THE BI-EMBEDDABILITY RELATION FOR FINITELY GENERATED GROUPS

SIMON THOMAS AND JAY WILLIAMS

Abstract. There does not exist a Borel selection of an isomorphism class within each bi-embeddability class of finitely generated groups.

1. Introduction

Two groups G, H are said to be *bi-embeddable*, written $G \approx_{em} H$, if there exist embeddings $G \hookrightarrow H$ and $H \hookrightarrow G$. In this paper, we will consider the bi-embeddability relation on the space $\mathcal{G}_{fg}$ of finitely generated groups. Here $\mathcal{G}_{fg}$ denotes the Polish space of finitely generated groups introduced by Grigorchuk [3]; i.e., the elements of $\mathcal{G}_{fg}$ are the isomorphism types of *marked groups* $\langle G, \bar{c} \rangle$, where G is a finitely generated group and $\bar{c}$ is a finite sequence of generators. (For a clear account of the basic properties of the space $\mathcal{G}_{fg}$, see either Champetier [1] or Grigorchuk [4].) Since each finitely generated group G has only countably many finitely generated subgroups, it follows that $\approx_{em}$ is a countable Borel equivalence relation; i.e. that every $\approx_{em}$-class is countable. Consequently, since the isomorphism relation $\cong$ on $\mathcal{G}_{fg}$ is a universal countable Borel equivalence relation [15], there exists a Borel reduction from $\approx_{em}$ to $\cong$; i.e. a Borel map $f : \mathcal{G}_{fg} \to \mathcal{G}_{fg}$ such that if G, $H \in \mathcal{G}_{fg}$, then

$$G \approx_{em} H \iff f(G) \cong f(H).$$

However, the only known proof of the existence of such a Borel reduction relies ultimately upon the Feldman-Moore Theorem [2] and this proof does not produce an explict example of such a reduction.

Open Problem 1.1. Find an explicit (preferably "group-theoretic") example of a Borel reduction from $\approx_{em}$ to $\cong$.

The research of the first author was partially supported by NSF Grants DMS 1101597.

Of course, one approach to this problem would be to seek a Borel map which selects an isomorphism class within each bi-embeddability class of finitely generated groups. However, the main result of this paper shows that no such map exists.

Theorem 1.2. *There does* not *exist a Borel reduction* $f : \mathcal{G}_{fg} \to \mathcal{G}_{fg}$ *from* $\approx_{em}$ *to* $\cong$ *such that* $f(G) \approx_{em} G$ *for all* $G \in \mathcal{G}_{fg}$.

This paper is organized as follows. In Section 2, we will recall some basic notions and results concerning Borel equivalence relations and group theory. In Sections 3 and 4, we will discuss the structure of an uncountable family of finitely generated groups which was first introduced by B.H. Neumann [9] and we will make an easy observation concerning embeddings between products of suitably chosen pairs of these groups. In Section 5, we will discuss a smooth group-theoretic invariant that will play a key role in the proof of Theorem 1.2. In Section 6, we will use the Neumann groups to give a simple proof that there does not exist a Borel selection of an isomorphism class within each commensurability class of finitely generated groups. (This result was first proved in Thomas [14] via a significantly more complicated argument.) In Section 7, we will use products of suitably chosen pairs of Neumann groups to prove Theorem 1.2, modulo two technical results which will be proved in Sections 8 and 9. (The argument in Section 8 makes essential use of the work of P.M. Neumann [10] on the structure of finitary permutation groups.) Finally, in Section 10, we will briefly discuss a few of the many open problems suggested by the material in this paper.

2. Preliminaries

In this section, we will recall some basic notions and results concerning Borel equivalence relations and group theory.

2.1. **Borel equivalence relations.** Suppose that $(X, \mathcal{B})$ is a measurable space; i.e. that $\mathcal{B}$ is a σ-algebra of subsets of the set X. Then $(X, \mathcal{B})$ is said to be a *standard Borel space* if there exists a Polish topology $\mathcal{T}$ on X such that $\mathcal{B}$ is the σ-algebra of Borel subsets of $(X, \mathcal{T})$. If X, Y are standard Borel spaces, then a map $f : X \to Y$ is *Borel* if $f^{-1}(Z)$ is a Borel subset of X for each Borel subset $Z \subseteq Y$. Equivalently, $f : X \to Y$ is Borel if graph(f) is a Borel subset of $X \times Y$.

If X is a standard Borel space, then a *Borel equivalence relation* on X is an equivalence relation $E \subseteq X^2$ which is a Borel subset of X^2. If E, F are Borel equivalence relations on the standard Borel spaces X, Y respectively, then a Borel map $f : X \to Y$ is said to be a *homomorphism* from E to F if for all $x, y \in X$,

$$x \, E \, y \quad \Longrightarrow \quad f(x) \, F \, f(y).$$

If f satisfies the stronger property that for all $x, y \in X$,

$$x \, E \, y \quad \Longleftrightarrow \quad f(x) \, F \, f(y),$$

then f is said to be a *Borel reduction* and we write $E \leq_B F$. If both $E \leq_B F$ and $F \leq_B E$, then we say that E and F are *Borel bireducible* and write $E \sim_B F$. Finally we write $E <_B F$ if both $E \leq_B F$ and $F \nleq_B E$.

The Borel equivalence relation E on the standard Borel space X is said to be *smooth* if E is Borel reducible to the identity relation Id_Z on some (equivalently every) uncountable standard Borel space Z. Let E_0 be the Borel equivalence relation on the Cantor space $2^{\mathbb{N}}$, which is defined by

$$x \, E_0 \, y \quad \Longleftrightarrow \quad x(n) = y(n) \text{ for all but finitely many } n.$$

Then, by Harrington-Kechris-Louveau [6], if E is any Borel equivalence relation, then E is nonsmooth if and only if $E_0 \leq_B E$. The following "generic ergodicity" result will play a key role in the proof of Theorem 1.2. (For example, see Hjorth [7, Theorem 3.2].)

Proposition 2.1. *Suppose that E is a smooth Borel equivalence relation on the standard Borel space X. If $f : 2^{\mathbb{N}} \to X$ is a Borel homomorphism from E_0 to E, then there exists a comeager subset $C \subseteq 2^{\mathbb{N}}$ such that f maps C to a single E-class.*

In this paper, we will mainly work with the following slight variant of E_0. Suppose that $\mathbb{A}$ is an infinite subset of $\mathbb{N}$. Then we can clearly identify E_0 with the corresponding equivalence relation on $2^{\mathbb{A}}$. Let $\mathrm{Inc}(\mathbb{A})$ be the set of strictly increasing sequences $\mathbf{a} = \langle a_0, a_1, \cdots, a_n, \cdots \rangle$ of elements of $\mathbb{A}$. Then we can identify each $\mathbf{a} \in \mathrm{Inc}(\mathbb{A})$ with the corresponding infinite subset of A,

$$S_{\mathbf{a}} = \{ a_0, a_1, \cdots, a_n, \cdots \} \in [\mathbb{A}]^{\mathbb{N}}.$$

Finally if we identify each $S \in [\mathbb{A}]^{\mathbb{N}}$ with its characteristic function $\chi_S \in 2^{\mathbb{A}}$, then $\operatorname{Inc}(\mathbb{A})$ becomes identified with the dense G_δ-subset of $2^{\mathbb{A}}$ consisting of the functions x such that $x(a) = 1$ for infinitely many $a \in \mathbb{A}$. Notice that $E_0 \restriction \operatorname{Inc}(\mathbb{A})$ corresponds to the Borel equivalence relation E_t on the original space $\operatorname{Inc}(\mathbb{A})$ of strictly increasing sequences of elements of $\mathbb{A}$ defined by

$$\langle\, a_n | n \in \mathbb{N} \,\rangle \; E_t \; \langle\, b_n | n \in \mathbb{N} \,\rangle \quad \Longleftrightarrow \quad (\, \exists k \,)\, (\, \exists \ell \,)\, (\, \forall\, m \,)\; a_{k+m} = b_{\ell+m}.$$

Of course, it is clear that E_t is Borel bireducible with E_0.

2.2. **Some basic group theory.** Throughout this paper, permutation groups will always act on the left. Thus, for example, we will have that

$$(\,1 \;\; 2 \;\; 3\,)\,(\,1 \;\; 3 \;\; 5 \;\; 7\,)\,(\,1 \;\; 2 \;\; 3\,)^{-1} = (\,2 \;\; 1 \;\; 5 \;\; 7\,)$$

If Ω is a nonempty set, then $\operatorname{Sym}(\Omega)$ denotes the group of all permutations of Ω. A permutation $\pi \in \operatorname{Sym}(\Omega)$ is said to be *finitary* if $\operatorname{supp}(\pi)$ is finite, where $\operatorname{supp}(\pi) = \{\, \alpha \in \Omega \mid \pi(\alpha) \neq \alpha \,\}$. The group of even finitary permutations of Ω is denoted by $\operatorname{Alt}(\Omega)$.

If G is a group, then the *FC-radical* is the characteristic subgroup

$$\Delta(G) = \{\, a \in G \mid a^G \text{ is finite } \}.$$

Here a^G denotes the conjugacy class $\{\, g\, a\, g^{-1} \mid g \in G \,\}$. The group G is said to be an *FC-group* if $\Delta(G) = G$ and is said to be *FC-by-finite* if $[\, G : \Delta(G) \,] < \infty$.

By Dicman's Lemma [16], if $g_1, \cdots, g_t$ are elements of G, each having finite order and each having only finitely many conjugates in G, then there exists a finite normal subgroup $N \trianglelefteq G$ such that $g_1, \cdots g_t \in N$. Hence we can define a characteristic subgroup $\Delta^+(G)$ of G by

$$\Delta^+(G) = \{ g \in G \mid g \text{ is contained in a finite normal subgroup of } G \,\}.$$

It is clear that $\Delta^+(G)$ is a (possibly trivial) periodic FC-subgroup of G and that $\Delta^+(G) \leqslant \Delta(G)$. In fact, $\Delta^+(G) = \{\, g \in \Delta(G) \mid g \text{ has finite order } \}$.

3. A construction of B.H. Neumann

In this section, we will begin our discussion of a family of finitely generated groups which was originally introduced by B.H. Neumann [9] in order to prove the existence of uncountably many finitely generated groups. Throughout this paper, $\mathrm{Alt}(d)$ will denote the alternating group on the set $\{1, 2, \cdots, d\}$. It is well-known that if $d \geq 5$ is odd, then $\mathrm{Alt}(d)$ is simple and is generated by the permutations $a_d = (1\ 2\ 3\ \cdots\ d)$ and $b_d = (1\ 2\ 3)$. Let $\mathbb{O} = \{ d \in \mathbb{N} \mid d \geq 5 \text{ is an odd integer } \}$ and let $\mathrm{Inc}(\mathbb{O})$ be the set of strictly increasing sequences $\mathbf{d} = \langle d_0, d_1, \cdots, d_n, \cdots \rangle$ of elements of $\mathbb{O}$.

Definition 3.1 (B.H. Neumann [9])**.** For each $\mathbf{d} \in \mathrm{Inc}(\mathbb{O})$, let $G_{\mathbf{d}}$ be the subgroup of $\prod_{n\in\mathbb{N}} \mathrm{Alt}(d_n)$ generated by the two elements $a_{\mathbf{d}} = \prod_{n\in\mathbb{N}} a_{d_n}$ and $b_{\mathbf{d}} = \prod_{n\in\mathbb{N}} b_{d_n}$.

We will present a brief but reasonably complete account of the structure of the groups $G_{\mathbf{d}}$ in the next section. In this section, we will prove a bi-embeddability result for suitably chosen products of these groups, which will play a key role in the proof of Theorem 1.2. For the remainder of this paper, let $\mathbf{e} = \langle 5, 7, \cdots \rangle$ be the increasing enumeration of the set of *all* odd integers $d \geq 5$.

Definition 3.2. For each $\mathbf{d} \in \mathrm{Inc}(\mathbb{O})$, let $H_{\mathbf{d}} = G_{\mathbf{d}} \times G_{\mathbf{e}}$.

Proposition 3.3. *If* $\mathbf{c}$, $\mathbf{d} \in \mathrm{Inc}(\mathbb{O})$ *and* $\mathbf{c}\, E_t\, \mathbf{d}$, *then* $H_{\mathbf{c}}$ *and* $H_{\mathbf{d}}$ *are bi-embeddable.*

Proposition 3.3 is an easy consequence of the following two lemmas.

Lemma 3.4. *For each* $j \geq 1$, *there exists a* fixed *word* $w_j(x, y)$ *such that for each* $n \geq 2j + 5$,

$$w_j(a_n, b_n) = (1\ 2\ 3\ \cdots\ n - 2j).$$

Proof. It is enough to prove the result when $j = 1$, for then an easy induction yields the general result. Since

$$(1\ n\ n-1)(1\ 2\ 3\ \cdots\ n) = (1\ 2\ 3\ \cdots\ n-2).$$

and $a_n^{-2}\, b_n^{-1}\, a_n^2 = (1\ n\ n-1)$, it follows that we can take $w_1(x, y) = x^{-2}\, y^{-1}\, x^3$. □

Lemma 3.5. *For each* $2j + 1 \geq 5$,

$$\mathrm{Alt}(5) \times \mathrm{Alt}(7) \times \cdots \times \mathrm{Alt}(2j + 1) \times G_{\mathbf{e}} \hookrightarrow G_{\mathbf{e}}.$$

Proof. For each odd integer $\ell \geq 5$, let $\mathrm{Alt}(\ell)$ be the subgroup of $G_{\mathbf{e}} = \prod_{n \in \mathbb{N}} \mathrm{Alt}(e_n)$ consisting of the elements $\theta = \prod_{n \in \mathbb{N}} \theta_n$ such that $\theta_n = 1$ for all $n \neq \ell$. Then, by Neumann [9], we have that each $\mathrm{Alt}(\ell)$ is a subgroup of $G_{\mathbf{e}}$. Hence, letting $\mathbf{c} = \langle\, 2j+3, 2j+5, \cdots \rangle$ enumerate the odd integers $\ell \geq 2j+3$, it follows that

$$G_{\mathbf{e}} = \mathrm{Alt}(5) \times \mathrm{Alt}(7) \times \cdots \times \mathrm{Alt}(2j+1) \times G_{\mathbf{c}};$$

and so it is enough to show that $G_{\mathbf{e}} \hookrightarrow G_{\mathbf{c}}$. To see this, note that Lemma 3.4 implies that the map $\varphi : \{\, a_{\mathbf{e}}, b_{\mathbf{e}} \,\} \to G_{\mathbf{c}}$, defined by $\varphi(a_{\mathbf{e}}) = w_{j-1}(a_{\mathbf{c}}, b_{\mathbf{c}})$ and $\varphi(b_{\mathbf{e}}) = b_{\mathbf{c}}$, extends to an embedding of $G_{\mathbf{e}}$ into $G_{\mathbf{c}}$. □

Proof of Proposition 3.3. Suppose that k, ℓ are such that $c_{k+m} = d_{\ell+m}$ for all m and let

$$\mathbf{a} = \langle\, c_k, c_{k+1} \cdots \rangle = \langle\, d_\ell, d_{\ell+1} \cdots \rangle.$$

Then clearly

$$H_{\mathbf{c}} = \mathrm{Alt}(c_0) \times \mathrm{Alt}(c_1) \times \cdots \times \mathrm{Alt}(c_{k-1}) \times G_{\mathbf{a}} \times G_{\mathbf{e}}$$

embeds into

$$G_{\mathbf{a}} \times \mathrm{Alt}(5) \times \mathrm{Alt}(7) \times \cdots \times \mathrm{Alt}(c_{k-1}) \times G_{\mathbf{e}},$$

which by Lemma 3.5 embeds into $G_{\mathbf{a}} \times G_{\mathbf{e}}$. Of course, it is clear that $G_{\mathbf{a}} \times G_{\mathbf{e}}$ embeds into

$$H_{\mathbf{d}} = \mathrm{Alt}(d_0) \times \mathrm{Alt}(d_1) \times \cdots \times \mathrm{Alt}(d_{\ell-1}) \times G_{\mathbf{a}} \times G_{\mathbf{e}}.$$

Similarly, $H_{\mathbf{d}}$ embeds into $H_{\mathbf{c}}$. □

As we will explain in the next section, Proposition 3.3 easily implies that the bi-embeddability relation is nonsmooth on the space $\mathcal{G}_{am}$ of finitely generated amenable groups. In [17], Williams proved that the bi-embeddability relation on the space $\mathcal{G}_{fg}$ of all finitely generated groups is countable universal.

Question 3.6. Is the bi-embeddability relation on $\mathcal{G}_{am}$ countable universal?

4. The structure of the Neumann groups

In the remaining sections of this paper, it will be convenient to work with the following "more symmetric" realization of the Neumann groups. For each odd integer $d = 2\ell + 1 \geq 5$, let

$$\Omega_d = \{ -\ell,\ -(\ell-1),\ \cdots,\ -1,\ 0,\ 1, \cdots,\ \ell-1,\ \ell \}$$

and let α_d, $\beta_d \in \mathrm{Alt}(\Omega_d)$ be the permutations defined by:

- $\alpha_d = (\, -\ell \;\; -(\ell-1) \;\; \cdots \;\; -1 \;\; 0 \;\; 1 \;\; \cdots \;\; \ell-1 \;\; \ell \,)$
- $\beta_d = (\, -1 \;\; 0 \;\; 1 \,)$

If $\mathbf{d} \in \mathrm{Inc}(\mathbb{O})$, then $G_{\mathbf{d}}$ is clearly isomorphic to the subgroup $\tilde{G}_{\mathbf{d}}$ of $\prod_{n\in\mathbb{N}} \mathrm{Alt}(\Omega_{d_n})$ generated by the two elements $\alpha_{\mathbf{d}} = \prod_{n\in\mathbb{N}} \alpha_{d_n}$ and $\beta_{\mathbf{d}} = \prod_{n\in\mathbb{N}} \beta_{d_n}$. In the remainder of this paper, we will identify $G_{\mathbf{d}}$ with $\tilde{G}_{\mathbf{d}}$. Also, in order to simplify notation, if $\theta \in \prod_{n\in\mathbb{N}} \mathrm{Alt}(\Omega_{d_n})$, then we will write $\theta = \prod \theta_n$ instead of $\theta = \prod_{n\in\mathbb{N}} \theta_n$

Next we will give a brief account of the main structural features of $G_{\mathbf{d}}$. (For a more detailed account, see de la Harpe [5, III.B.35].) For each $\ell \in \mathbb{N}$, identify $\mathrm{Alt}(\Omega_{d_\ell})$ with the subgroup of $\prod_{n\in\mathbb{N}} \mathrm{Alt}(\Omega_{d_n})$ consisting of the elements $\theta = \prod \theta_n$ such that $\theta_n = 1$ for all $n \neq \ell$; and let

$$P_{\mathbf{d}} = \bigoplus_{n\in\mathbb{N}} \mathrm{Alt}(\Omega_{d_n})$$

be the restricted direct product consisting of those $\sigma = \prod \sigma_n \in \prod_{n\in\mathbb{N}} \mathrm{Alt}(\Omega_{d_n})$ such that $\sigma_n = 1$ for all but finitely many $n \in \mathbb{N}$. Then, by Neumann [9], we have that $\Delta^+(G_{\mathbf{d}}) = P_{\mathbf{d}}$ and $G_{\mathbf{d}}/P_{\mathbf{d}}$ is isomorphic to the subgroup B_∞ of $\mathrm{Sym}(\mathbb{Z})$ generated by $\mathrm{Alt}(\mathbb{Z})$ and the shift map $z \mapsto z+1$. In this paper, the normal subgroup of $G_{\mathbf{d}}$ consisting of the elements of finite order will be denoted by $F_{\mathbf{d}}$. Thus $F_{\mathbf{d}}/P_{\mathbf{d}} \cong \mathrm{Alt}(\mathbb{Z})$ and $G_{\mathbf{d}}/F_{\mathbf{d}} \cong \mathbb{Z}$ with $\alpha_{\mathbf{d}}$ mapping to the generator of $\mathbb{Z}$. In particular, the following result holds.

Theorem 4.1 (B.H. Neumann [9]). $\{ G_{\mathbf{d}} \mid \mathbf{d} \in \mathrm{Inc}(\mathbb{O}) \}$ *is an uncountable family of pairwise nonisomorphic finitely generated amenable groups.*

The following result is now an easy consequence of Proposition 3.3.

Proposition 4.2. *The bi-embeddability relation $\approx_{em}$ on the space $\mathcal{G}_{am}$ of finitely generated amenable groups is nonsmooth.*

Proof. By Proposition 3.3 and Theorem 4.1, the map $f : \mathrm{Inc}(\mathbb{O}) \to \mathcal{G}_{am}$, defined by $f(\mathbf{d}) = H_{\mathbf{d}}$, is an injective Borel homomorphism from E_t to $\approx_{em}$. In particular, since each $\approx_{em}$-class is countable, no comeager subset $C \subseteq \mathrm{Inc}(\mathbb{O})$ is mapped to a single $\approx_{em}$-class. Hence the result follows from Proposition 2.1. □

In the remainder of this section, we will record a few basic properties of the Neumann groups, which are needed in the proof of Theorem 1.2.

Proposition 4.3. *If $N \trianglelefteq G_{\mathbf{d}}$ is a subgroup of finite index, then there exists a finite (possibly trivial) cyclic group C and a finite (possibly empty) subset $T \subseteq \mathbb{N}$ such that $G_{\mathbf{d}}/N \cong C \times \prod_{\ell \in T} \mathrm{Alt}(\Omega_{d_\ell})$.*

Proof. It is easily seen that $N \cap P_{\mathbf{d}} = \bigoplus_{d \in S} \mathrm{Alt}(\Omega_{d_n})$ for some subset $S \subseteq \mathbb{N}$. Furthermore, since $[\, P_{\mathbf{d}} : N \cap P_{\mathbf{d}} \,] < \infty$, it follows that S must be a cofinite subset of $\mathbb{N}$. Let $\mathbf{s} = \langle s_n \mid n \in \mathbb{N} \rangle$ be the increasing enumeration of S and let $T = \mathbb{N} \smallsetminus S$.

Next suppose that $g = \prod g_n \in N$ and that $g_n \neq 1$. Then there exists an element $h \in \mathrm{Alt}(\Omega_{d_n})$ such that $h g_n h^{-1} \neq g_n$ and it follows that

$$1 \neq hgh^{-1}g^{-1} \in N \cap \mathrm{Alt}(\Omega_{d_n}).$$

This implies that $\mathrm{Alt}(\Omega_{d_n}) \leqslant N$ and so $n \in S$. Thus $P_{\mathbf{s}} \leqslant N \leqslant G_{\mathbf{s}}$ and hence

$$G_{\mathbf{d}}/N \cong (\, G_{\mathbf{s}}/N \,) \times \prod_{\ell \in T} \mathrm{Alt}(\Omega_{d_\ell}).$$

Thus it is enough to show that $G_{\mathbf{s}}/N$ is a finite (possibly trivial) cyclic group. To see this, let $p : G_{\mathbf{s}} \to G_{\mathbf{s}}/P_{\mathbf{s}}$ be the canonical surjective homomorphism and let $M = p[\, N \,]$. Then identifying $G_{\mathbf{s}}/P_{\mathbf{s}}$ with $B_\infty = \mathrm{Alt}(\mathbb{Z}) \rtimes \mathbb{Z}$, it follows that $[\, \mathrm{Alt}(\mathbb{Z}) : M \cap \mathrm{Alt}(\mathbb{Z}) \,] < \infty$ and hence $\mathrm{Alt}(\mathbb{Z}) \leqslant M$. This implies that B_∞/M is a finite (possibly trivial) cyclic group and hence the same is true of $G_{\mathbf{s}}/N$. □

The following observation will be used repeatedly in the proof of Theorem 1.2. (In fact, we have switched to the current "more symmetric" realization of the Neumann groups in order to obtain the following clean statement of Lemma 4.4.)

Lemma 4.4. *Suppose that $\sigma = \prod \sigma_n \in G_{\mathbf{d}}$ has infinite order. Then there exist integers n_0, k, $c \in \mathbb{N}$ and a fixed (possibly empty) subset $F \subseteq \mathbb{Z}$ such that for all $n \geq n_0$,*

(i) *$F \subseteq \Omega_{d_n}$ is σ_n-invariant; and*

(ii) *if* $\omega_n \in \Omega_{d_n} \smallsetminus F$*, then the orbit of* ω_n *under the action of* σ_n *has length at least* $(d_n - c)/k$*.*

Proof. If $\sigma = \prod \sigma_n \in G_{\mathbf{d}}$ has infinite order, then there exist integers $n_0 \in \mathbb{N}$ and $\ell \in \mathbb{Z} \smallsetminus \{ 0 \}$, together with a fixed permutation $\pi \in \mathrm{Alt}(\mathbb{Z})$ such that $\mathrm{supp}(\pi) \subseteq \Omega_{d_n}$ and $\sigma_n = \alpha_{\mathbf{d}}^{\ell} \pi$ for all $n \geq n_0$. A moment's thought now shows that there exists a subset $F \subseteq \{ m \in \mathbb{Z} \mid \min \mathrm{supp}(\pi) \leq m \leq \max \mathrm{supp}(\pi) \}$ such that the result holds with $k = |\ell|$. □

The following is the first of many consequences of Lemma 4.4.

Proposition 4.5. *If* $\sigma \in G_{\mathbf{d}}$ *is an element of infinite order, then* $C_{F_{\mathbf{d}}}(\sigma)$ *is an* FC*-by-finite group.*

Proof. By Lemma 4.4, if $\sigma = \prod \sigma_n \in G_{\mathbf{d}}$ has infinite order, then there exists an integer $n_0 \in \mathbb{N}$ and a fixed (possibly empty) subset $F \subseteq \mathbb{Z}$ such that:

(i) $F \subseteq \Omega_{d_n}$ is invariant under the action of σ_n for all $n \geq n_0$.

(ii) If $(\omega_n) \in \prod_{n \geq n_0} \Omega_{d_n} \smallsetminus F$, then the lengths of the orbits of ω_n under the action of σ_n are unbounded as $n \to \infty$.

Suppose that $\tau = \prod \tau_n \in C_{F_{\mathbf{d}}}(\sigma)$. Then there exists an integer $n_1 \geq n_0$ and a fixed permutation $\pi_\tau \in \mathrm{Alt}(\mathbb{Z})$ such that $\tau_n = \pi_\tau$ for all $n \geq n_1$. Since τ_n commutes with σ_n for all $n \in \mathbb{N}$, it follows that $\mathrm{supp}\, \pi_\tau \subseteq F$. Thus the map $\tau \mapsto \pi_\tau$ is a homomorphism from $C_{F_{\mathbf{d}}}(\sigma)$ to $\mathrm{Sym}(F)$ with kernel $C_{F_{\mathbf{d}}}(\sigma) \cap P_{\mathbf{d}}$. The result follows. □

5. A smooth Borel invariant

The general strategy of the proof of Theorem 1.2 can be described as follows. Recall that we have already shown that the bi-embeddability relation $\approx_{em}$ on the space $\{ G_{\mathbf{d}} \mid \mathbf{d} \in \mathrm{Inc}(\mathbb{O}) \}$ is a nonsmooth countable Borel equivalence relation. We will next define a suitable *smooth Borel invariant* on the space $\mathcal{G}$ of countable groups; i.e. a Borel homomorphism $\Phi : \mathcal{G} \to X$ from the isomorphism relation $\cong$ on $\mathcal{G}$ to a smooth Borel equivalence relation E on some standard Borel space X. Now suppose that $f : \mathcal{G}_{fg} \to \mathcal{G}_{fg}$ is a Borel map which selects an isomorphism class within each bi-embeddability class. Then the Borel map $\mathbf{d} \mapsto (\Phi \circ f)(G_{\mathbf{d}})$ is a homomorphism from E_t to the smooth Borel equivalence relation E; and hence by Proposition 2.1, there exists a comeager subset $C \subseteq \operatorname{Inc}(\mathbb{O})$ which is mapped to a single E-class. Of course, Φ will be chosen so that this leads to a contradiction.

Definition 5.1. If G is any group, then:

(i) let $\mathcal{S}_G$ be the set of nonabelian finite simple normal subgroups of G; and

(ii) let $\Phi(G)$ be the subgroup of G generated by $\bigcup \mathcal{S}_G$.

(If $\mathcal{S}_G = \emptyset$, then we define $\Phi(G) = 1$.)

It is easily seen that $\Phi(G) = \bigoplus \{ S \mid S \in \mathcal{S}_G \}$ is the direct sum of the subgroups $S \in \mathcal{S}_G$. Furthermore, it is clear that $\Phi(G)$ is a characteristic subgroup of G and that $\Phi(G) \leqslant \Delta^+(G)$. For example, if $\mathbf{d} \in \operatorname{Inc}(\mathbb{O})$, then

$$\Phi(G_{\mathbf{d}}) = \Delta^+(G_{\mathbf{d}}) = \bigoplus_{n \in \mathbb{N}} \operatorname{Alt}(\Omega_{d_n}).$$

On the other hand, if $n \geq 2$ and

$$G = \operatorname{Alt}(5) \operatorname{wr} \operatorname{Sym}(n) = [\underbrace{\operatorname{Alt}(5) \times \cdots \times \operatorname{Alt}(5)}_{n! \text{ times}}] \rtimes \operatorname{Sym}(n),$$

then $\Phi(G) = 1$ and $\Delta^+(G) = G$.

Next let $\mathcal{G}_{ss}$ be the standard Borel space of countable groups R which can be expressed as a direct sum

$$R = \bigoplus\nolimits_{n \in I} T_n \tag{5.1}$$

of nonabelian finite simple groups T_n, where I is either $\mathbb{N}$ or else some finite (possibly empty) initial segment of $\mathbb{N}$. Since $\{ T_n \mid n \in I \}$ is exactly the set of minimal nontrivial normal subgroups of R, it follows that the direct sum decomposition (5.1) is essentially unique; in the sense that if

$$R = \bigoplus\nolimits_{n \in J} T'_n \tag{5.2}$$

is a second such decomposition, then $J = I$ and there is a permutation $\sigma \in \operatorname{Sym}(I)$ such that $T'_n = T_{\sigma(n)}$ for all $n \in I$. Hence, letting $\{ S_\ell \mid \ell \in \mathbb{N} \}$ be a fixed enumeration of the distinct nonabelian finite simple groups up to isomorphism, it follows that R is determined up to isomorphism by the Borel invariant

$$\operatorname{tp}(R) = \langle m_0(R), m_1(R), \cdots, m_\ell(R), \cdots \rangle \in (\mathbb{N} \cup \{ \infty \})^{\mathbb{N}}$$

defined by

$$m_\ell(R) = \text{ the number of } n \in I \text{ such that } T_n \cong S_\ell.$$

Thus the isomorphism relation on $\mathcal{G}_{ss}$ is smooth.

Finally, note if $\theta : G \to H$ is a group isomorphism, then $\theta[\,\Phi(G)\,] = \Phi(H)$. Hence the map $G \mapsto \Phi(G)$ is a Borel homomorphism from the isomorphism relation on $\mathcal{G}$ to the isomorphism relation on $\mathcal{G}_{ss}$.

6. The commensurability relation for finitely generated groups

In this section, as a warm-up exercise for the proof of Theorem 1.2, we will use the Neumann groups to give a simple proof that there does not exist a Borel selection of an isomorphism class within each commensurability class of finitely generated groups. (This result was first proved in Thomas [14] via a significantly more complicated argument.)

Recall that two groups G_1, G_2 are said to be (abstractly) *commensurable*, written $G_1 \approx_C G_2$, if there exist subgroups $H_i \leqslant G_i$ of finite index such that $H_1 \cong H_2$.

Theorem 6.1. *There does* not *exist a Borel reduction $h : \mathcal{G}_{fg} \to \mathcal{G}_{fg}$ from $\approx_C$ to $\cong$ such that $h(G) \approx_C G$ for all $G \in \mathcal{G}_{fg}$.*

The proof of Theorem 6.1 makes use of the smooth invariant $\Delta^+_{ss}(G)$ on the space $\mathcal{G}$ of countable groups defined by

$$\Delta^+_{ss}(G) = \Phi(\,\Delta^+(G)\,).$$

For example, if $\mathbf{d} \in \mathrm{Inc}(\mathbb{O})$, then

$$\Delta^+_{ss}(G_{\mathbf{d}}) = \Phi(G_{\mathbf{d}}) = \bigoplus_{n \in \mathbb{N}} \mathrm{Alt}(\Omega_{d_n}).$$

However, the following result is *not* true if we replace $\Delta^+_{ss}(G)$ by $\Phi(G)$.

Lemma 6.2. *If G, H are (not necessarily finitely generated) groups such that $G \approx_C H$, then $\Delta^+_{ss}(G) \approx_C \Delta^+_{ss}(H)$.*

Proof. It is enough to prove the result for the special case when $H \leqslant G$ is a subgroup of finite index. Assuming this, we will first show that

$$\Delta^+(H) = \Delta^+(G) \cap H.$$

Obviously $\Delta^+(G) \cap H \leqslant \Delta^+(H)$. Conversely, suppose that $h \in H \smallsetminus \Delta^+(G)$. If h has infinite order, then clearly $h \notin \Delta^+(H)$. So we can suppose that h has finite order and thus $|h^G| = [\, G : C_G(h)\,] = \infty$. Since $[\, G : H\,] < \infty$ and

$$[\, G : H\,]\,[\, H : C_H(h)\,] = [\, G : C_H(h)\,] \geq [\, G : C_G(h)\,] = \infty,$$

it follows that $[\, H : C_H(h)\,] = \infty$ and so $h \notin \Delta^+(H)$. This completes the proof that $\Delta^+(H) = \Delta^+(G) \cap H$. In particular, it follows that $\Delta^+(H)$ is a subgroup of finite index in $\Delta^+(G)$.

Next recall that $\Delta^+_{ss}(G) = \bigoplus\{\, S \mid S \in \mathcal{S}_{\Delta^+(G)}\,\}$ is the direct sum of the subgroups $S \in \mathcal{S}_{\Delta^+(G)}$. Since $[\, \Delta^+(G) : \Delta^+(H)\,] < \infty$, it follows that there is a cofinite subset $\mathcal{S}_0 \subseteq \mathcal{S}_{\Delta^+(G)}$ such that $\bigoplus\{\, S \mid S \in \mathcal{S}_0\,\} \leqslant \Delta^+(H)$ and it is clear that $\mathcal{S}_0 \subseteq \mathcal{S}_{\Delta^+(H)}$. Finally, let $g_1, \cdots, g_t$ be a set of coset representatives of $\Delta^+(H)$ in $\Delta^+(G)$. Since

$$[\, \Delta^+_{ss}(H) : C_{\Delta^+_{ss}(H)}(g_1, \cdots, g_t)\,] < \infty,$$

it follows easily that there is a cofinite subset $\mathcal{S}_1 \subseteq \mathcal{S}_{\Delta^+(H)}$ such that $\mathcal{S}_1 \subseteq \mathcal{S}_{\Delta^+(G)}$. It now follows that $\Delta^+_{ss}(G) \approx_C \Delta^+_{ss}(H)$. □

Lemma 6.3. *If* $\mathbf{c}$, $\mathbf{d} \in \mathrm{Inc}(\mathbb{O})$, *then the following statements are equivalent:*

(i) $\mathbf{c}\, E_t\, \mathbf{d}$;
(ii) $G_{\mathbf{c}} \approx_C G_{\mathbf{d}}$;
(iii) $\Delta^+_{ss}(\, G_{\mathbf{c}}\,) \approx_C \Delta^+_{ss}(\, G_{\mathbf{d}}\,)$.

Proof. It is clear that (i) implies (ii); and, applying Lemma 6.2, it follows that (ii) implies (iii). To see that (iii) implies (i), simply note that if $\mathbf{c}$, $\mathbf{d} \in \mathrm{Inc}(\mathbb{O})$ are not E_t-equivalent, then $\Delta^+_{ss}(\, G_{\mathbf{c}}\,) \cong \bigoplus_{n\in\mathbb{N}} \mathrm{Alt}(c_n)$ and $\Delta^+_{ss}(\, G_{\mathbf{d}}\,) \cong \bigoplus_{n\in\mathbb{N}} \mathrm{Alt}(d_n)$ are clearly not commensurable. □

Proof of Theorem 6.1. Suppose that $h : \mathcal{G}_{fg} \to \mathcal{G}_{fg}$ is a Borel reduction from $\approx_C$ to $\cong$ such that $h(G) \approx_C G$ for all $G \in \mathcal{G}_{fg}$. Let $\varphi : \mathrm{Inc}(\mathbb{O}) \to \mathcal{G}_{ss}$ be the Borel map defined by $\varphi(\mathbf{d}) = \Delta^+_{ss}(\, h(G_{\mathbf{d}})\,)$. Then, by Lemma 6.3, φ is a Borel homomorphism from E_t to the isomorphism relation $\cong$ on $\mathcal{G}_{ss}$. Hence, by Proposition 2.1, there exists a comeager subset $C \subseteq \mathrm{Inc}(\mathbb{O})$ such that φ maps C to a single $\cong$-class. Let $\mathbf{c}$, $\mathbf{d} \in C$ be such that $\mathbf{c}$ and $\mathbf{d}$ are not E_t-equivalent. Then, since $h(G_{\mathbf{c}}) \approx_C G_{\mathbf{c}}$

and $h(G_{\mathbf{d}}) \approx_C G_{\mathbf{d}}$, it follows from Lemma 6.2 that

$$\Delta^+_{ss}(\,G_{\mathbf{c}}\,) \approx_C \Delta^+_{ss}(\,h(G_{\mathbf{c}})\,) \cong \Delta^+_{ss}(\,h(G_{\mathbf{d}})\,) \approx_C \Delta^+_{ss}(\,G_{\mathbf{d}}\,).$$

But then, applying Lemma 6.3 once more, it follows that $\mathbf{c}\, E_t\, \mathbf{d}$, which is a contradiction. □

7. The heart of the matter

In this section, we will present the proof of Theorem 1.2, modulo two technical results which will be proved in Sections 8 and 9. For the remainder of this paper $\mathbb{A} \subseteq \mathbb{N}$ will be a fixed infinite set of odd integers such that if $\mathbb{A} = \{\, a_n \mid n \in \mathbb{N}\}$ is the increasing enumeration, then:

(i) $a_0 \geq 5$;
(ii) if $m < n$, then $a_m \,|\, (a_n - 2)$; and
(iii) the sequence $\langle\, a_n \mid n \in \mathbb{N}\,\rangle$ is "sufficiently fast growing".

Here "sufficiently fast growing" means sufficiently fast growing for the various steps in the proof of Theorem 1.2 to go through. In a similar fashion, we will use the notation $n \ll m$ to indicate that m is "much larger" than n. Condition (ii) has been included in order to ensure that the following result holds.

Lemma 7.1. *If* $\mathbf{d} \in \mathrm{Inc}(\mathbb{A})$*, then for all* $n \in \mathbb{N}$*,*

$$1 \neq [\,\alpha_{\mathbf{d}}^{d_n - 2}\,\beta_{\mathbf{d}}\,\alpha_{\mathbf{d}}^{-(d_n-2)},\, \beta_{\mathbf{d}}\,] \in \mathrm{Alt}(\Omega_{d_n}).$$

Proof. If $d \geq 5$ is odd, then easy calculations show that $[\,\alpha_d^{\ell}\,\beta_d\,\alpha_d^{-\ell},\, \beta_d\,] = 1$ if either $\ell \in \{\,3, 4, \cdots, d-3\,\}$ or $d \,|\, \ell$; and that $[\,\alpha_d^{\ell}\,\beta_d\,\alpha_d^{-\ell},\, \beta_d\,] \neq 1$ if $\ell = d-2$. The result follows. □

Suppose that $f : \mathcal{G}_{fg} \to \mathcal{G}_{fg}$ is a Borel reduction from $\approx_{em}$ to $\cong$ such that $f(G) \approx_{em} G$ for all $G \in \mathcal{G}_{fg}$. Consider the Borel map $h : \mathrm{Inc}(\mathbb{A}) \to \mathcal{G}_{ss}$ defined by $h(\mathbf{d}) = \Phi(\,f(H_{\mathbf{d}})\,)$. By Proposition 3.3, if $\mathbf{c} E_t \mathbf{d}$, then $H_{\mathbf{c}} \approx_{em} H_{\mathbf{d}}$ and so h is a Borel homomorphism from E_t to the isomorphism relation on $\mathcal{G}_{ss}$. Applying Proposition 2.1, since the isomorphism relation on $\mathcal{G}_{ss}$ is smooth, it follows that there exists a comeager subset $C \subseteq \mathrm{Inc}(\mathbb{A})$ such that h maps C to a single isomorphism class. In order to show that this is impossible, it is necessary to analyze the structure of groups of the form $\Phi(K)$, where $K \in \mathcal{G}_{fg}$ satisfies $K \approx_{em} H_{\mathbf{d}}$ for some $\mathbf{d} \in \mathrm{Inc}(\mathbb{A})$.

Since $H_{\mathbf{d}} \hookrightarrow K \hookrightarrow H_{\mathbf{d}}$, it is enough to consider the case of finitely generated groups K such that $\varphi(H_{\mathbf{d}}) \leqslant K \leqslant H_{\mathbf{d}}$ for some embedding $\varphi : H_{\mathbf{d}} \to H_{\mathbf{d}}$. Thus most of our effort in this section will be devoted to an analysis of the possibilities for the embedding φ. From now on, let $p_{\mathbf{d}} : G_{\mathbf{d}} \times G_{\mathbf{e}} \to G_{\mathbf{d}}$ and $p_{\mathbf{e}} : G_{\mathbf{d}} \times G_{\mathbf{e}} \to G_{\mathbf{e}}$ be the canonical projections.

First consider the element $\varphi(\alpha_{\mathbf{d}}) = (\sigma, \sigma') \in H_{\mathbf{d}} = G_{\mathbf{d}} \times G_{\mathbf{e}}$. Then at least one of the elements σ, σ' has infinite order. Suppose that both σ and σ' have infinite order. Then, applying Proposition 4.5, it follows that

$$C_{F_{\mathbf{d}} \times F_{\mathbf{e}}}(\varphi(\alpha_{\mathbf{d}})) = C_{F_{\mathbf{d}}}(\sigma) \times C_{F_{\mathbf{e}}}(\sigma')$$

is an FC-by-finite group. However, since

$$\varphi(F_{\mathbf{e}}) \leqslant \varphi(C_{F_{\mathbf{d}} \times F_{\mathbf{e}}}(\alpha_{\mathbf{d}})) \leqslant C_{F_{\mathbf{d}} \times F_{\mathbf{e}}}(\varphi(\alpha_{\mathbf{d}}))$$

and $F_{\mathbf{e}}/P_{\mathbf{e}} \cong \mathrm{Alt}(\mathbb{Z})$, this is impossible. Thus exactly one of the elements σ, σ' has infinite order.

Lemma 7.2. *σ has infinite order.*

Proof. Suppose that σ' has infinite order and let $\sigma' = \prod \sigma'_n$. Applying Lemma 4.4, there exist integers n_0, k, $c \in \mathbb{N}$ and a fixed (possibly empty) subset $F \subseteq \mathbb{Z}$ such that for all $n \geq n_0$,

(i) $F \subseteq \Omega_{e_n}$ is σ'_n-invariant; and
(ii) if $\omega_n \in \Omega_{e_n} \smallsetminus F$, then the orbit of ω_n under the action of σ'_n has length at least $(e_n - c)/k$.

Let $\varphi(\beta_{\mathbf{d}}) = (\pi, \pi')$ and let $\pi' = \prod \pi'_n$. Then we can also suppose that n_0 has been chosen such that for all $n \geq n_0$,

(iii) there exists a fixed permutation $\theta \in \mathrm{Alt}(\mathbb{Z})$ such that $\pi'_n = \theta$.

Since $\varphi(\beta_{\mathbf{d}})$ has infinitely many conjugates under the action of $\varphi(\alpha_{\mathbf{d}})$ and the group $D = \langle \sigma, \pi \rangle \leq F_{\mathbf{d}}$ is finite, it follows that $\mathrm{supp}(\theta) \nsubseteq F$. Finally, since the sequence $\langle a_n \mid n \in \mathbb{N} \rangle$ is "sufficiently fast growing", there exists an integer $n \geq n_0$ such that for some $m \in \mathbb{N}$,

(iv) $d_m \ll e_n \ll d_{m+1}$; and
(v) $\prod_{\ell \leq m} |\, \mathrm{Alt}(\Omega_{d_\ell})| < \frac{e_n - c}{k |\, \mathrm{supp}(\theta)|}$.

Now consider the homomorphism $\psi : G_{\mathbf{d}} \to \mathrm{Sym}(\Omega_{e_n})$ obtained by projecting $\varphi(G_{\mathbf{d}})$ into $\mathrm{Sym}(\Omega_{e_n})$. Then clearly $\psi(\alpha_{\mathbf{d}}) = \sigma'_n$ and $\psi(\beta_{\mathbf{d}}) = \theta$. Applying Proposition 4.3, there exists a finite (possibly trivial) cyclic group C and a finite (possibly empty) subset $T \subseteq \mathbb{N}$ such that $\psi(\, G_{\mathbf{d}}\,) \cong C \times \prod_{\ell \in T} \mathrm{Alt}(\Omega_{d_\ell})$. Since $e_n < d_{m+1}$, it follows that $T \subseteq \{\, 0, 1, \cdots, m \,\}$. In particular, each conjugacy class of $\psi(\, G_{\mathbf{d}}\,)$ has cardinality at most $\prod_{\ell \leq m} |\, \mathrm{Alt}(\Omega_{d_\ell})|$. However, since $\mathrm{supp}(\theta) \nsubseteq F$ and each orbit of $\psi(\alpha_{\mathbf{d}}) = \sigma'_n$ on $\Omega_{e_n} \smallsetminus F$ has length at least $(e_n - c)/k$, the conjugacy class of $\psi(\beta_{\mathbf{d}}) = \theta$ in $\psi(\, G_{\mathbf{d}}\,)$ has cardinality at least $\frac{e_n - c}{k|\,\mathrm{supp}(\theta)|}$, which is a contradiction. □

Lemma 7.3. *There exists an integer $n_0 \in \mathbb{N}$ such that $\varphi(\, \mathrm{Alt}(\Omega_{d_n})\,) = \mathrm{Alt}(\Omega_{d_n})$ for all $n \geq n_0$.*

Proof. We will continue to write $\varphi(\alpha_{\mathbf{d}}) = (\, \sigma, \sigma'\,)$. Let $\sigma = \prod \sigma_n$. Then, applying Lemma 4.4, there exist integers n_0, k, $c \in \mathbb{N}$ and a fixed (possibly empty) subset $F \subseteq \mathbb{Z}$ such that for all $n \geq n_0$,

(i) $F \subseteq \Omega_{d_n}$ is σ_n-invariant; and

(ii) if $\omega_n \in \Omega_{d_n} \smallsetminus F$, then the orbit of ω_n under the action of σ_n has length at least $(d_n - c)/k$.

Let $\varphi(\beta_{\mathbf{d}}) = (\, \pi, \pi'\,)$. Since σ', $\pi' \in F_{\mathbf{e}}$, it follows that the group $D = \langle\, \sigma', \pi'\,\rangle \leq F_{\mathbf{e}}$ is finite. We can also suppose that n_0 has been chosen so that

(iii) $|\, \mathrm{Alt}(\Omega_{d_{n_0}})| > \max\{\, |\, \mathrm{Alt}(F)|, |D|\,\}$.

Let $\pi = \prod \pi_n$. Then we can also suppose that n_0 has been chosen such that for all $n \geq n_0$,

(iv) there exists a fixed permutation $\eta \in \mathrm{Alt}(\mathbb{Z})$ such that $\pi_n = \eta$.

Finally, since the sequence $\langle\, a_n \mid n \in \mathbb{N}\,\rangle$ is "sufficiently fast growing", we can also suppose that n_0 has been chosen such that:

(v) if $n > \ell \geq n_0$, then $|\, \mathrm{Alt}(\Omega_{d_\ell})| < \frac{d_n - c}{4k|\,\mathrm{supp}(\eta)|}$.

Now let $\ell \geq n_0$ and let $g_\ell = [\, \alpha_{\mathbf{d}}^{d_\ell - 2} \beta_{\mathbf{d}} \alpha_{\mathbf{d}}^{-(d_\ell - 2)}, \beta_{\mathbf{d}}\,]$. Then, by Lemma 7.1, we have that $1 \neq g_\ell \in \mathrm{Alt}(\Omega_{d_\ell})$; and hence the conjugates of g_ℓ in $G_{\mathbf{d}}$ generate $\mathrm{Alt}(\Omega_{d_\ell})$. Let $\varphi(g_\ell) = (\, \prod \tau_n, \prod \tau'_n\,)$. If $\tau'_n \neq 1$ for some $n \in \mathbb{N}$, then we can define a nontrivial homomorphism from $\mathrm{Alt}(\Omega_{d_\ell})$ into D by $(\, p_{\mathbf{e}} \circ \varphi\,) \restriction \mathrm{Alt}(\Omega_{d_\ell})$, which is impossible since $|\, \mathrm{Alt}(\Omega_{d_\ell})| > |D|$. It follows similarly that $\tau_n = 1$ for all $n < \ell$. Finally, suppose that $\tau_n \neq 1$ for some $n > \ell$. First note that if $\mathrm{supp}(\tau_n) \subseteq F$, then there exists a nontrivial homomorphism from $\mathrm{Alt}(\Omega_{d_\ell})$ into $\mathrm{Alt}(F)$, which is impossible. Thus $\mathrm{supp}(\tau_n) \cap (\Omega_{d_n} \smallsetminus F) \neq \emptyset$. Next notice that since $\pi_n = \eta$, it follows that $|\mathrm{supp}(\tau_n)| \leq 4|\mathrm{supp}(\eta)|$. But this means that τ_n has at least $\frac{d_n - c}{4k|\mathrm{supp}(\eta)|}$ distinct conjugates under the action of σ_n and hence g_ℓ has more than $|\mathrm{Alt}(\Omega_{d_\ell})|$ distinct conjugates under the action of $\varphi(\alpha_{\mathbf{d}})$, which is impossible. It follows that $\varphi(g_\ell) \in \mathrm{Alt}(\Omega_{d_\ell})$ and this implies that $\varphi(\mathrm{Alt}(\Omega_{d_\ell})) = \mathrm{Alt}(\Omega_{d_\ell})$. □

Next let $\varphi(\alpha_{\mathbf{e}}) = (\psi', \psi)$. Then, arguing as above, we see that exactly one of the elements ψ', ψ has infinite order.

Lemma 7.4. *ψ has infinite order.*

Proof. Clearly $\varphi(F_{\mathbf{d}} \times F_{\mathbf{e}}) = \varphi(H_{\mathbf{d}}) \cap (F_{\mathbf{d}} \times F_{\mathbf{e}})$ and so φ induces an embedding

$$\bar{\varphi} : H_{\mathbf{d}}/(F_{\mathbf{d}} \times F_{\mathbf{e}}) \hookrightarrow H_{\mathbf{d}}/(F_{\mathbf{d}} \times F_{\mathbf{e}}) \cong \mathbb{Z} \oplus \mathbb{Z}.$$

It follows that $\mathrm{im}\,\bar{\varphi}$ is a subgroup of finite index in $H_{\mathbf{d}}/(F_{\mathbf{d}} \times F_{\mathbf{e}})$. Combined with Lemma 7.2, this implies that ψ must have infinite order. □

Let $\psi = \prod \psi_n$. Applying Lemma 4.4 once again, there exist integers n_1, k', c' with $n_1 \geq n_0$ and a fixed (possibly empty) subset $F' \subseteq \mathbb{Z}$ such that for all $n \geq n_1$,

- $F' \subseteq \Omega_{e_n}$ is ψ_n-invariant; and
- if $\omega'_n \in \Omega_{e_n} \smallsetminus F'$, then the orbit of ω'_n under the action of ψ_n has length at least $(e_n - c')/k'$.

Lemma 7.5. *If K is an* arbitrary *subgroup such that $\varphi(H_{\mathbf{d}}) \leqslant K \leqslant H_{\mathbf{d}}$, then $[\Delta^+(p_{\mathbf{e}}(K)) : \Delta^+(p_{\mathbf{e}}(K)) \cap P_{\mathbf{e}}] < \infty$.*

Proof. If $g = \prod g_n \in \Delta^+(p_{\mathbf{e}}(K))$, then there exists $n_g \geq n_1$ and a fixed permutation $\eta_g \in \mathrm{Alt}(\mathbb{Z})$ such that $g_n = \eta_g$ for all $n \geq n_g$. Since the conjugacy class of g in $p_{\mathbf{e}}(K)$ is finite and $\varphi(\alpha_{\mathbf{e}}) \in K$, it follows that $\mathrm{supp}(\eta_g) \subseteq F'$. Thus $g \mapsto \eta_g$ is a homomorphism from $\Delta^+(p_{\mathbf{e}}(K))$ to $\mathrm{Sym}(F')$ with kernel $\Delta^+(p_{\mathbf{e}}(K)) \cap P_{\mathbf{e}}$. □

Until further notice, we will fix some finitely generated group K such that $\varphi(H_{\mathbf{d}}) \leqslant K \leqslant H_{\mathbf{d}}$. Let $U \leqslant K$ be the subgroup defined by

$$U = \{ (a, b) \in K \mid b \in P_{\mathbf{e}} \}.$$

Then, applying Suzuki [11, 4.19], we have that

$$p_{\mathbf{e}}(U)/(\,U \cap G_{\mathbf{e}}\,) \cong p_{\mathbf{d}}(U)/(\,U \cap G_{\mathbf{d}}\,). \tag{7.6}$$

Note that $p_{\mathbf{e}}(U)/(\,U \cap G_{\mathbf{e}}\,) = p_{\mathbf{e}}(U)/(\,U \cap P_{\mathbf{e}}\,)$ is a homomorphic image of a subgroup of $P_{\mathbf{e}}$. In particular, it follows that $p_{\mathbf{e}}(U)/(\,U \cap G_{\mathbf{e}}\,)$ is a periodic FC-group.

Lemma 7.6. *There exist only finitely many integers $m \geq 5$ such that* $\mathrm{Alt}(m)$ *embeds into $p_{\mathbf{e}}(U)/(\,U \cap G_{\mathbf{e}}\,)$.*

In view of the isomorphism (7.6), it is enough to show that there exist only finitely many integers $m \geq 5$ such that $\mathrm{Alt}(m)$ embeds into $p_{\mathbf{d}}(U)/(\,U \cap G_{\mathbf{d}}\,)$. To see this, first recall that there exists an integer n_0 such that

$$\bigoplus_{n \geq n_0} \mathrm{Alt}(\Omega_{d_n}) \leqslant \varphi(H_{\mathbf{d}}) \cap G_{\mathbf{d}} \leqslant U \cap G_{\mathbf{d}}.$$

Let $\mathbf{d}' = \langle\, d_{n_0+\ell} \mid \ell \in \mathbb{N}\,\rangle$. Then $P = p_{\mathbf{d}}(U) \cap G_{\mathbf{d}'}$ is a subgroup of finite index in $p_{\mathbf{d}}(U)$ and hence

$$P(\,U \cap G_{\mathbf{d}}\,)/(\,U \cap G_{\mathbf{d}}\,) \cong P/(\,P \cap U \cap G_{\mathbf{d}}\,)$$

is a subgroup of finite index in $p_{\mathbf{d}}(U)/(\,U \cap G_{\mathbf{d}}\,)$. Hence it is enough to show that there exist only finitely many integers $m \geq 5$ such that $\mathrm{Alt}(m)$ embeds into $P/(\,P \cap U \cap G_{\mathbf{d}}\,)$. Notice that $P \cap U \cap G_{\mathbf{d}} = P \cap U \cap G_{\mathbf{d}'}$ and that

$$\bigoplus_{n \geq n_0} \mathrm{Alt}(\Omega_{d_n}) \leqslant P \cap U \cap G_{\mathbf{d}'} \leqslant P \leqslant G_{\mathbf{d}'}.$$

We claim that P contains an element of infinite order. Since $[\,p_{\mathbf{d}}(U) : P\,] < \infty$, it is enough to show that $p_{\mathbf{d}}(U)$ contains an element of infinite order. To see this, recall that $\varphi(\alpha_{\mathbf{d}}) = (\,\sigma, \sigma'\,)$ for some $\sigma \in G_{\mathbf{d}}$ of infinite order and some $\sigma' \in G_{\mathbf{e}}$ of finite order. Hence if r is the order of σ', then $\varphi(\alpha_{\mathbf{d}})^r = (\,\sigma^r, 1\,) \in p_{\mathbf{d}}(U)$ is an element of infinite order.

Thus there exist a subgroup $\Gamma \leqslant B_\infty = \mathrm{Alt}(\mathbb{Z}) \rtimes \mathbb{Z}$ with $\Gamma \nleqslant \mathrm{Alt}(\mathbb{Z})$ and a normal subgroup $N \trianglelefteq \Gamma$ such that $\Gamma/N \cong P/(\,P \cap U \cap G_{\mathbf{d}}\,)$. Hence Lemma 7.6 is a consequence of the following result, which will be proved in Section 8.

Proposition 7.7. *Suppose that $\Gamma \leqslant B_\infty = \mathrm{Alt}(\mathbb{Z}) \rtimes \mathbb{Z}$ and that $\Gamma \nleqslant \mathrm{Alt}(\mathbb{Z})$. If $N \trianglelefteq \Gamma$ is a normal subgroup such that Γ/N is an FC-group, then there exist only finitely many integers $m \geq 5$ such that $\mathrm{Alt}(m)$ embeds into Γ/N.*

For each countable group G and integer $m \geq 5$, let $\mathrm{mult}_G(m) \in \mathbb{N} \cup \{ \infty \}$ be the multiplicity with which $\mathrm{Alt}(m)$ occurs in $\Phi(G)$. (As expected, in the statement of the following result, we define $\infty + 1$ to be ∞.)

Proposition 7.8. *For all but finitely many $m \geq 5$,*

$$\mathrm{mult}_K(m) = \begin{cases} \mathrm{mult}_{p_{\mathbf{e}}(K)}(m) + 1, & \text{if } m \in \{ d_n \mid n \in \mathbb{N} \}; \\ \mathrm{mult}_{p_{\mathbf{e}}(K)}(m), & \text{otherwise.} \end{cases} \tag{7.8}$$

Proof. Suppose that $m \geq 5$ is "sufficiently large". First suppose that $S \trianglelefteq K$ and that $S \cong \mathrm{Alt}(m)$. Since m is sufficiently large, there are no nontrivial homomorphisms from $\mathrm{Alt}(m)$ into $\mathrm{Alt}(\Omega_{d_\ell})$ for $\ell < n_0$; and hence if $p_{\mathbf{d}}(S) \neq 1$, then

$$1 \neq p_{\mathbf{d}}(S) \leqslant \prod_{n \geq n_0} \mathrm{Alt}(\Omega_{d_n}).$$

Let $1 \neq s \in S$ and let $p_{\mathbf{d}}(s) = \prod s_n \in \prod_{n \geq n_0} \mathrm{Alt}(\Omega_{d_n})$. Then there exists $n \geq n_0$ such that $s_n \neq 1$. Let $g \in \mathrm{Alt}(\Omega_{d_n})$ be an element such that $[\, s_n, g \,] \neq 1$. Since $\mathrm{Alt}(\Omega_{d_n}) \leqslant K$ and $S \trianglelefteq K$, it follows that

$$1 \neq [\, s, g \,] = s \,(\, g\, s^{-1} g^{-1}) \in S \cap \mathrm{Alt}(\Omega_{d_n}),$$

and this implies that $S = \mathrm{Alt}(\Omega_{d_n})$. On the other hand, if $p_{\mathbf{d}}(S) = 1$, then $S \leqslant G_{\mathbf{e}}$ and so $S = p_{\mathbf{e}}(S) \trianglelefteq p_{\mathbf{e}}(K)$. It follows that the left-hand side of equation (7.8) is less than or equal to the right-hand side.

Next suppose that $T \trianglelefteq p_{\mathbf{e}}(K)$ and that $T \cong \mathrm{Alt}(m)$. Since m is sufficiently large, Lemma 7.5 implies that $T \leqslant p_{\mathbf{e}}(K) \cap P_{\mathbf{e}} = p_{\mathbf{e}}(U)$; and hence Lemma 7.6 implies that $T \leqslant U \cap G_{\mathbf{e}} \leqslant K \cap G_{\mathbf{e}}$. Thus $T \trianglelefteq K$. Finally if $m \in \{ d_n \mid n \in \mathbb{N} \}$, then $m = d_n$ for some $n \geq n_0$ and so $\mathrm{Alt}(m) \cong \mathrm{Alt}(\Omega_{d_n}) \leqslant K$. Hence the right-hand side of equation (7.8) is also less than or equal to the left-hand side. □

Of course, in order for this result to be useful, it is necessary that $\mathrm{mult}_{p_{\mathbf{e}}(K)}(m)$ should not take the value ∞ too many times. This is confirmed by the following result, which will be proved in Section 9. (To see that $p_{\mathbf{e}}(K)$ satisfies the hypotheses of Proposition 7.9, note that $p_{\mathbf{e}}(K)$ is obviously finitely generated and that we have already shown that the subgroup $p_{\mathbf{e}}(U)$ contains an element of infinite order.)

Proposition 7.9. *If* $\mathbf{c} \in \mathrm{Inc}(\mathbb{O})$ *and* $L \leqslant G_{\mathbf{c}}$ *is a finitely generated non-periodic subgroup, then there exist only finitely many* $m \in \mathbb{N}$ *such that* $\mathrm{Alt}(m)$ *occurs with infinite multiplicity in* $\Phi(L)$.

At this point, it is easy to complete the proof of Theorem 1.2. So suppose that $f : \mathcal{G}_{fg} \to \mathcal{G}_{fg}$ is a Borel reduction from $\approx_{em}$ to $\cong$ such that $f(G) \approx_{em} G$ for all $G \in \mathcal{G}_{fg}$. Let $h : \mathrm{Inc}(\mathbb{A}) \to \mathcal{G}_{ss}$ be the Borel homomorphism from E_t to the isomorphism relation on $\mathcal{G}_{ss}$ defined by $h(\mathbf{d}) = \Phi(\, f(H_{\mathbf{d}})\,)$. Then there exists a comeager subset $C \subseteq \mathrm{Inc}(\mathbb{A})$ such that h maps C to a single isomorphism class; and hence there is a *fixed* function $\nu : \mathbb{O} \to \mathbb{N} \cup \{\, \infty \,\}$ such that $\mathrm{mult}_{f(H_{\mathbf{d}})} = \nu$ for all $\mathbf{d} \in C$. Let $\{\, L_k \mid k \in \mathbb{N} \,\}$ be the set of finitely generated non-periodic subgroups of $G_{\mathbf{e}}$; and for each $k \in \mathbb{N}$, let $C_k \subseteq C$ be the set of those $\mathbf{d} \in C$ such that for all but finitely many $m \geq 5$,

$$\mathrm{mult}_{f(H_{\mathbf{d}})}(m) = \begin{cases} \mathrm{mult}_{L_k}(m) + 1, & \text{if } m \in \{\, d_n \mid n \in \mathbb{N} \,\}; \\ \mathrm{mult}_{L_k}(m), & \text{otherwise.} \end{cases}$$

Applying Proposition 7.9, it follows that if $\mathbf{d} \in C_k$, then for all but finitely many $m \geq 5$,

$$m \in \{\, d_n \mid n \in \mathbb{N} \,\} \quad \iff \quad \nu(m) \neq \mathrm{mult}_{L_k}(m);$$

and this clearly implies that C_k is countable. However, applying Proposition 7.8, we obtain that $C = \bigcup_{k \in \mathbb{N}} C_k$, which is a contradiction.

8. Finitary permutation groups

In this section, we will present the proof of Proposition 7.7, making use of P.M. Neumann's work [10] on the structure of finitary permutation groups, together with the following easy observation. (Recall that G is said to be an *ICC-group* if for each $1 \neq g \in G$, the corresponding conjugacy class g^G is infinite.)

Proposition 8.1. *If* G *is a transitive group of finitary permutations of an infinite set* Ω*, then* G *is an ICC-group.*

Proof. Suppose that $1 \neq \varphi \in G$. Since G acts transitively on Ω, it follows that

$$\Omega = \bigcup_{g \in G} g\,(\,\mathrm{supp}(\varphi)\,) = \bigcup_{g \in G} \mathrm{supp}(g\,\varphi\,g^{-1}).$$

Since $\operatorname{supp}(\varphi)$ is a finite subset of Ω, this implies that φ has infinitely many distinct conjugates. □

Proof of Proposition 7.7. Let $p : \operatorname{Alt}(\mathbb{Z}) \rtimes \mathbb{Z} \to \mathbb{Z}$ be the canonical surjective homomorphism. Then $p[\,\Gamma\,] = k\mathbb{Z}$ for some $k \geq 1$. Choose some $t \in \Gamma$ with $p(t) = k$ and let $T = \langle\, t\,\rangle$. Then clearly $\Gamma = G \rtimes T$, where $G = \operatorname{Alt}(\mathbb{Z}) \cap \Gamma$. Let $\tau \in B_\infty$ be the shift map, $z \overset{\tau}{\mapsto} z+1$. Then there exists a permutation $\pi \in \operatorname{Alt}(\mathbb{Z})$ such that $t = \pi\,\tau^k$. It follows that T has finitely many finite orbits and finitely many infinite orbits on $\mathbb{Z}$, and this implies that the same is true of Γ. Let

$$\mathbb{Z} = F_1 \sqcup \cdots \sqcup F_r \sqcup \Delta_1 \sqcup \cdots \sqcup \Delta_s$$

be the corresponding decomposition into Γ-orbits, where each F_i is finite and each Δ_i is infinite. Fix some $1 \leq i \leq s$ and consider the action of Γ on Δ_i. Since $G \trianglelefteq \Gamma$, it follows that Γ acts transitively on the set $\mathcal{B}_i$ of G-orbits on Δ_i; and, in particular, each G-orbit on Δ_i has the same cardinality. It follows that there is an integer n such that if F is a finite orbit for the action of G on $\mathbb{Z}$, then $|F| \leq n$.

We claim that if $m \geq 5$ is an integer such that $m > |\operatorname{Sym}(n)|$, then $\operatorname{Alt}(m)$ does not embed into Γ/N. To see this, suppose that $\theta : \operatorname{Alt}(m) \to \Gamma/N$ is an embedding. Then, since the quotient of Γ/N by its normal subgroup GN/N is abelian, it follows that $\theta(\operatorname{Alt}(m)) \leqslant GN/N$. Slightly abusing notation, since $GN/N \cong G/(G \cap N)$, we can suppose that θ is an embedding of $\operatorname{Alt}(m)$ into $G/(G \cap N)$.

Let Ω_0 be the union of the finite G-orbits and let Ω_1 be the union of the infinite G-orbits. Let K be the kernel of the action of G on Ω_0. Then G/K acts faithfully as a group of finitary permutations of Ω_0 and all its orbits have cardinality at most n. Thus G/K is isomorphic to a subgroup of a restricted direct product $\bigoplus_{j \in J} H_j$ of finite groups, each of which embeds into $\operatorname{Sym}(n)$. In particular, it follows that the order of each element of G/K is a divisor of $|\operatorname{Sym}(n)|$. Since the quotient Q of $G/(G\cap N)$ by $K(G\cap N)/(G\cap N)$ is a homomorphic image of G/K, the order of each element of Q is also a divisor of $|\operatorname{Sym}(n)|$ and this implies that $\operatorname{Alt}(m)$ does not embed into Q. It follows that $\theta(\operatorname{Alt}(m)) \leqslant K(G \cap N)/(G \cap N)$. Slightly abusing notation once again, since $K(G \cap N)/(G \cap N) \cong K/(K \cap N)$, we can suppose that θ is an embedding of $\operatorname{Alt}(m)$ into $K/(K \cap N)$.

Since $G/(G \cap N)$ embeds into Γ/N, it follows that $G/(G \cap N)$ is also a periodic FC-group. In particular, by Proposition 8.1, $G/(G \cap N)$ cannot act as a transitive group of finitary permutations on an infinite set. By Neumann [10, Lemma 2.1], it follows that $G \cap N$ acts transitively on every infinite orbit of G. Applying Neumann [10, Theorem 4.1], this implies that $[\,K, K\,] \leqslant (\,G \cap N\,)$. But this means that $[\,K, K\,] \leqslant (\,K \cap N\,)$ and hence $K/(\,K \cap N\,)$ is abelian, which is a contradiction. □

9. An application of Jordan's Theorem

In this section, we will present the proof of Proposition 7.9, making use of the following classical theorem of Jordan on finite primitive permutation groups containing nontrivial elements of small support. Recall that if Ω is a nonempty (not necessarily finite) set and $G \leqslant \mathrm{Sym}(\Omega)$, then G is said to act *imprimitively* on Ω if:

(i) G acts transitively on Ω; and
(ii) there exists a subset $A \subseteq \Omega$ with $1 < |A| < |\Omega|$ such that for every $g \in G$, either $g(A) = A$ or $g(A) \cap A = \emptyset$.

In this case, A is said to be a nontrivial *block of imprimitivity*; and we can define a corresponding nontrivial G-invariant equivalence relation E on Ω by

$$x \, E \, y \quad \Longleftrightarrow \quad (\,\exists g \in G\,)\ x, y \in g(A).$$

As expected, a transitive subgroup $G \leqslant \mathrm{Sym}(\Omega)$ is said to act *primitively* on Ω if there do not exist any nontrivial blocks of imprimitivity.

Theorem 9.1 (Jordan [8])**.** *There exists a nondecreasing function $c : \mathbb{N} \to \mathbb{N}$ such that whenever Ω is a finite set with $|\Omega| > c(d)$ and $H \leqslant \mathrm{Sym}(\Omega)$ is a subgroup such that:*

(i) *H acts primitively on Ω; and*
(ii) *H contains a nontrivial element h with $|\operatorname{supp}(h)| \leq d$;*

then H is either $\mathrm{Alt}(\Omega)$ *or* $\mathrm{Sym}(\Omega)$.

Proof of Proposition 7.9. Let $p : G_{\mathbf{c}} \to \mathbb{Z}$ be the canonical surjective homomorphism. Then there exists an integer $k \geq 1$ such that $p(L) = k\mathbb{Z}$. Choose $g \in L$ such that $p(g) = k$. Then $g = \alpha_{\mathbf{c}}^{k}\,\tau$ for some $\tau \in F_{\mathbf{c}}$. Let $\{\,\theta_1, \cdots, \theta_t\,\}$ be a finite generating set for L, chosen so that $\theta_1 = g$. Suppose that $2 \leq i \leq t$ and that $\theta_i = \alpha_{\mathbf{c}}^{\ell_i}\tau_i$ for some $\ell_i \neq 0$ and $\tau_i \in F_{\mathbf{c}}$. Then there exists $0 \neq m_i \in \mathbb{Z}$ such that

$\ell_i = m_i k$ and clearly $\theta_1^{-m_i}\theta_i \in F_{\mathbf{c}}$. Hence, replacing each such θ_i by $\theta_1^{-m_i}\theta_i$, we can suppose that $\theta_2, \cdots, \theta_t \in F_{\mathbf{c}}$.

Let $\theta_1 = \prod \sigma_n$ and let $\theta_i = \prod \theta_{i,n}$ for each $2 \leq i \leq t$. Then there exist integers n_0, $d \in \mathbb{N}$ and a fixed (possibly empty) subset $F \subseteq \mathbb{Z}$ such that for all $n \geq n_0$,

(i) $F \subseteq \Omega_{c_n}$ is σ_n-invariant; and

(ii) if $\omega_n \in \Omega_{c_n} \smallsetminus F$, then the orbit of ω_n under the action of σ_n has length at least $(c_n - d)/k$.

We can also suppose that n_0 has been chosen so that for all $2 \leq i \leq t$, there exists a fixed permutation $\pi_i \in \mathrm{Alt}(\mathbb{Z})$ such that:

(iii) $\theta_{i,n} = \pi_i$ for all $n \geq n_0$.

Finally we can also suppose that n_0 has been chosen sufficiently large with respect to $\max_{2 \leq i \leq t} |\operatorname{supp}(\pi_i)|$ to allow the following argument to go through.

Suppose that $m \gg \max\{\, |\operatorname{supp}(\pi_i)| \mid 2 \leq i \leq t\,\} \cup \{\, |F|\,\}$ and that $\mathrm{Alt}(m)$ occurs with infinite multiplicity in $\Phi(L)$. Then there exists a subgroup $T \in \mathcal{S}_L$ such that $T \cong \mathrm{Alt}(m)$ and such that T projects nontrivially into $\mathrm{Alt}(\Omega_{c_n})$ for some $n \geq n_0$ with $m \ll c_n$. Let S be the projection of T into $\mathrm{Alt}(\Omega_{c_n})$ and let $\Gamma = \langle\, \sigma_n, \pi_2, \cdots, \pi_t \,\rangle \leqslant \mathrm{Alt}(\Omega_{c_n})$. Then $S \trianglelefteq \Gamma$ and $S \cong \mathrm{Alt}(m)$. In the remainder of this proof, we will focus on the action of Γ on Ω_{c_n}.

Let Δ be a Γ-orbit on Ω_{c_n} such that S acts nontrivially and hence faithfully on Δ. Since $m > |F|$, it follows that Δ intersects $\Omega_{c_n} \smallsetminus F$ nontrivially and hence $|\Delta| \geq (c_n - d)/k$. Since $m \ll c_n$, it follows that S does not act transitively on Δ; and since $S \trianglelefteq \Gamma$, it follows that Γ permutes transitively the set of S-orbits on Δ. Let E be the corresponding nontrivial Γ-invariant equivalence relation on Δ defined by

$$x \, E \, y \quad \Longleftrightarrow \quad x, y \text{ lie in the same } S\text{-orbit};$$

and let $\Delta/E = \{\, X_j \mid j \in J\,\}$ be the corresponding set of S-orbits. Then if $j \neq k \in J$, the permutation groups $(\, S, X_j\,)$ and $(\, S, X_k\,)$ are isomorphic; and S acts on Δ as a "diagonal subgroup" of $\prod_{j \in J} \mathrm{Sym}(X_j)$. Furthermore, since

$$m \gg \max\{\, |\operatorname{supp}(\pi_i)| \mid 2 \leq i \leq t\,\},$$

it follows that each $|X_j| > \max\{\, |\operatorname{supp}(\pi_i)| \mid 2 \leq i \leq t\,\}$ and this implies that each π_i acts trivially on Δ/E.

Let X be an S-orbit on Δ and let $\Gamma_{\{X\}}$ be the subgroup of Γ which fixes X setwise. Let $A \subsetneq X$ be a maximal proper block of imprimitivity for the action of $\Gamma_{\{X\}}$ on X. (If the induced action of $\Gamma_{\{X\}}$ on X is primitive, then we take A to be a trivial singleton block.) Then A is also a block of imprimitivity for the action of Γ on Δ. Let E' be the corresponding Γ-invariant equivalence relation on Δ. If every π_i acts trivially on the set Δ/E' of E'-classes, then Γ acts as a transitive cyclic group on Δ/E', which is a contradiction since S acts transitively and hence faithfully on the E'-classes contained in X. Thus some π_{i_0} acts nontrivially on Δ/E'. For each S-orbit Y, let Y/E' denote the set of E'-classes which are contained in Y. Let Z be an S-orbit such that π_{i_0} moves some E'-equivalence class $B \in Z/E'$. Then B is a maximal proper block of imprimitivity for the action of $\Gamma_{\{Z\}}$ on Z; and so the induced action of $\Gamma_{\{Z\}}$ on Z/E' is primitive. Applying Jordan's Theorem, since the support for the nontrivial action of π_{i_0} on Z/E' is small, it follows that $\Gamma_{\{Z\}}$ induces at least $\mathrm{Alt}(Z/E')$ on Z/E'. Furthermore, since $\Gamma_{\{Z\}}$ normalizes S, it follows that S also induces $\mathrm{Alt}(Z/E')$ on Z/E'. Thus $|Z/E'| = m$ and S acts on Δ/E' as a "diagonal subgroup" of $\prod_{j\in J} \mathrm{Alt}(X_j/E')$. Next note that since the support of π_{i_0} is small and the number of E-classes is large, it follows that there exists an S-orbit Z' such that π_{i_0} acts trivially on Z'/E'. Let $\psi \in S$ be an element such that the permutation induced by ψ on Z/E' does not commute with the permutation induced by π_{i_0} on Z/E'. Then the commutator $[\,\psi, \pi_{i_0}\,]$ acts nontrivially on Z/E' and acts trivially on Z'/E'. But since Γ normalizes S, it follows that

$$[\,\psi, \pi_{i_0}\,] = \psi\,(\,\pi_{i_0}\psi^{-1}\pi_{i_0}^{-1}\,) \in S,$$

which contradicts the fact that S acts on Δ/E' as a "diagonal subgroup" of $\prod_{j\in J} \mathrm{Alt}(X_j/E')$. This completes the proof of Proposition 7.9. □

10. Some Open Problems

In this section, we will discuss a few of the many open questions suggested by the material in this paper, including the question of whether there exists a purely "group-theoretic" Borel reduction from $\approx_{em}$ to $\cong$. By this, we mean a construction which only involves purely group-theoretic notions such as wreath products, free products with amalgamation, HNN-extensions, etc. In each case that we have considered, such a construction induces a continuous map on the space of marked finitely generated groups. Thus it is natural to ask whether various group-theoretic problems have continuous solutions.

Conjecture 10.1. There does not exist a continuous Borel reduction $f : \mathcal{G}_{fg} \to \mathcal{G}_{fg}$ from the bi-embeddability relation $\approx_{em}$ to the isomorphism relation $\cong$.

Conjecture 10.2. There does not exist a continuous Borel reduction $f : \mathcal{G}_{fg} \to \mathcal{G}_{fg}$ from the commensurability relation $\approx_C$ to the isomorphism relation $\cong$.

It is perhaps worth mentioning that if S is a fixed infinite finitely generated simple group, then the map $f : \mathcal{G}_{fg} \to \mathcal{G}_{fg}$, defined by

$$G \overset{f}{\mapsto} (\mathrm{Alt}(5) \text{ wr } G) \text{ wr } S,$$

is a continuous Borel reduction from $\cong$ to $\approx_C$. (See Thomas [14].) On the other hand, it is not known whether or not there exists a continuous Borel reduction from $\cong$ to $\approx_{em}$.

Very few nontrivial natural examples are currently known of pairs E, F of countable Borel equivalence relations with $E \leq_B F$ such that there does not exist a continuous reduction from E to F. In fact, as far as we are aware, the only known examples are variants of those presented in Thomas [13]. In particular, no such examples are known when E and F are natural "group-theoretic" equivalence relations on the space $\mathcal{G}_{fg}$ of finitely generated groups.

It is also interesting to ask whether the analog of Theorem 1.2 holds for the quasi-isometry relation $\approx_{QI}$ on the space of finitely generated groups. (A clear account of the basic properties of the quasi-isometry relation for finitely generated groups can be found in de la Harpe [5].)

Conjecture 10.3. There does not exist a Borel selection of an isomorphism class within each quasi-isometry class of finitely generated groups.

In Thomas [12], it was conjectured that the quasi-isometry relation $\approx_{QI}$ on the space of finitely generated groups is a universal K_σ equivalence relation. Of course, if this is true, then there does not exist a Borel reduction from $\approx_{QI}$ to $\cong$ and so Conjecture 10.3 holds.

References

[1] C. Champetier, *L'espace des groupes de type fini*, Topology **39** (2000), 657–680.

[2] J. Feldman and C. C. Moore, *Ergodic equivalence relations, cohomology and von Neumann algebras, I*, Trans. Amer. Math. Soc. **234** (1977), 289–324.

[3] R. I. Grigorchuk, *Degrees of growth of finitely generated groups and the theory of invariant means*, Math. USSR-Izv. **25** (1985), 259–300.

[4] R. I. Grigorchuk, *Solved and Unsolved Problems Around One Group*, in *Infinite Groups: Geometric, Combinatorial and Dynamical Aspects*, (Birkhäuser, Boston, 2005), pp. 117–218.

[5] P. de la Harpe, *Topics in Geometric Group Theory*, Chicago Lectures in Mathematics Series, (University of Chicago Press, 2000).

[6] L. Harrington, A. S. Kechris and A. Louveau, *A Glimm-Effros dichotomy for Borel equivalence relations*, J. Amer. Math. Soc. **3** (1990), 903–927.

[7] G. Hjorth, *Classification and orbit equivalence relations*, Math. Surveys Monogr. **75**, American Mathematical Society, Providence, 2000.

[8] C. Jordan, *Théorèmes sur les groupes primitifs*, J. Math. Pure Appl. **16** (1871), 383–408.

[9] B. H. Neumann, *Some remarks on infinite groups*, J. London Math. Soc. **12** (1937), 120-127.

[10] P. M. Neumann, *The structure of finitary permutation groups*, Arch. Math. **27** (1976), 3–17.

[11] M. Suzuki, *Group theory I*, Grundlehren der Mathematischen Wissenschaften **247**, Springer-Verlag, Berlin-New York, 1982.

[12] S. Thomas, *On the complexity of the quasi-isometry and virtual isomorphism problems for finitely generated groups*, Groups Geom. Dyn. **2** (2008), 281–307.

[13] S. Thomas, *Continuous versus Borel Reductions*, Arch. Math. Logic **48** (2009), 761–770.

[14] S. Thomas, *The commensurability relation for finitely generated groups*, J. Group Theory **12** (2009), 901–909.

[15] S. Thomas and B. Velickovic, *On the complexity of the isomorphism relation for finitely generated groups*, J. Algebra **217** (1999), 352–373.

[16] M. J. Tomkinson, FC-*groups*, Research Notes in Mathematics **96**, Pitman, Boston, 1984.

[17] J. Williams, *Countable Borel Quasi-Orders*, Ph. D. dissertation, Rutgers University, 2012.

Mathematics Department, Rutgers University, 110 Frelinghuysen Road, Piscataway, NJ 08854

E-mail address: sthomas@math.rutgers.edu

Department of Mathematics, California Institute of Technology, Pasadena, CA 91125

E-mail address: jaywill@caltech.edu