COMPLETE GROUPS ARE COMPLETE CO-ANALYTIC

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ABSTRACT. The set of complete groups is a complete co-analytic subset of the standard Borel space of countably infinite groups.

1. Introduction

If G is a centerless group, then we can define an embedding $G \hookrightarrow \operatorname{Aut}(G)$ by $g \mapsto i_g$, where i_g is the corresponding inner automorphism defined by

$$i_g(x) = g\,x\,g^{-1}, \qquad x \in G.$$

It is easily checked that $\operatorname{Inn}(G) = \{ i_g \mid g \in G \}$ is a normal subgroup of $\operatorname{Aut}(G)$. Let $\operatorname{Out}(G) = \operatorname{Aut}(G)/\operatorname{Inn}(G)$ be the associated quotient group.

Definition 1.1. A group G is *complete* if G is centerless and $\operatorname{Aut}(G) = \operatorname{Inn}(G)$.

Let $\mathcal{G}$ be the standard Borel space of countably infinite groups. Then it is clear that $\mathcal{G}_{cmp} = \{ G \in \mathcal{G} \mid G \text{ is complete} \}$ is a co-analytic subset of $\mathcal{G}$; and the main result of this paper shows that $\mathcal{G}_{cmp}$ is not a Borel subset of $\mathcal{G}$.

Theorem 1.2. *The set $\mathcal{G}_{cmp}$ of complete groups is a complete co-analytic subset of the standard Borel space $\mathcal{G}$ of countably infinite groups.*

This paper is organized as follows. In Section 2, we will present the proof of Theorem 1.2, and we will also briefly consider three other natural co-analytic sets of countably infinite groups. Then, in Section 3, we will define a natural $\mathbf{\Pi}^1_1$-rank on the collection of complete groups $\mathcal{G}_{cmp} \subseteq \mathcal{G}$.

Acknowledgments

As usual, I would like to thank Alekos Kechris for interesting discussions on the material in this paper.

Research partially supported by NSF Grant DMS 1362974.

2. The proof of Theorem 1.2

In this section, we will prove that the set $\mathcal{G}_{cmp}$ of complete groups is a complete co-analytic subset of the standard Borel space $\mathcal{G}$ of countably infinite groups. The idea of the proof is easily explained. Suppose that K is a field such that $|K| > 3$. Then $PGL(2,K)$ is a centerless group; and, by Schreier and van der Waerden [6],

$$\mathrm{Aut}(PGL(2,K)) = \mathrm{Inn}(PGL(2,K)) \rtimes \mathrm{Aut}(K).$$

Hence $\mathrm{Out}(PGL(2,K)) \cong \mathrm{Aut}(K)$, and so $PGL(2,K)$ is a complete group if and only if $\mathrm{Aut}(K) = 1$. Thus, in order to prove that the set of complete groups is complete co-analytic, it is enough to show that there exists a Borel map $T \mapsto K_T$ from the standard Borel space $\mathcal{T}$ of countable trees to the standard Borel space $\mathcal{F}$ of countable fields such that

$$T \text{ is well-founded} \qquad \Longleftrightarrow \qquad \mathrm{Aut}(K_T) = 1.$$

As we will explain below, the existence of such a map follows easily from results of Harrison [3] and Friedman-Stanley [2] on pseudo-well orderings, together with a theorem of Fried-Kollar [1] which provides a suitable coding of an arbitary structure within a field of characteristic 0.

Definition 2.1. A *pseudo-well ordering* is a linear ordering R of ω which has no infinite descending sequence which is hyperarithmetic in R.

Lemma 2.2 (Harrison [3])**.** *If R is a non-well-founded pseudo-well ordering, then R has order-type $(\omega_1^R \times (1 + \mathbb{Q})) + \alpha$ for some $\alpha < \omega_1^R$.*

Thus if R is a pseudo-well ordering, then R is not a well-ordering if and only if $\mathrm{Aut}(R) \neq 1$. Let $\mathcal{L}$ be the standard Borel space of countable linear orders.

Lemma 2.3 (Friedman-Stanley [2])**.** *There exists a Borel map $\varphi : \mathcal{T} \to \mathcal{L}$ such that for each $T \in \mathcal{T}$:*

(a) *$\varphi(T)$ is a pseudo-well ordering.*
(b) *$\varphi(T)$ is a well-ordering if and only if T is a well-founded tree.*

Let $\mathcal{F}_0$ be the standard Borel space of countable fields of characteristic 0.

Lemma 2.4. *There exists a Borel map* $\psi : \mathcal{L} \to \mathcal{F}_0$ *such that* $\mathrm{Aut}(L) \cong \mathrm{Aut}(\psi(L))$ *for each* $L \in \mathcal{L}$.

Sketch proof. The proof proceeds in two steps. First, as a very special case of Hodges [4, Theorem 5.1.1], there exists an explicit construction which to any infinite linear order L associates a connected graph Γ_L of the same cardinality such that $\mathrm{Aut}(\Gamma_L) \cong \mathrm{Aut}(L)$.

Secondly, by Fried-Kollár [1], there exists an explicit construction which to any infinite connected graph Γ associates a field K_Γ of characteristic 0 of the same cardinality such that $\mathrm{Aut}(K_\Gamma) \cong \mathrm{Aut}(\Gamma)$. Since both constructions are explicit, we easily obtain a Borel map $\psi : \mathcal{L} \to \mathcal{F}_0$ such that $\mathrm{Aut}(L) \cong \mathrm{Aut}(\psi(L))$ for each $L \in \mathcal{L}$. □

For each tree $T \in \mathcal{T}$, let $K_T = (\psi \circ \varphi)(T)$. Then $T \mapsto PGL(2, K_T)$ is a Borel map from $\mathcal{T}$ to $\mathcal{G}$; and, by construction, for each tree $T \in \mathcal{T}$,

$$T \text{ is well-founded} \quad \Longleftrightarrow \quad \mathrm{Out}(PGL(2, K_T)) \cong \mathrm{Aut}(K_T) = 1.$$

This completes the proof that the set $\mathcal{G}_{cmp}$ of complete groups is a complete co-analytic subset of $\mathcal{G}$.

Remark 2.5. It is easily shown that each of the groups $G_T = PGL(2, K_T)$ is *pseudo-complete* in the sense that there do not exist any outer automorphisms $\pi \in \mathrm{Aut}(G_T) \smallsetminus \mathrm{Inn}(G_T)$ which are hyperarithmetic in G_T.

In the remainder of this section, we will briefly consider three other natural co-analytic sets of countably infinite groups. First let

$$\mathcal{G}_{cnt} = \{\, G \in \mathcal{G} \mid \mathrm{Aut}(G) \text{ is countable} \,\}.$$

By Kueker [5], if G is a countable group, then $\mathrm{Aut}(G)$ is countable if and only if there exist $g_1, \cdots, g_n \in G$ such that the identity map Id_G is the only automorphism $\pi \in \mathrm{Aut}(G)$ satisfying $\pi(g_i) = g_i$ for all $1 \leq i \leq n$. It follows that $\mathcal{G}_{cnt}$ is a co-analytic subset of $\mathcal{G}$. Since the Borel map $T \mapsto PGL(2, K_T)$ also satisfies

$$T \text{ is well-founded} \quad \Longleftrightarrow \quad \mathrm{Aut}(PGL(2, K_T)) \text{ is countable},$$

we obtain the following result.

Theorem 2.6. *$\mathcal{G}_{cnt}$ is a complete co-analytic subset of $\mathcal{G}$.*

Next recall that a group G is said to be *Hopfian* if every surjective homomorphism $\pi : G \to G$ is injective; and that a group G is said to be *co-Hopfian* if every injective homomorphism $\pi : G \to G$ is surjective. Let

- $\mathcal{G}_{Hop} = \{ G \in \mathcal{G} \mid G \text{ is Hopfian} \}$;
- $\mathcal{G}_{coH} = \{ G \in \mathcal{G} \mid G \text{ is co-Hopfian} \}$.

Then $\mathcal{G}_{Hop}$ and $\mathcal{G}_{coH}$ are clearly co-analytic subsets of $\mathcal{G}$. Unfortunately, our earlier argument tells us nothing about the questions of whether $\mathcal{G}_{Hop}$ and $\mathcal{G}_{coH}$ are complete co-analytic subsets of $\mathcal{G}$, since it is easily shown that for every tree $T \in \mathcal{T}$, the group $PGL(2, K_T)$ is Hopfian and not co-Hopfian.

Question 2.7. Are $\mathcal{G}_{Hop}$ and $\mathcal{G}_{coH}$ are complete co-analytic subsets of $\mathcal{G}$?

3. A $\mathbf{\Pi}^1_1$-rank on the set of complete groups

In this section, we will define a natural $\mathbf{\Pi}^1_1$-rank on the collection of complete groups $\mathcal{G}_{cmp} \subseteq \mathcal{G}$. We will make use of the following observation.

Definition 3.1. If G is any group, then an automorphism $\varphi \in \operatorname{Aut}(G)$ is said to be *locally inner* if for every finite subset $X \subset G$, there exists $g \in G$ such that $\varphi(x) = gxg^{-1}$ for every $x \in X$.

Proposition 3.2. *If G is a countably infinite centerless group, then the following statements are equivalent.*

(a) *There exist $2^{\aleph_0}$ locally inner autmorphisms $\varphi \in \operatorname{Aut}(G) \smallsetminus \operatorname{Inn}(G)$.*
(b) *There exist a locally inner autmorphism $\varphi \in \operatorname{Aut}(G) \smallsetminus \operatorname{Inn}(G)$.*
(c) *$C_G(X) \neq 1$ for every finite subset $X \subset G$.*

Proof. Clearly (a) implies (b). To see that (b) implies (c), let $\varphi \in \operatorname{Aut}(G) \smallsetminus \operatorname{Inn}(G)$ be locally inner and let $X \subseteq G$ be any finite subset. Then there exists $g \in G$ such that $\varphi(x) = gxg^{-1}$ for every $x \in X$. Since $\varphi \notin \operatorname{Inn}(G)$, there exists $y \in G$ such that $\varphi(y) \neq gyg^{-1}$. Let $Z = X \cup \{ y \}$ and let $h \in G$ be such that $\varphi(z) = hzh^{-1}$ for every $z \in Z$. Then $1 \neq h^{-1}g \in C_G(X)$.

Finally suppose that $C_G(X) \neq 1$ for every finite subset $X \subset G$. Then we can express $G = \bigcup_{n \in \mathbb{N}} G_n$ as the union of a strictly increasing chain of finitely generated subgroups G_n such that for every $n \in \mathbb{N}$, there exists an element $g_n \in G_n$ such that $g_0 \in G_0 \smallsetminus Z(G_0)$ and $g_{n+1} \in C_{G_{n+1}}(G_n) \smallsetminus Z(G_{n+1})$. For each $\varepsilon \in \{ 0, 1 \}$, let

$$g_n^\varepsilon = \begin{cases} g_n, & \text{if } \varepsilon = 1; \\ 1, & \text{if } \varepsilon = 0. \end{cases}$$

Then for each binary sequence $s = (\varepsilon_n) \in 2^{\mathbb{N}}$, we can define a corresponding locally inner automorphism φ_s by setting

$$\varphi_s(x) = (g_n^{\varepsilon_n} \cdots g_0^{\varepsilon_0}) x (g_n^{\varepsilon_n} \cdots g_0^{\varepsilon_0})^{-1}, \qquad x \in G_n.$$

If $s \neq s' = (\varepsilon'_n) \in 2^{\mathbb{N}}$ and n is the least natural number such that $\varepsilon_n \neq \varepsilon'_n$, then $\varphi_s \restriction G_n \neq \varphi_{s'} \restriction G_n$. Since G has only countably many inner automorphisms, it follows that there exist $2^{\aleph_0}$ locally inner autmorphisms $\varphi \in \mathrm{Aut}(G) \smallsetminus \mathrm{Inn}(G)$. □

Definition 3.3. A group G is *strongly centerless* if there exists a finite subset $X \subset G$ such that $C_G(X) = 1$.

Corollary 3.4. *If G is a countably infinite complete group, then G is strongly centerless.*

Convention 3.5. From now on, we will regard each element of $\mathcal{G}$ as an ordered pair (G, e), where G is a countably infinite group and $e = (g_n \mid n \in \mathbb{N})$ is an enumeration of G such that $e_0 = 1$.

Definition 3.6. For each $(G, e) \in \mathcal{G}$, let T_G^e be the tree consisting of the finite injective partial functions $f : G \to G$, partially ordered by inclusion, such that either $f = (1, 1)$, or else

$$\operatorname{dom} f = \{ g_0, \cdots, g_n \} \cup \{ f^{-1}(g_0), \cdots, f^{-1}(g_n) \}$$

for some $n \in \mathbb{N}$ and the following conditions hold:

(i) The partial function f extends to an embedding of the subgroup generated by $\operatorname{dom} f$ into G.

(ii) For all $h \in G$, there exists $a \in \operatorname{dom} f$ such that $f(a) \neq h^{-1} a h$.

Lemma 3.7. *If $(G, e) \in \mathcal{G}$ is strongly centerless, then G is complete if and only if T_G^e is well-founded.*

Proof. First suppose that G is not complete and let $\varphi \in \mathrm{Aut}(G) \smallsetminus \mathrm{Inn}(G)$ be an outer automorphism. For each $n \in \mathbb{N}$, let

$$\varphi_n = \varphi \restriction \{ g_0, \cdots, g_n, \varphi^{-1}(g_0), \cdots \varphi^{-1}(g_n) \}.$$

By Proposition 3.2, the outer automorphism φ is not locally inner, and hence there exists $n_0 \in \mathbb{N}$ such that $\varphi_n \in T^e_G$ for all $n \geq n_0$. Thus T^e_G is not well-founded.

Conversely, suppose that T^e_G is not well-founded and let B be an infinite branch of T^e_G. Then it is clear that $\varphi = \bigcup B$ is an automorphism of G. Furthermore, by considering any element $f \in B$, we see that $\varphi \notin \mathrm{Inn}(G)$. □

Definition 3.8. Let $\rho : \mathcal{G}_{cmp} \to \mathrm{Ord}$ be the function defined by $\rho(G, e) = \mathrm{rank}(T^e_G)$.

Theorem 3.9. *$\rho : \mathcal{G}_{cmp} \to \mathrm{Ord}$ is a $\mathbf{\Pi}^1_1$-rank.*

Proof. If $(H, c) \in \mathcal{G}_{cmp}$ and $(G, e) \in \mathcal{G}$, then the following conditions are equivalent:

(i) $(G, e) \in \mathcal{G}_{cmp}$ and $\rho(G, e) \leq \rho(H, c)$.

(ii) G is strongly centerless and $\mathrm{rank}(T^e_G) \leqslant \mathrm{rank}(T^c_H)$.

It is easily checked that the map $(G, e) \mapsto T^e_G$ is Borel. The result follows. □

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