d(G_n) < \infty$, then $\prod G_n$ does not have the automatic continuity property and hence does not have the Steinhaus property.

Once again, let $\mathcal{U}$ be a nonprincipal ultrafilter over ω and let $K = \prod_{\mathcal{U}} \mathbb{F}_{p_n}$ be the corresponding ultraproduct of the fields $\mathbb{F}_{p_n}$ of order p_n. Our arguments depend upon the existence of a suitable valuation $\upsilon : K \to \mathbb{Q} \cup \{\,\infty\,\}$.

Definition 3.1. Let F be a field and let t be an indeterminate over F. Then $F((t))$ denotes the corresponding field of formal power series; and

$$\mathbf{P}(F) = \bigcup_{n \geq 1} F((t^{1/n}))$$

denotes the corresponding *field of Puiseux series*. Let $\upsilon_F : \mathbf{P}(F) \to \mathbb{Q} \cup \{\,\infty\,\}$ be the valuation such that if

$$0 \neq a = \sum_{k \geq M}^{\infty} a_k t^{k/n} \in \mathbf{P}(F)$$

where $a_k \in F$, $a_M \neq 0$, k, $M \in \mathbb{Z}$ and $n \geq 1$, then $\upsilon_F(a) = M/n$. (As usual, we set $\upsilon_F(0) = \infty$.)

It is well-known that if F is an algebraically closed field of characteristic 0, then $\mathbf{P}(F)$ is algebraically closed. (For example, see Chevalley [4].) In particular, if $\overline{\mathbb{Q}}$ is the field of algebraic numbers, then $\mathbf{P}(\overline{\mathbb{Q}})$ is an algebraically closed field of cardinality $2^{\aleph_0}$. Hence, since $K = \prod_{\mathcal{U}} \mathbb{F}_{p_n}$ is a field of characteristic 0 and cardinality $2^{\aleph_0}$, we can suppose that K is a subfield of $\mathbf{P}(\overline{\mathbb{Q}})$. Furthermore, since K is uncountable and the automorphism group of $\mathbf{P}(\overline{\mathbb{Q}})$ acts transitively on non-algebraic elements, we can suppose that $t \in K$. From now on, we let $v = v_{\overline{\mathbb{Q}}} \restriction K$ denote the corresponding valuation of K and let $R = \{\, a \in K \mid v(a) \geq 0 \,\}$ be the corresponding valuation ring. We will make use of the following result, which was proved in Thomas [29, Section 2].

Theorem 3.2. $[\, SL(d, K) : SL(d, R) \,] = \omega$.

Corollary 3.3. *There exists a non-continuous homomorphism of* $\prod SL(d, p_n)$ *into* $\mathrm{Sym}(\omega)$.

Proof. Let $\pi : \prod SL(d, p_n) \to SL(d, K)$ be the canonical surjective homomorphism and let $H = \pi^{-1}(\, SL(d, R)\,)$. Then $[\prod SL(d, p_n) : H\,] = \omega$ and the action of $\prod SL(d, p_n)$ on the cosets of H induces a homomorphism

$$\varphi : \prod SL(d, p_n) \to \mathrm{Sym}(\omega)$$

such that $\varphi(H)$ is the stabilizer of 0 in $\varphi(\,\prod SL(d, p_n)\,)$. If S is the stabilizer of 0 in $\mathrm{Sym}(\omega)$, then S is an open subgroup of $\mathrm{Sym}(\omega)$ and $\varphi^{-1}(S) = H$. Since H is clearly not an open subgroup of $\prod SL(d, p_n)$, it follows that φ is not continuous. □

Next we will prove that $\prod SL(d, p_n)$ is not Cayley bounded. By the following easy observation, it is enough to show that $SL(d, K)$ is not Cayley bounded.

Lemma 3.4. *Suppose that G is a group and that $N \trianglelefteq G$ is a normal subgroup. If G is Cayley bounded, then $H = G/N$ is also Cayley bounded.*

Proof. Suppose that the symmetric generating set $S \subseteq H$ witnesses that H is not Cayley bounded. Let $\pi : G \to H$ be the canonical surjective homomorphism and let $T = \pi^{-1}(S)$. Then T witnesses that G is not Cayley bounded. □

From now on, in order to simplify notation, we will suppose that $d = 2$. Recall that after identifying K with its image under a suitable embedding into the field

$\mathbf{P}(\overline{\mathbb{Q}})$ of Puiseux series in the indeterminate t, we have that $t \in K$. Also note that $\upsilon(t) = 1$ and that $\upsilon(t^{-1}) = -1$. For each $k \in K^* = K \smallsetminus \{0\}$, let

$$x(k) = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} \qquad y(k) = \begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix} \qquad d(k) = \begin{pmatrix} k & 0 \\ 0 & k^{-1} \end{pmatrix}$$

Then it is well-known that $T = \{\, x(k) \mid k \in K^* \,\} \cup \{\, y(k) \mid k \in K^* \,\}$ generates $SL(2,K)$. (For example, see Lang [18, Lemma XIII.8.1].) Let

$$U = \{\, d(t), d(t^{-1}) \,\} \cup \{\, x(k) \mid 0 \leq \upsilon(k) \leq 2\} \cup \{\, y(k) \mid 0 \leq \upsilon(k) \leq 2\}.$$

Since $\upsilon(-k) = \upsilon(k)$ for all $k \in K$, it follows that U is a symmetric subset of $SL(2,K)$. We claim that U generates $SL(2,K)$. To see this, note that

$$d(t)x(k)d(t)^{-1} = x(t^2k) \qquad d(t)^{-1}x(k)d(t) = x(t^{-2}k)$$

and that

$$\upsilon(t^2k) = \upsilon(t^2) + \upsilon(k) = \upsilon(k) + 2 \qquad \upsilon(t^{-2}k) = \upsilon(t^{-2}) + \upsilon(k) = \upsilon(k) - 2.$$

Hence if $k \in K^*$, then there exists $m \in \mathbb{Z}$ such that $d(t)^m x(k) d(t)^{-m} \in U$; and similarly, there exists $m \in \mathbb{Z}$ such that $d(t)^m y(k) d(t)^{-m} \in U$. It follows that $T \subseteq \langle\, U \,\rangle$ and hence $\langle\, U \,\rangle = SL(2,K)$. Next for each matrix

$$A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \in SL(2,K),$$

we define

$$\tau(A) = \min\{\, \upsilon(a_i) \mid 1 \leq i \leq 4 \,\}.$$

Notice that since

$$\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} = \begin{pmatrix} a_1b_1 + a_2b_3 & a_1b_2 + a_2b_4 \\ a_3b_1 + a_4b_3 & a_3b_2 + a_4b_4 \end{pmatrix}$$

and since, for example,

$$\begin{aligned} \upsilon(a_1b_1 + a_2b_3) &\geq \min\{\, \upsilon(a_1b_1), \upsilon(a_2b_3) \,\} \\ &= \min\{\, \upsilon(a_1) + \upsilon(b_1), \upsilon(a_2) + \upsilon(b_3) \,\}, \end{aligned}$$

it follows that $\tau(AB) \geq \tau(A) + \tau(B)$ for all A, $B \in SL(2,K)$. Finally recall that for each $m \in \mathbb{N}$, we have that $\upsilon(t^{-m}) = -m$ and so $\tau(d(t^m)) = -m$. It now follows easily that for each $n \in \mathbb{N}$, there exists $m \in \mathbb{N}$ such that $d(t^m)$ is not a product of

n elements of $U \cup \{1\}$. Thus $SL(2, K)$ is not Cayley bounded and it follows that $\prod SL(2, p_n)$ is also not Cayley bounded.

The remainder of this section is devoted to the proof of Theorem 2.3. Suppose that $(G_n \mid n \in \omega)$ is a sequence of nontrivial finite groups with $\liminf d(G_n) < \infty$. Then there exists an infinite subset $I \subseteq \omega$ and a fixed $d \geq 1$ such that for each $n \in I$ there exists a nontrivial homomorphism

$$\varphi_n : G_n \to GL(d, F_n)$$

for some field F_n. In order to simplify notation, we will suppose that $I = \omega$. Let $\mathcal{U}$ be a nonprincipal ultrafilter over ω and let

$$\varphi : \prod\nolimits_{\mathcal{U}} G_n \to \prod\nolimits_{\mathcal{U}} GL(d, F_n)$$

be the homomorphism defined by $(g_n)_{\mathcal{U}} \mapsto (\varphi_n(g_n))_{\mathcal{U}}$. Let $F = \prod_{\mathcal{U}} F_n$ and $H = \varphi(\prod_{\mathcal{U}} G_n)$. By Thomas [29, Theorem 2.1], since F is a field of cardinality at most $2^{\aleph_0}$ and

$$1 \neq H \leqslant \prod\nolimits_{\mathcal{U}} GL(d, F_n) \cong GL(d, F),$$

it follows that there exists a proper subgroup $H_0 < H$ such that $1 < [H : H_0] \leq \omega$. (As with our earlier arguments, the proof of Thomas [29, Theorem 2.1] involves defining a suitable valuation on F.) Let $L = \varphi^{-1}(H_0)$. Then L is a proper subgroup of $\prod_{\mathcal{U}} G_n$ of countable (possibly finite) index. Let $\pi : \prod G_n \to \prod_{\mathcal{U}} G_n$ be the canonical surjective homomorphism and let $M = \pi^{-1}(L)$. Then M is a proper subgroup of $\prod_{\mathcal{U}} G_n$ of countable (possibly finite) index. If M has countably infinite index, then arguing as in the proof of Corollary 3.3, it follows that there exists a non-continuous homomorphism of $\prod G_n$ into $\mathrm{Sym}(\omega)$. So suppose that $[\prod G_n : M] = \ell > 1$ is finite. Since $\mathcal{U}$ is nonprincipal, it follows that if P is *any* open subgroup of $\prod G_n$, then $\pi(P) = \prod_{\mathcal{U}} G_n$. In particular, M is not an open subgroup and hence there exists a non-continuous homomorphism from $\prod G_n$ into the finite group $\mathrm{Sym}(\ell)$. This completes the proof of Theorem 2.3.

4. On the existence of nonprincipal ultrafilters

In this section, working with the axiom system $ZF + DC$, we will prove that the failure of the Bergman property for suitably chosen infinite products $\prod H_n$ of finite groups implies the existence of a nonprincipal ultrafilter over ω. It is currently not known whether failures of the Steinhaus property also imply the existence of a nonprincipal ultrafilter over ω. However, we will show that failures of a weak form of the Steinhaus property do indeed imply the existence of such an ultrafilter.

Theorem 4.1 *(ZF + DC).* *Let $(H_n \mid n \in \omega)$ be a sequence of nontrivial finite groups which satisfies the following condition:*

- (†) *There is a* fixed *integer $t \geq 1$ such that for all $n \in \omega$, there is a conjugacy class $C_n \subseteq H_n$ such that $C_n^t = H_n$.*

If $\prod H_n$ does not have the Bergman property, then there exists a nonprincipal ultrafilter over ω.

Proof. Suppose that $G = \prod H_n$ does not have the Bergman property. Then we can express $G = \bigcup_{k\in\omega} U_k$ as the union of a strictly increasing chain of symmetric proper subsets such that $U_k U_k \subseteq U_{k+1}$ for all $k \in \omega$. Consider

$$\mathcal{I} = \{ A \subseteq \omega \mid \prod_{n\in A} H_n \subseteq U_k \text{ for some } k \in \omega \}.$$

Then clearly $\mathcal{I}$ is an ideal which contains all the finite subsets of ω. Hence it is enough to prove that there exists a set $B \notin \mathcal{I}$ such that $\mathcal{I} \cap \mathcal{P}(B)$ is a prime ideal over B.

Suppose that no such set B exists. Then for each $A \notin \mathcal{I}$, there exists $A' \subseteq A$ such that $A' \notin \mathcal{I}$ and $A \smallsetminus A' \notin \mathcal{I}$; and hence we can inductively find pairwise disjoint subsets $\{ A_k \mid k \in \omega \}$ of ω such that $A_k \notin \mathcal{I}$ and $\omega \smallsetminus \bigcup_{\ell\leq k} A_\ell \notin \mathcal{I}$ for all $k \in \omega$.

Claim 4.2. *There exists $k \in \omega$ such that for every $h \in \prod_{n\in A_k} H_n$, there exists $g \in U_k$ such that $g \restriction A_k = h$.*

Proof of Claim 4.2. If not, then there exists $h \in G$ such that for all $k \in \omega$ and $g \in U_k$, we have that $g \restriction A_k \neq h \restriction A_k$. But this means that $h \notin \bigcup_{k\in\omega} U_k$, which is a contradiction. □

Fix some such $k \in \omega$. For each $n \in A_k$, let C_n be the conjugacy class of H_n given by condition (†) and let $h = (h_n) \in \prod_{n\in A_k} H_n$ be such that $h_n \in C_n$ for all $n \in A_k$. Let $h \in U_\ell$ and let $m = \max\{ k, \ell \}$. Then it follows that the conjugacy class C of h in $\prod_{n\in A_k} H_n$ is contained in U_m^3; and hence $\prod_{n\in A_k} H_n$ is contained in

U_m^{3t}. But this means that $\prod_{n \in A_k} H_n \subseteq U_s$ for some $s \geq m$, which contradicts the fact that $A_k \notin \mathcal{I}$. This completes the proof of Theorem 4.1. □

Clearly Theorem 2.9 is an immediate consequence of Theorem 4.1, together with the following result.

Proposition 4.3 (Ellers-Gordeev-Herzog [10]). *Suppose that K is any field such that $|K| > 5$ and that C is any noncentral conjugacy class of $SL(d, K)$.*

(i) *If $d = 2$, then $C^8 = SL(2, K)$.*
(ii) *If $d > 2$, then $C^{2d} = SL(d, K)$.*

In the remainder of this section, we will consider the following weak form of the Steinhaus property.

Definition 4.4. The Polish group G is said to have the *weak Steinhaus property* if for every symmetric countably syndetic subset $W \subseteq G$, there exists an integer $k \geq 1$ such that W^k contains an open neighborhood of the identity element 1_G.

For example, if the Polish group G has a non-open subgroup of countable index, then clearly G does not have the weak Steinhaus property. In particular, if we work with ZFC, then the results of Section 3 show that $\prod SL(d, p_n)$ does not have the weak Steinhaus property. The rest of this section is devoted to the proof of the following result.

Theorem 4.5 ($ZF + DC$). *Suppose that $d \geq 2$ and that $(p_n \mid n \in \omega)$ is an increasing sequence of primes. If $\prod SL(d, p_n)$ does not have the weak Steinhaus property, then there exists a nonprincipal ultrafilter over ω.*

Notice that Theorem 1.5 is an easy consequence of Theorem 4.5. For suppose that $\varphi : \prod SL(d, p_n) \to \mathrm{Sym}(\omega)$ is a non-continuous homomorphism. Then there exists an open subgroup $U \leqslant \mathrm{Sym}(\omega)$ such that $\varphi^{-1}(U)$ is not open in $\prod SL(d, p_n)$. Since U has countable index in $\mathrm{Sym}(\omega)$, it follows that $\varphi^{-1}(U)$ is a non-open subgroup of countable index in $\prod SL(d, p_n)$ and hence $\prod SL(d, p_n)$ does not have the weak Steinhaus property.

Most of our effort will go into proving the following special case of Theorem 4.5.

Theorem 4.6 *(ZF + DC)*. *Suppose that $d \geq 2$ and that $(\, p_n \mid n \in \omega\,)$ is an increasing sequence of primes. If there exists a subgroup $H < \prod SL(d, p_n)$ such that $[\,\prod SL(d, p_n) : H\,] = \omega$, then there exists a nonprincipal ultrafilter over ω.*

The proof of Theorem 4.6 makes use of some of the basic properties of primitive permutation groups. Recall that if Ω is any nonempty set and $G \leqslant \mathrm{Sym}(\Omega)$, then G is said to act *primitively* on Ω if:

(i) G acts transitively on Ω; and
(ii) there does not exist a nontrivial G-invariant equivalence relation on Ω.

It is well-known that if $G \leqslant \mathrm{Sym}(\Omega)$ is a transitive subgroup, then G acts primitively on Ω if and only if the stabilizer $G_\alpha = \{\, g \in G \mid g(\alpha) = \alpha \,\}$ is a maximal subgroup of G for some (equivalently every) $\alpha \in \Omega$. Also if G acts primitively on Ω and $1 \neq N \trianglelefteq G$ is a nontrivial normal subgroup, then it follows that N must act transitively on Ω. (For example, see Cameron [3, Theorem 1.7].)

The proof of Theorem 4.6 also makes use of the following easy consequence of Proposition 4.3.

Lemma 4.7 *(ZF + DC)*. *Suppose that $d \geq 2$ and that $(\, p_n \mid n \in \omega\,)$ is an increasing sequence of primes. Then every normal subgroup N of countable index in $\prod SL(d, p_n)$ is open.*

Proof. Let $G = \prod SL(d, p_n)$ and let $\mathcal{F} = \{\, g_\tau = (\, g_\tau(n)\,) \mid \tau \in 2^{\mathbb{N}} \,\} \subseteq G$ be a family such that for each $\tau \neq \sigma \in 2^{\mathbb{N}}$, there exists an integer $n_{\tau,\sigma} \geq 0$ such that

- $g_\tau(n) = g_\sigma(n)$ for all $n < n_{\tau,\sigma}$; and
- $g_\tau(n)^{-1} g_\sigma(n)$ is a noncentral element of $SL(d, p_n)$ for all $n \geq n_{\tau,\sigma}$.

Since $[\, G : N \,] \leq \omega$, there exist $\tau \neq \sigma \in 2^{\mathbb{N}}$ such that $g_\tau N = g_\sigma N$ and hence $g = g_\tau^{-1} g_\sigma \in N$. Since N is a normal subgroup, the conjugacy class $C = g^G$ is contained in N. Applying Proposition 4.3, it follows easily that N contains the open subgroup $\prod_{n \geq n_{\tau,\sigma}} SL(d, p_n)$ and hence N is open. □

Proof of Theorem 4.6. Let $G = \prod SL(d, p_n)$ and let $\{\, P_j \mid j \in J \,\}$ be the set of open subgroups of G such that $H \leqslant P_j$. Since $H \leqslant \bigcap_{j \in J} P_j$ and the intersection of infinitely many open subgroups of G has index 2^ω, it follows that J is finite. Let

$$G' = \prod_{n \geq n_0} SL(d, p_n) \leqslant \bigcap_{j \in J} P_j.$$

Then after replacing G by G' and H by its projection H' into G' if necessary, we can suppose that H is not contained in any proper open subgroups of G.

Let $G = \bigsqcup_{n\in\omega} g_n H$ be the coset decomposition of H in G. Then we can construct a strictly increasing chain H_n of proper subgroups of G as follows.

- $H_0 = H$.
- Suppose inductively that H_n has been defined and that $H \leqslant H_n < G$. If H_n is a maximal proper subgroup of G, then the construction terminates with H_n. Otherwise, let k_n be the least integer k such that $H_n < \langle H_n, g_k \rangle < G$ and let $H_{n+1} = \langle H_n, g_{k_n} \rangle$.

First suppose that there exists an integer n such that H_n is a maximal proper subgroup of G. Then we claim that $[\, G : H_n \,] = \omega$. Otherwise, $[\, G : H_n \,] < \omega$ and hence $N = \bigcap_{g\in G} gH_n g^{-1}$ is a normal subgroup of G such that $N \leqslant H_n$ and $[\, G : N \,] < \omega$. Applying Lemma 4.7, it follows that N is an open subgroup of G and hence H_n is also an open subgroup of G. But this contradicts the fact that H is not contained in any proper open subgroups of G. Next suppose that the construction does not terminate after finitely many steps and let $H_\omega = \bigcup_{n\in\omega} H_n$. Then either $H_\omega = G$ or else H_ω is a maximal proper subgroup of G. In the former case, G has countable cofinality and hence, by Theorem 2.9, there exists a nonprincipal ultrafilter over ω. Thus we can suppose that H_ω is a maximal proper subgroup of G and our earlier argument shows that $[\, G : H_\omega \,] = \omega$.

In order to simplify notation, we will suppose that H is a maximal subgroup of G. Hence, by considering the left translation action of G on the set $\{\, g_n H \mid n \in \mathbb{N} \,\}$, we obtain a homomorphism

$$\psi : G \to \mathrm{Sym}(\omega)$$

such that $\psi(G)$ acts primitively on ω. It follows that if $N \trianglelefteq G$ is any normal subgroup, then either $\psi(N) = 1$ or else $\psi(N)$ acts transitively on $\mathbb{N}$. Let

$$\mathcal{I} = \{\, A \subseteq \omega \mid \psi(\prod_{n\in A} SL(d, p_n)) = 1 \,\}.$$

Then $\mathcal{I}$ is clearly an ideal on ω. Furthermore, if $F \subseteq \omega$ is a finite subset, then $\psi(\prod_{n\in F} SL(d, p_n))$ cannot act transitively on $\mathbb{N}$ and so $F \in \mathcal{I}$. We will show that $\mathcal{I}$ is a prime ideal.

So suppose that there exists a subset $A \subseteq \omega$ such that both $A \notin \mathcal{I}$ and $\omega \smallsetminus A \notin \mathcal{I}$. Let $P = \prod_{n \in A} SL(d, p_n)$ and let $Q = \prod_{n \in \omega \smallsetminus A} SL(d, p_n)$. Then both $\psi(P)$ and $\psi(Q)$ act transitively on $\mathbb{N}$. Suppose that $g \in P$ is such that $\psi(g)$ fixes some integer $n \in \mathbb{N}$. If $k \in \mathbb{N}$ is arbitrary, then there exists $h \in Q$ such that $\psi(h)(n) = k$; and since g and h commute, it follows that

$$\psi(g)(k) = (\,\psi(g) \circ \psi(h)\,)(n) = (\,\psi(h) \circ \psi(g)\,)(n) = \psi(h)(n) = k.$$

Thus $g \in \ker \psi$. It follows that $N = \ker \psi \cap P$ is a normal subgroup of P such that $[\,P : N\,] = \omega$, which contradicts Lemma 4.7. □

Proof of Theorem 4.5. Let $G = \prod SL(d, p_n)$ and suppose that the symmetric countably syndetic subset $W \subseteq G$ witnesses the failure of the weak Steinhaus property. Let $H = \langle\, W \,\rangle$ be the subgroup generated by W. Then clearly $[\,G : H\,] \leq \omega$. If $[\,G : H\,] = \omega$, then the result follows from Theorem 4.6 and so we can suppose that $[\,G : H\,] < \omega$. Applying Lemma 4.7, it follows easily that H is an open subgroup of G. Let

$$G' = \prod_{n \geq n_0} SL(d, p_n) \leqslant H$$

and let $\pi : G \to G'$ be the canonical projection. Consider the set $W' = \pi(W)$ of generators of G'. If W' witnesses that G' is not Cayley bounded, then the result follows from Theorem 2.9. Hence we can suppose that there exists an integer $k \geq 1$ such that $(W')^k = G'$. Let $g = (g_n) \in G'$ be such that g_n is a noncentral element of $SL(d, p_n)$ for all $n \geq n_0$. Then $g \in W^\ell$ for some $\ell \geq 1$; and Proposition 4.3 implies that

$$G' \subseteq \underbrace{W^k W^\ell W^k \cdots W^k W^\ell W^k}_{m \text{ times}} = W^{2km + \ell m},$$

where $m = 8$ if $d = 2$ and $m = 2d$ if $d > 2$. But this contradicts the assumption that W witnesses the failure of the weak Steinhaus property. □

5. The Bergman and Steinhaus Properties in $L(\mathbb{R})[\mathcal{U}]$

In this section, we will present a partition property PP for products of finite sets with measures; and we will show that $ZF{+}DC{+}PP$ implies that if $(\,H_n \mid n \in \omega\,)$ is a sequence of nontrivial finite groups such that $(\,d_{\mathbb{C}}(H_n) \mid n \in \omega\,)$ grows sufficiently fast, then $\prod H_n$ has both the Bergman property and the Steinhaus property.

The Partition Property (PP)**.** If $(\langle a_n, \mu_n \rangle \mid n \in \omega)$ is a sufficiently fast growing sequence of finite sets a_n with measures μ_n, then for every partition

$$\prod a_n = \bigsqcup_{m \in \omega} X_m,$$

there exists an integer $m \in \omega$ such that $\prod b_n \subseteq X_m$ for some sequence of subsets $b_n \subseteq a_n$ such that $\lim_{n \to \infty} \mu_n(b_n) = \infty$.

Here the words "sufficiently fast growing" should be interpreted in the sense that there is a *fixed* function f that assigns a natural number to every finite sequence of finite sets with measures $(\langle a_m, \mu_m \rangle \mid m < n)$ and that an infinite sequence $(\langle a_n, \mu_n \rangle \mid n \in \omega)$ is sufficiently fast growing if

$$\mu_n(a_n) > f((\langle a_m, \mu_m \rangle \mid m < n))$$

for all $n \in \omega$. The exact formula for the function f is immaterial for the purposes of this paper. We will only mention that it is primitive recursive with a growth rate approximately that of a tower of exponentials of linear height.

The partition property PP fails in ZFC, since the Axiom of Choice can be used to construct highly irregular partitions. However, it does hold in ZFC if we restrict our attention to partitions into Borel sets; and it also holds for arbitrary partitions in many models of set theory in which the Axiom of Choice fails. In particular, in Section 7, we will prove the following result, which extends the work of Di Prisco-Todorcevic [8, Section 7].

Theorem 5.1 (LC)**.** *$L(\mathbb{R})[\mathcal{U}]$ satisfies PP.*

We will also make use of the following recent result of Babai-Nikolov-Pyber [1] in the newly flourishing area of "arithmetic combinatorics". Recall that if H is a nontrivial finite group and K is a field, then $d_K(H)$ denotes the minimal dimension of a nontrivial K-representation of H; i.e. the least d such that there exists a nontrivial homomorphism $\theta : H \to GL(d, K)$.

Theorem 5.2 (Babai-Nikolov-Pyber [1])**.** *Let H be a nontrivial finite group and let k be an integer such that $1 \leq k^3 \leq d_{\mathbb{C}}(H)$. If $A \subseteq H$ is a subset such that $|A| \geq |H|/k$, then $A^3 = H$.*

Proof. By Babai-Nikolov-Pyber [1, Corollary 2.6], if $1 \leq k^3 \leq d_{\mathbb{R}}(H)$ and $A \subseteq H$ with $|A| \geq |H|/k$, then $A^3 = H$. Since $d_{\mathbb{C}}(H) \leq d_{\mathbb{R}}(H)$, the result follows. □

Remark 5.3. If H is a nontrivial finite group, then either $d_{\mathbb{R}}(H) = d_{\mathbb{C}}(H)$ or else $d_{\mathbb{R}}(H) = 2d_{\mathbb{C}}(H)$. For the purposes of this paper, it does not matter whether we work with $d_{\mathbb{C}}(H)$ or $d_{\mathbb{R}}(H)$. Since most of the literature on the representation theory of finite groups deals with complex representations, we have chosen to state our results in terms of $d_{\mathbb{C}}(H)$.

Theorem 5.4 $(ZF + DC + PP)$**.** *If* $\langle H_n \mid n \in \omega \rangle$ *is a sequence of nontrivial finite groups such that* $\langle d_{\mathbb{C}}(H_n) \mid n \in \omega \rangle$ *grows sufficiently fast, then* $\prod H_n$ *has both the Bergman property and the Steinhaus property.*

Proof. For each $n \in \omega$, let $k_n = \lfloor d_{\mathbb{C}}(H_n)^{1/3} \rfloor$ and let μ_n be the measure on H_n defined by $\mu_n(A) = k_n\,(\,|A|/|H_n|\,)$. To see that $G = \prod H_n$ has the Steinhaus property, suppose that $W \subseteq G$ is a symmetric countably syndetic subset and let $G = \bigcup_{m\in\omega} g_m W$. Since $\mu_n(H_n) = k_n$ grows sufficiently fast, PP implies that there exists $m \in \omega$ such that $\prod A_n \subseteq g_m W$ for some sequence of subsets $A_n \subseteq H_n$ such that $\lim_{n\to\infty} \mu_n(A_n) = \infty$; and after replacing $\prod A_n$ by $g_m^{-1} \prod A_n$, we can suppose that $\prod A_n \subseteq W$. Let $n_0 \in \omega$ be such that $\mu_n(A_n) \geq 1$ and hence $|A_n| \geq |H_n|/k_n$ for all $n \geq n_0$. Clearly we can suppose that $A_n = \{\, a_n \,\}$ is a singleton for each $n < n_0$. Applying Theorem 5.2, it follows that $W^3 \supseteq gG'$, where

- $g = (\, a_0^3, \cdots, a_{n_0-1}^3, 1, 1, \cdots)$ and
- G' is the open subgroup $\prod_{n\geq n_0} H_n$.

Since W is symmetric, it follows that $W^6 \supseteq (gG')^{-1} gG' = G'$. This completes the proof that $\prod H_n$ has the Steinhaus property.

To see that $G = \prod H_n$ has the Bergman property, suppose that $G = \bigcup_{m\in\omega} U_m$ is the union of an increasing chain of symmetric subsets such that $U_m U_m \subseteq U_{m+1}$ for all $m \in \omega$. Arguing as above, it follows that there exists $m \in \omega$ such that U_m^6 contains an open subgroup G' and hence $G' \subseteq U_{m+3}$. Since $[\,G : G'\,] < \omega$, this implies that there exists $k \in \omega$ such that $G = U_k$, as required. □

It is perhaps worth pointing out that the proof of following corollary does *not* make use of the classification of the finite simple groups.

Corollary 5.5 (*LC*)**.** *If* $(S_n \mid n \in \omega)$ *is a sufficiently fast growing sequence of nonabelian finite simple groups, then* $\prod S_n$ *has both the Bergman property and the Steinhaus property in* $L(\mathbb{R})[\mathcal{U}]$.

Proof. By Jordan's Theorem, there exists a function $\varphi : \mathbb{N} \to \mathbb{N}$ such that if H is a finite subgroup of $GL(n, \mathbb{C})$, then H contains an abelian normal subgroup N with $[\, H : N \,] \leq \varphi(n)$. (For example, see Curtis-Reiner [5, Theorem 36.13].) Hence if $|S_n|$ grows sufficiently fast, then $d_{\mathbb{C}}(S_n)$ also grows sufficiently fast. □

Corollary 5.6 (*LC*)**.** *If* $d \geq 2$ *and* $(p_n \mid n \in \omega)$ *is a sufficiently fast growing sequence of primes, then* $\prod SL(d, p_n)$ *has both the Bergman property and the Steinhaus property in* $L(\mathbb{R})[\mathcal{U}]$.

Proof. Recall that if $p \geq 5$ is a prime, then $d_{\mathbb{C}}(\, SL(2,p)\,) = (p-1)/2$; and that if $d > 2$, then $d_{\mathbb{C}}(\, SL(d,p)\,) = (p^d - p)/(p-1)$.

□

The next result suggests that the fast growth conditions in the statements of Corollary 5.5 and Corollary 5.6 are almost certainly not necessary.

Theorem 5.7 ($ZF + DC + PP$)**.** *Suppose that* $(H_n \mid n \in \omega)$ *is a sequence of nontrivial finite groups which satisfies the following conditions:*

(i) $\lim_{n\to\infty} d_{\mathbb{C}}(H_n) = \infty$.
(ii) *There is a* fixed *integer* $t \geq 1$ *such that for all* $n \in \omega$*, there exists a conjugacy class* $C_n \subseteq H_n$ *such that* $C_n^t = H_n$.

Then $\prod H_n$ *has both the Bergman property and the weak Steinhaus property.*

The proof of Theorem 5.7 makes use of the following two simple observations.

Lemma 5.8. *If* $H_1, \cdots H_s$ *are nontrivial finite groups and* $H = H_1 \times \cdots \times H_s$*, then* $d_{\mathbb{C}}(H) = \min\{d_{\mathbb{C}}(H_i) \mid 1 \leq i \leq s\}$.

Proof. Let $m = \min\{d_{\mathbb{C}}(H_i) \mid 1 \leq i \leq s\}$. Then it is clear that $d_{\mathbb{C}}(H) \leq m$. So suppose that $d < m$ and that $\theta : H \to GL(d, \mathbb{C})$ is a homomorphism. Then $\theta \restriction H_i$ is the trivial homomorphism for each $1 \leq i \leq s$ and hence θ is trivial. □

Lemma 5.9. *Let* $(H_n \mid n \in \omega)$ *be a sequence of nontrivial finite groups such that* $\lim_{n\to\infty} d_{\mathbb{C}}(H_n) = \infty$. *Then there exists an increasing sequence of integers* $0 = a_0 < a_1 < \cdots < a_n < \cdots$ *such that if*

$$P_n = \prod_{a_{2n} \leq i < a_{2n+1}} H_i \quad \textit{and} \quad Q_n = \prod_{a_{2n+1} \leq i < a_{2n+2}} H_i,$$

then both $(d_{\mathbb{C}}(P_n) \mid n \in \omega)$ *and* $(d_{\mathbb{C}}(Q_n) \mid n \in \omega)$ *grow sufficiently fast.*

Proof. First let $a_1 = 1$. Now suppose that $n \geq 1$ and that a_ℓ has been defined for all $\ell \leq n$. Suppose, for example, that $n = 2m+1$ is odd, so that the groups $P_0, \cdots, P_m$ have already been determined. Then we can choose a_{2m+2} so that $d_{\mathbb{C}}(H_{a_{2m+2}})$ is sufficiently large with respect to $(d_{\mathbb{C}}(P_0), \cdots, d_{\mathbb{C}}(P_m))$ and such that $d_{\mathbb{C}}(H_{a_{2m+2}}) \leq d_{\mathbb{C}}(H_i)$ for all $i \geq a_{2m+2}$. Applying Lemma 5.8, it follows that for any choice of a_{2m+3}, we will have that $d_{\mathbb{C}}(P_{m+1}) = d_{\mathbb{C}}(H_{a_{2m+2}})$ is sufficiently large with respect to $(d_{\mathbb{C}}(P_0), \cdots, d_{\mathbb{C}}(P_m))$. □

Proof of Theorem 5.7 (ZF + DC + PP). Let $(H_n \mid n \in \omega)$ be a sequence of nontrivial finite groups which satisfies conditions (5.7)(i) and (5.7)(ii). First suppose that $\prod H_n$ does not have the Bergman property and express $\prod H_n = \bigcup_{k\in\omega} U_k$ as the union of a strictly increasing chain of symmetric subsets such that $U_k U_k \subseteq U_{k+1}$ for all $k \in \omega$. Then

$$\mathcal{I} = \{ A \subseteq \omega \mid \textstyle\prod_{n\in A} H_n \subseteq U_k \text{ for some } k \in \omega \}.$$

is a proper ideal over ω. Let $(a_n \mid n \in \omega)$ be the increasing sequence of natural numbers given by Lemma 5.9. Then we can suppose that

$$A = \{ i \mid a_{2n} \leq i < a_{2n+1} \text{ for some } n \in \omega \} \notin \mathcal{I}.$$

Since $(d_{\mathbb{C}}(P_n) \mid n \in \omega)$ grows sufficiently fast, it follows that

$$\prod P_n = \textstyle\prod_{n\in A} H_n$$

has the Bergman property. For each $k \in \omega$, let

$$W_k = \{ g \restriction A \mid g \in U_k \} \subseteq \textstyle\prod_{n\in A} H_n.$$

Then there exists $k \in \omega$ such that $W_k = \prod_{n\in A} H_n$. Let $t \geq 1$ be the integer given by condition (5.7)(ii). Then for each $n \in A$, there exists a conjugacy class $C_n \subseteq H_n$

such that $C_n^t = H_n$. Let $h = (h_n) \in \prod_{n \in A} H_n$ be such that $h_n \in C_n$ and let $\ell \geq k$ be such that $h \in U_\ell$. Then clearly

$$\prod_{n \in A} H_n \subseteq \underbrace{U_k U_\ell U_k \cdots U_k U_\ell U_k}_{t \text{ times}} \subseteq U_\ell^{3t};$$

and so $\prod_{n \in A} H_n \subseteq U_s$ for some $s \geq \ell$, which contradicts the fact that $A \notin \mathcal{I}$. Thus $\prod H_n$ has the Bergman property.

To show that H_n has the weak Steinhaus property, we will first prove that $\prod H_n$ has no subgroups H such that $[\prod H_n : H] = \omega$. So suppose that such a subgroup H exists. Then, arguing as in the proof of Theorem 4.6 and using the fact that $\prod H_n$ has the Bergman property, we can suppose that H is a maximal proper subgroup. Hence, by considering the left translation action of $\prod H_n$ on the set of cosets of H in $\prod H_n$, we obtain a homomorphism

$$\psi : \prod H_n \to \mathrm{Sym}(\omega)$$

such that $\psi(\prod H_n)$ acts primitively on ω. In particular, it follows that if $N \trianglelefteq \prod H_n$ is any normal subgroup, then either $\psi(N) = 1$ or else $\psi(N)$ acts transitively on ω. Clearly

$$\mathcal{I} = \{ A \subseteq \omega \mid \psi(\prod_{n \in A} H_n) = 1 \}.$$

is a proper ideal over ω. Arguing as in the previous paragraph, it follows that there exists a subset $A \notin \mathcal{I}$ such that $\prod_{n \in A} H_n$ has the Steinhaus property. But since $\psi(\prod_{n \in A} H_n)$ acts transitively on ω, there exists a subgroup K such that $[\prod_{n \in A} H_n : K] = \omega$, which is a contradiction.

At this point, we know that $\prod H_n$ has the Bergman property and that $\prod H_n$ has no subgroups H with $[\prod H_n : H] = \omega$. Arguing as in the proof of Theorem 4.5, it follows easily that $\prod H_n$ has the weak Steinhaus property. □

Corollary 5.10 (*LC*)**.** *If $d \geq 2$ and $(p_n \mid n \in \omega)$ is an increasing sequence of primes, then $\prod SL(d, p_n)$ has both the Bergman property and the weak Steinhaus property in $L(\mathbb{R})[\mathcal{U}]$.*

Proof. We have already seen that $(SL(d, p_n) \mid n \in \omega)$ satisfies conditions (5.7)(i) and (5.7)(ii). □

In contrast to Corollary 5.5, the proof of the following result does make use of the classification of the finite simple groups. (More precisely, the proof of Shalev [26, Corollary 2.3] makes use of the classification.)

Corollary 5.11 (LC)**.** *If* $(S_n \mid n \in \omega)$ *is a sequence of distinct nonabelian finite simple groups, then* $\prod S_n$ *has both the Bergman property and the weak Steinhaus property in* $L(\mathbb{R})[\mathcal{U}]$.

Proof. Arguing as in the proof of Corollary 5.5, it follows that $(S_n \mid n \in \omega)$ satisfies condition (5.7)(i). By Shalev [26, Corollary 2.3], there exists a constant N such that if S is a nonabelian finite simple group with $|S| \geq N$, then there exists a conjugacy class $C \subseteq S$ such that $C^3 = S$. It follows that $(S_n \mid n \in \omega)$ also satisfies condition (5.7)(ii). □

We will conclude this section with a result which shows that it is necessary to impose some condition on the growth rate of the sequence $(d_{\mathbb{C}}(H_n) \mid n \in \omega)$ if we wish to obtain the conclusion of Theorem 5.4.

Theorem 5.12 ($ZF + DC$)**.** *Suppose that there exists a nonprincipal ultrafilter over* ω*. Then whenever* $(H_n \mid n \in \omega)$ *is a sequence of nontrivial finite groups such that* $\liminf d_{\mathbb{C}}(H_n) < \infty$*, then* $\prod H_n$ *does not have the automatic continuity property and hence does not have the Steinhaus property.*

Proof. Recall that every complex representation of a finite group is similar to a unitary representation. (For example, see Curtis-Reiner [5, Exercise 10.6].) Hence there exists an infinite subset $I \subseteq \omega$ and a fixed integer $d \geq 1$ such that for each $n \in I$, there exists a nontrivial homomorphism $\varphi_n : H_n \to \mathrm{U}(d, \mathbb{C})$, where $\mathrm{U}(d, \mathbb{C})$ denotes the compact group of $d \times d$ unitary matrices. In order to simplify notation, we will suppose that $I = \omega$.

For each $g_n \in H_n$ and $1 \leq i, j \leq d$, let $\varphi_n(g_n)_{ij}$ denote the ij entry of the matrix $\varphi_n(g_n) \in \mathrm{U}(d, \mathbb{C})$. Then if $\mathcal{U}$ is a nonprincipal ultrafilter over ω, we can define a homomorphism

$$\psi : \prod H_n \to \mathrm{U}(d, \mathbb{C})$$
$$(g_n) \mapsto (z_{ij}),$$

where $z_{ij} = \lim_{\mathcal{U}} \varphi_n(g_n)_{ij}$. We claim that ψ is not continuous. To see this, suppose that ψ is continuous and let $W \subseteq \mathrm{U}(d, \mathbb{C})$ be an open neighborhood of the identity element which contains no nontrivial subgroups. (For the existence of such a neighborhood, see Helgason [11, II.B.5].) Then there exists an open subgroup $H \subseteq \prod H_n$ such that $\psi(H) \subseteq W$ and hence $H \leqslant \ker \psi$. In particular, there exists a cofinite subset $A \subseteq \omega$ such that $\prod_{k \in A} H_k \leqslant \ker \psi$. For each $k \in A$, choose $g_k \in H_k$ such that $\varphi_k(g_k) \notin W$. Then, letting $g = (g_k) \in \prod_{k \in A} H_k$, we have that $\psi(g) \notin W$, which is a contradiction. □

Remark 5.13. Recall that de Cornulier [6] has shown that if G is a product of infinitely many copies of a *fixed* finite perfect group H, then G has the Bergman property. Thus the analogue of Theorem 5.12 is false for the Bergman property.

6. The Bergman and Steinhaus Properties in V

Suppose that $(S_n \mid n \in \omega)$ is a sequence of distinct nonabelian finite simple groups. Then, in the previous section, assuming the existence of suitable large cardinals, we proved that $\prod S_n$ has the Bergman property in $L(\mathbb{R})[\mathcal{U}]$; and we proved that if $(S_n \mid n \in \omega)$ is sufficiently fast growing, then $\prod S_n$ also has the Steinhaus property in $L(\mathbb{R})[\mathcal{U}]$. In this section, we will briefly discuss the question of when $\prod S_n$ has either the Bergman property or the Steinhaus property in the actual set-theoretic universe V. In particular, throughout this section, we will work with the usual ZFC axioms of set theory.

Recall that the classification of the finite simple groups says that if S is a nonabelian finite simple group, then one of the following cases must hold.

(i) S is one of the 26 sporadic finite simple groups.
(ii) S is an alternating group $\mathrm{Alt}(n)$ for some $n \geq 5$.
(iii) S is a group $L(q)$ of (possibly twisted) Lie type L over a finite field $\mathbb{F}_q$ for some prime power q.

The following condition is the key to understanding when the product $\prod S_n$ has countable cofinality.

Definition 6.1. A sequence $(S_n \mid n \in \omega)$ of nonabelian finite simple groups satisfies the *Malcev condition* if there exists an infinite subset I of ω such that the following properties hold.

(a) There exists a fixed (possibly twisted) Lie type L such that for all $n \in I$, $S_n = L(q_n)$ for some prime power q_n.
(b) If n, $m \in I$ and $n < m$, then $q_n < q_m$.

Arguing as in the proof of Theorem 2.7(a), it follows easily that if $(S_n \mid n \in \omega)$ satisfies the Malcev condition, then $\prod S_n$ has countable cofinality. Conversely, by Saxl-Shelah-Thomas [25, Theorem 1.9], if $(S_n \mid n \in \omega)$ does not satisfy the Malcev condition, then $\prod S_n$ has uncountable cofinality. Furthermore, as we mentioned earlier, Zalan Gyenis has recently checked that the arguments of Saxl-Shelah-Thomas [25] can be modified to prove that an infinite product $\prod S_n$ of finite simple groups has the Bergman property if and only if $\prod S_n$ has uncountable cofinality. Consequently, we have the following classification of the infinite products $\prod S_n$ satisfying the Bergman property.

Theorem 6.2. *If* $(\, S_n \mid n \in \omega \,)$ *is a sequence of nonabelian finite simple groups, then the following are equivalent:*

(a) $(\, S_n \mid n \in \omega \,)$ *does not satisfy the Malcev condition.*
(b) $\prod S_n$ *has the Bergman property.*

The proof of Theorem 2.3 shows that if $(S_n \mid n \in \omega)$ satisfies the Malcev condition, then there exists a subgroup $H \leqslant \prod S_n$ with $[\,\prod S_n : H\,] = \omega$ and hence $\prod S_n$ does not have the Steinhaus property. Also, it is clear that if $(S_n \mid n \in \omega)$ satisfies the following condition, then $\prod S_n$ has a non-open subgroup of finite index and so once again the Steinhaus property fails. (See Example 1.1.)

Definition 6.3. A sequence $(S_n \mid n \in \omega)$ of nonabelian finite simple groups satisfies the *Saxl-Wilson condition* if there exists an infinite subset I of ω and a fixed group S such that $S_n = S$ for all $n \in I$.

In Thomas [29], it was shown that $\prod S_n$ has a non-open subgroup H such that $[\,\prod S_n : H\,] < 2^{\aleph_0}$ if and only if $(S_n \mid n \in \omega)$ satisfies either the Malcev condition or the Saxl-Wilson condition. Consequently, it seems natural to make the following conjecture.

Conjecture 6.4. If $(\, S_n \mid n \in \omega \,)$ is a sequence of nonabelian finite simple groups, then the following are equivalent:

(a) $(S_n \mid n \in \omega)$ satisfies neither the Malcev condition nor the Saxl-Wilson condition.

(b) $\prod S_n$ has the Steinhaus property.

Remark 6.5. Using the classification of the finite simple groups, it is easily seen that condition (a) is equivalent to:

(a)′ $\liminf d(S_n) = \infty$.

7. The Partition Property (PP)

Suppose that the Ramsey ultrafilter $\mathcal{U}$ is $L(\mathbb{R})$-generic for for the notion of forcing $\mathcal{P}(\omega)/\mathrm{Fin}$. In this section, assuming the existence of suitable large cardinals, we will prove that the Partition Property (PP) holds in $L(\mathbb{R})[\mathcal{U}]$. More specifically, we will make use of the following large cardinal assumption.

(LC) *There exist infinitely many Woodin cardinals below a measurable cardinal.*

If we merely want to prove the consistency of $ZF + DC + PP$, then it is only necessary to assume the existence of an inaccessible cardinal. In more detail, suppose that $\kappa \in V$ is an inaccessible cardinal and that $\mathrm{Coll}(\omega, < \kappa)$ is the usual Lévy collapse. (For the basic properties of the Lévy collapse, see Jech [14, Chapter 26].) Let $G \subseteq \mathrm{Coll}(\omega, < \kappa)$ be a V-generic filter and let $\bar{\mathbb{R}}$ be the set of reals in the generic extension $V[G]$. Then the corresponding *Solovay model* $V(\bar{\mathbb{R}})$ consists of the sets $z \in V[G]$ which are hereditarily definable in $V[G]$ from parameters in $\bar{\mathbb{R}} \cup V$. Let the Ramsey ultrafilter $\bar{\mathcal{U}}$ be $V[G]$-generic for $\mathcal{P}(\omega)/\mathrm{Fin}$. Then clearly $\bar{\mathcal{U}}$ is also $V(\bar{\mathbb{R}})$-generic for $\mathcal{P}(\omega)/\mathrm{Fin}$. Most of our effort in this section will be devoted to proving the following result.

Theorem 7.1. *With the above hypotheses, $V(\bar{\mathbb{R}})[\bar{\mathcal{U}}]$ satisfies PP.*

To transfer this result from $V(\bar{\mathbb{R}})[\bar{\mathcal{U}}]$ to $L(\mathbb{R})[\mathcal{U}]$, we will make use of the fact that, assuming LC, if $\kappa \in V$ is the least inaccessible cardinal, then the theory of $L(\mathbb{R})[\mathcal{U}]$ is not altered by forcing with $\mathrm{Coll}(\omega, < \kappa)$. In more detail, assuming LC, the theory of $L(\mathbb{R})$ is not altered by forcing with $\mathrm{Coll}(\omega, < \kappa)$; i.e. $L(\bar{\mathbb{R}})$ is elementarily equivalent to $L(\mathbb{R})$. (For example, see Larson [19, Corollary 3.1.16].) Since $\mathcal{P}(\omega)/\mathrm{Fin}$ is a homogeneous notion of forcing, it follows that if φ is any sentence in the language of set theory, then

$$\begin{aligned} L(\mathbb{R})[\mathcal{U}] \models \varphi &\iff L(\mathbb{R}) \models \mathcal{P}(\omega)/\mathrm{Fin} \Vdash \varphi \\ &\iff L(\bar{\mathbb{R}}) \models \mathcal{P}(\omega)/\mathrm{Fin} \Vdash \varphi \\ &\iff L(\bar{\mathbb{R}})[\bar{\mathcal{U}}] \models \varphi. \end{aligned}$$

Proof of Theorem 5.1 (LC). Let κ be the least inaccessible cardinal and let $V(\bar{\mathbb{R}})[\bar{\mathcal{U}}]$ be as in Theorem 7.1. Then the Ramsey ultrafilter $\bar{\mathcal{U}}$ is also $L(\bar{\mathbb{R}})$-generic for $\mathcal{P}(\omega)/\mathrm{Fin}$. Working inside $L(\bar{\mathbb{R}})[\bar{\mathcal{U}}]$, suppose that $(\langle a_n, \mu_n \rangle \mid n \in \omega)$ is a sufficiently fast growing sequence of finite sets a_n with measures μ_n and that

$$\prod a_n = \bigsqcup_{m \in \omega} X_m$$

is any partition. Since $V(\bar{\mathbb{R}})[\bar{\mathcal{U}}]$ satisfies PP, there exists an integer $m \in \omega$ and a sequence of subsets $(b_n \subseteq a_n \mid n \in \omega) \in V(\bar{\mathbb{R}})[\bar{\mathcal{U}}]$ such that $\prod b_n \subseteq X_m$ and $\lim_{n\to\infty} \mu_n(b_n) = \infty$. Since $L(\bar{\mathbb{R}})[\bar{\mathcal{U}}]$ and $V(\bar{\mathbb{R}})[\bar{\mathcal{U}}]$ have the same reals, it follows that $(b_n \subseteq a_n \mid n \in \omega) \in L(\bar{\mathbb{R}})[\bar{\mathcal{U}}]$. Thus $L(\bar{\mathbb{R}})[\bar{\mathcal{U}}]$ satisfies PP. Finally since the theory of $L(\mathbb{R})[\mathcal{U}]$ is not altered by forcing with $\mathrm{Coll}(\omega, < \kappa)$, it follows that $L(\mathbb{R})[\mathcal{U}]$ satisfies also PP. □

The remainder of this section will be devoted to the proof of Theorem 7.1. As usual, we will identify the notion of forcing $\mathcal{P}(\omega)/\mathrm{Fin}$ with the quasi-order $([\omega]^\omega, \subseteq^*)$ of infinite subsets of ω, quasi-ordered by $c \subseteq^* d$ if and only if $|c \smallsetminus d| < \omega$. The key element of the proof is the work of Shelah-Zapletal [28] showing that for every sufficiently fast growing sequence $(\langle a_n, \mu_n \rangle \mid n \in \omega)$ of finite sets a_n with measures μ_n, there is a notion of forcing $\mathbb{P}$ with the following properties:

(1) $\mathbb{P}$ adds a new element $\dot{x} \in \prod a_n$.
(2) $\mathbb{P}$ is proper, ${}^\omega\omega$-bounding and adds no independent reals.
(3) $\mathbb{P}$ is defined in a way which depends only on the reals; i.e. if $M \subseteq N$ are transitive models of set theory with the same reals, then $\mathbb{P}^M = \mathbb{P}^N$.
(4) Suppose that M is a transitive model of set theory such that $\mathcal{P}(\mathcal{P}(\mathbb{R}))^M$ is countable. Then for every $p \in \mathbb{P}^M$, there exist a sequence of sets $(b_n \mid n \in \omega)$ with $b_n \subseteq a_n$ and $\mu_n(b_n) \to \infty$ such that the product $\prod b_n$ consists only of M-generic points for the poset $\mathbb{P}^M_p = \{ q \in \mathbb{P}^M \mid q \leq p \}$.

Here an *independent real* is an infinite subset $a \subseteq \omega$ in the generic extension such that neither a nor $\omega \setminus a$ contains an infinite ground model subset.

Let $\kappa \in V$ be an inaccessible cardinal and let $G \subseteq \mathrm{Coll}(\omega, <\kappa)$ be a V-generic filter. Suppose that $V(\bar{\mathbb{R}})$ is the corresponding Solovay model and that the Ramsey ultrafilter $\bar{\mathcal{U}}$ is $V[G]$-generic (and hence also $V(\bar{\mathbb{R}})$-generic) for $\mathcal{P}(\omega)/\mathrm{Fin}$. Let $(\langle a_n, \mu_n \rangle \mid n \in \omega) \in V(\bar{\mathbb{R}})[\bar{\mathcal{U}}]$ be a sufficiently fast growing sequence of finite sets a_n with measures μ_n and let $\prod a_n = \bigsqcup_{m \in \omega} X_m$ be a partition of the product into countably many pieces within the model $V(\bar{\mathbb{R}})[\bar{\mathcal{U}}]$. Working inside $V[G]$, let $c_0 \in [\omega]^\omega$ be any infinite subset of ω. Then it is enough to find a subset $c \in [\omega]^\omega$ with $c \subseteq^* c_0$ and a sequence of subsets $(\, b_n \subseteq a_n \mid n \in \omega\,) \in V[G]$ with $\lim_{n\to\infty} \mu_n(b_n) = \infty$ such that for some $m \in \omega$,

$$c \Vdash \prod b_n \subseteq \dot{X}_m.$$

Let $f : \prod a_n \to \omega$ be the function defined by

$$f(x) = m \iff x \in X_m$$

and let $\dot{f} \in V(\bar{\mathbb{R}})$ be a $\mathcal{P}(\omega)/\mathrm{Fin}$-name for f. By the standard homogeneity arguments with respect to the Lévy collapse $\mathrm{Coll}(\omega, <\kappa)$, we can assume that c_0, $(a_n \mid n \in \omega) \in V$ and that the $\mathcal{P}(\omega)/\mathrm{Fin}$-name $\dot{f}$ is definable from the elements of the ground model V. In particular, it follows that there exists a formula $\varphi(v_0, v_1, v_2)$ with parameters in V such that for every V-generic filter $H \subseteq \mathrm{Coll}(\omega, <\kappa)$ and every $c \in [\omega]^\omega$, $x \in \prod a_n$ and $m \in \omega$,

$$V[H] \vDash\ c \Vdash_{\mathcal{P}(\omega)/\mathrm{Fin}} \check{x} \in \dot{X}_m \iff V[H] \vDash\ \varphi(c, x, m). \tag{7.2}$$

Working inside the ground model V, consider the product of the forcing $\mathbb{P}$ with $\mathbb{Q} = \mathcal{P}(\omega)/\mathrm{Fin}$. Then the poset $\mathbb{Q}$ adds a Ramsey ultrafilter $\mathfrak{u}$ and $\mathbb{P}$ adds a point $x \in \prod a_n$. Since the definition of the forcing $\mathbb{P}$ only depends on the real numbers, it follows that $\mathbb{P}^V = \mathbb{P}^{V[\mathfrak{u}]}$. Hence if $\mathfrak{u}$,x are mutually generic, then x will be $\mathbb{P}^{V[\mathfrak{u}]}$-generic over the model $V[\mathfrak{u}]$.

Lemma 7.3. *In $V[\mathfrak{u}][x]$, $\mathfrak{u}$ still generates a Ramsey ultrafilter.*

Proof. By Shelah [27, VI.5.1], since $\mathbb{P}^V = \mathbb{P}^{V[\mathfrak{u}]}$ is proper and ${}^\omega\omega$-bounding in $V[\mathfrak{u}]$, it is enough to show that $\mathfrak{u}$ still generates an ultrafilter in $V[\mathfrak{u}][x]$. First note that since $\mathbb{Q}$ is σ-closed and $\mathbb{P}$ is proper, it follows that $\mathcal{P}(\omega) \cap V[\mathfrak{u}][x] = \mathcal{P}(\omega) \cap V[x]$. (Since $\mathbb{P}$ is proper, each real $r \in V[\mathfrak{u}][x]$ is obtained from a countable collection $\mathcal{C} = \{ C_n \mid n \in \omega \} \in V[\mathfrak{u}]$ of countable subsets $C_n \subseteq \mathbb{P}$ such that each C_n is predense below some condition $p \in \mathbb{P}$; and since $\mathbb{Q}$ is σ-closed, it follows that $\mathcal{C} \in V$ and hence $r \in V[x]$.) Now suppose that $p \in \mathbb{P}, q \in \mathbb{Q}$ are conditions and that $p \Vdash \tau \subseteq \omega$. Since $\mathbb{P}$ does not add any independent reals, there exists a condition $p' \leq p$ and an infinite subset $q' \subseteq q$ such that either $p' \Vdash q' \subseteq \tau$ or $p' \Vdash \tau \cap q' = \emptyset$. Hence either $\langle q', p' \rangle \Vdash \tau \in \dot{\mathfrak{u}}$ or $\langle q', p' \rangle \Vdash \omega \smallsetminus \tau \in \dot{\mathfrak{u}}$. It follows that $\mathfrak{u}$ still generates an ultrafilter in $V[\mathfrak{u}][x]$. □

From now on, fix some $\mathfrak{u} \in V[G]$ such that $\mathfrak{u}$ is V-generic for $\mathbb{Q}$ and $c_0 \in \mathfrak{u}$. Let $\mathbb{D} \in V[\mathfrak{u}]$ be the poset consisting of the conditions (s, S), where $s \in [\omega]^{<\omega}$ and $S \in \mathfrak{u}$, partially ordered by

$$(s, S) \leq (t, T) \iff s \supseteq t \text{ and } s \smallsetminus t \subseteq T.$$

Then $\mathbb{D}$ adjoins an infinite subset $\dot{c} \subseteq \omega$ which diagonalizes the Ramsey ultrafilter $\mathfrak{u}$; i.e. a subset $\dot{c}$ such that $|\dot{c} \smallsetminus S| < \omega$ for all $S \in \mathfrak{u}$. In fact, by Mathias [21], every set diagonalizing $\mathfrak{u}$ is $V[\mathfrak{u}]$-generic for the poset $\mathbb{D}$. By Lemma 7.3, if $\bar{\mathfrak{u}}$ is the upwards closure of $\mathfrak{u}$ in the model $V[\mathfrak{u}][x]$, then $\bar{\mathfrak{u}}$ is a Ramsey ultrafilter in $V[\mathfrak{u}][x]$. Hence if $\bar{\mathbb{D}} \in V[x, \mathfrak{u}]$ is the corresponding poset diagonalizing $\bar{\mathfrak{u}}$, then $\mathbb{D}$ is dense in $\bar{\mathbb{D}}$ and every set diagonalizing $\mathfrak{u}$ is $V[\mathfrak{u}][x]$-generic for both $\bar{\mathbb{D}}$ and $\mathbb{D}$. Let $\varphi(v_0, v_1, v_2)$ be the formula with parameters in V given by (7.2).

Lemma 7.4. *In $V[\mathfrak{u}][x]$, there exists a natural number $m \in \omega$ such that*

$$\langle 1, 1 \rangle \Vdash_{\mathbb{D} \times \mathrm{Coll}(\omega, <\kappa)} \varphi(\dot{c}, \check{x}, \check{m});$$

in other words, in $V[\mathfrak{u}][x]$, we have that

$$\langle 1, 1 \rangle \Vdash_{\mathbb{D} \times \mathrm{Coll}(\omega, <\kappa)} \dot{c} \Vdash_{\mathcal{P}(\omega)/\mathrm{Fin}} \check{x} \in \dot{X}_m.$$

Proof. Suppose not. Then there exist distinct numbers $m_0, m_1 \in \omega$ and conditions for the associated 3-step iteration

$$\langle r_0, s_0, \dot{d}_0 \rangle, \langle r_1, s_1, \dot{d}_1 \rangle \leq \langle 1, 1, \dot{c} \rangle$$

such that $\langle r_0, s_0, \dot{d}_0 \rangle \Vdash \check{x} \in \dot{X}_{m_0}$ and $\langle r_1, s_1, \dot{d}_1 \rangle \Vdash \check{x} \in \dot{X}_{m_1}$. Choose mutually $V[\mathfrak{u}][x]$-generic filters $H_0 \subseteq \mathbb{D}$, $K_0 \subseteq \mathrm{Coll}(\omega, < \kappa)$ such that $\langle r_0, s_0 \rangle \in H_0 \times K_0$; and note that, since $d = \dot{d}_0/(H_0 \times K_0) \subseteq^* \dot{c}/H_0$, it follows that d is $V[\mathfrak{u}][x]$-generic for the poset $\mathbb{D}$. Hence, after making a finite adjustment to the set d if necessary, we can find a $V[\mathfrak{u}][x]$-generic filter $H_1 \subseteq \mathbb{D}$ such that $r_1 \in H_1$ and $d = \dot{c}/H_1$. A standard homogeneity argument with respect to the Lévy collapse $\mathrm{Coll}(\omega, < \kappa)$ now shows that there exists a $V[\mathfrak{u}][x][H_1]$-generic filter $K_1 \subseteq \mathrm{Coll}(\omega, < \kappa)$ such that $s_1 \in K_1$ and $V[\mathfrak{u}][x][H_0 \times K_0] = V[\mathfrak{u}][x][H_1 \times K_1]$. Let $d' = \dot{d}_1/(H_1 \times K_1)$. Consider the notion of forcing $\mathcal{P}(\omega)/\mathrm{Fin}$ inside the model $V[\mathfrak{u}][x][H_0 \times K_0] = V[\mathfrak{u}][x][H_1 \times K_1]$. Working with the $V[\mathfrak{u}][x]$-generic filter $H_0 \times K_0$, it follows that $d \Vdash \check{x} \in \dot{X}_{m_0}$ and so $V[\mathfrak{u}][x][H_0 \times K_0] \vDash \varphi(d, x, m_0)$. Furthermore, since

$$d' = \dot{d}_1/(H_1 \times K_1) \subseteq^* \dot{c}/H_1 = d,$$

it follows that $V[\mathfrak{u}][x][H_0 \times K_0] \vDash \varphi(d', x, m_0)$ and so $V[\mathfrak{u}][x][H_0 \times K_0] \nvDash \varphi(d', x, m_1)$. On the other hand, working with the $V[\mathfrak{u}][x]$-generic filter $H_1 \times K_1$, it follows that $d' \Vdash \check{x} \in \dot{X}_{m_1}$ and so $V[\mathfrak{u}][x][H_1 \times K_1] \vDash \varphi(d', x, m_1)$, which is a contradiction. □

Next, working in the model $M = V[\mathfrak{u}]$, let $p \in \mathbb{P}^M$ be a condition that identifies the natural number m in the statement of Lemma 7.4. Then, since $\mathcal{P}(\mathcal{P}(\mathbb{R}))^M$ is countable in $V[G]$, there exists a sequence of sets $(b_n \mid n \in \omega) \in V[G]$ with $b_n \subseteq a_n$ and $\mu_n(b_n) \to \infty$ such that the product $\prod b_n$ consists only of M-generic points for the poset $\mathbb{P}^M_p = \{ q \in \mathbb{P}^M \mid q \leq p \}$. Let $c \in V[G]$ be an infinite subset of ω which diagonalizes the ultrafilter $\mathfrak{u} \in M$. Then clearly $c \subseteq^* c_0$ and we claim that $c \Vdash \prod b_n \subseteq \dot{X}_m$. To see this, suppose that $x \in \prod b_n$. Then $\mathfrak{u}$, x are mutually V-generic and c is $V[\mathfrak{u}][x]$-generic for $\mathbb{D}$. Hence, working in $V[G]$, Lemma 7.4 implies that the condition $c \in \mathcal{P}(\omega)/\mathrm{Fin}$ forces $\check{x} \in \dot{X}_m$. This completes the proof of Theorem 7.1.

References

[1] L. Babai, N. Nikolov and L. Pyber, *Product Growth and Mixing in Finite Groups* in: *Proc. 19th Annual ACM-SIAM Symposium on Discrete Algorithms*, ACM–SIAM 2008, pp. 248–257.

[2] G. M. Bergman, *Generating infinite symmetric groups*, Bull. London Math. Soc. **38** (2006), 429–440.

[3] P. J. Cameron, *Permutation groups*, Cambridge University Press, Cambridge, 1999.

[4] C. Chevalley, *Introduction to the theory of algebraic functions of one variable*, Amer. Math. Soc., New York, 1951.

[5] C. W. Curtis and I. Reiner, *Representation theory of finite groups and associative algebras*, Wiley, New York, 1962.

[6] Y. de Cornulier, *Strongly bounded groups and infinite powers of finite groups*, Comm. Algebra **34** (2006), 2337–2345.

[7] C. A. Di Prisco and S. Todorcevic, *Perfect set properties in* $L(\mathbb{R})[U]$, Adv. Math. **139** (1998), 240–259.

[8] C. A. Di Prisco and S. Todorcevic, *Souslin partitions of products of finite sets* Adv. Math. **176** (2003), 145–173.

[9] M. Droste and W. C. Holland, *Generating automorphism groups of chains*, Forum Math. **17** (2005), 699–710.

[10] E. W. Ellers, N. Gordeev and M. Herzog, *Covering numbers for Chevalley groups*, Israel J. Math. **111** (1999), 339–372.

[11] S. Helgason, *Differential Geometry, Lie Groups and Symmetric Spaces*, Graduate Studies in Mathematics vol. 34, American Mathematical Society, Providence, 2001.

[12] J. E. Humphreys, *Representations of* $\mathrm{SL}(2,p)$, Amer. Math. Monthly **82** (1975), 21–39.

[13] A. Ivanov, *Strongly bounded automorphism groups*, Colloq. Math. **105** (2006), 57–67.

[14] T. Jech, *Set Theory. The Third Millennium Edition, Revised and Expanded*, Springer-Verlag, Berlin, 2003.

[15] A. S. Kechris and C. Rosendal, *Turbulence, amalgamation, and generic automorphisms of homogeneous structures*, Proc. Lond. Math. Soc. **94** (2007), 302–350.

[16] J. Kittrell and T. Tsankov, *Topological properties of full groups*, Ergodic Theory Dynam. Systems **30** (2010), 525–545.

[17] S. Koppelberg and J. Tits, *Une propriété des produits directs infinis de groupes finis isomorphes*, C. R. Math. Acad. Sci. Paris, Sér. A **279** (1974), 583–585.

[18] S. Lang, *Algebra*, 2nd ed., Addison-Wesley, Reading, MA, 1984.

[19] P. B. Larson, *The Stationary Tower Forcing*, Univ. Lecture Ser. **32**, Amer. Math. Soc., Providence, RI, 2004.

[20] H. D. Macpherson and P. M. Neumann, *Subgroups of infinite symmetric groups*, J. London Math. Soc. **42** (1990), 64–84.

[21] A. R. D. Mathias, *Happy families*, Ann. Math. Logic **12** (1977), 59–111.

[22] Y. N. Moschovakis, *Descriptive Set Theory*, 2nd ed., Mathematical Surveys and Monographs **155**, Amer. Math. Soc., Providence, RI, 2009.

[23] C. Rosendal and S. Solecki, *Automatic continuity of homomorphisms and fixed points on metric compacta*, Israel J. Math. **162** (2007), 349–372.

[24] C. Rosendal, *Automatic continuity of group homomorphisms*, Bull. Symbolic Logic **15** (2009), 184–214.

[25] J. Saxl, S. Shelah and S.Thomas, *Infinite products of finite simple groups*, Trans. Amer. Math. Soc. **348** (1996), 4611–4641.

[26] A. Shalev, *Word maps, conjugacy classes, and a noncommutative Waring-type theorem*, Ann. of Math. **170** (2009), 1383–1416.

[27] S. Shelah, *Proper and improper forcing*, second ed., Perspectives in Mathematical Logic, Springer-Verlag, Berlin, 1998.

[28] S. Shelah and J. Zapletal, *Ramsey theorems for products of finite sets with submeasures*, Combinatorica **31** (2011), 225–244.

[29] S. Thomas, *Infinite products of finite simple groups II*, J. Group Theory **2** (1999), 401-434.

[30] P. H. Tiep and A. E. Zalesskii, *Minimal characters of the finite classical groups*, Comm. Algebra **24** (1996), 2093–2167.

[31] J. Zapletal, *Forcing Idealized*, Cambridge Tracts in Mathematics **174**, Cambridge University Press, Cambridge, 2008.

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