CONTINUOUS VS. BOREL REDUCTIONS

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Abstract. We present some natural examples of countable Borel equivalence relations E, F with $E \leq_B F$ such that there does not exist a continuous reduction from E to F.

1. Introduction

If E, E' are Borel equivalence relations on the standard Borel spaces X, X', then E is said to be Borel reducible to E', written $E \leq_B E'$, if there exists a Borel map $\theta : X \to Y$ such that $x \mathrel{E} y \Leftrightarrow \theta(x) \mathrel{E'} \theta(y)$. If there exists a continuous reduction $\theta : X \to Y$ from E to E', then we write $E \leq_c E'$. It has often been noted that it is difficult to find natural examples of Borel equivalence relations E, E' such that $E \leq_B E'$ but $E \nleq_c E'$. (For example, see Kanovei [6, Question 5.3.2].) In this paper, we shall use the following theorem to present some countable Borel equivalence relations with this property.

Throughout this paper, $\equiv_T$ denotes the *Turing equivalence relation* on $\mathcal{P}(\mathbb{N})$, which is identified with the Cantor space $2^{\mathbb{N}}$ by identifying subsets of $\mathbb{N}$ with their characteristic functions.

Theorem 1.1. *Suppose that G is a countable subgroup of* $\mathrm{Sym}(\mathbb{N})$ *and that E_G is the orbit equivalence relation of the action of G on $2^{\mathbb{N}}$. Then whenever $\theta : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ is a continuous homomorphism from $\equiv_T$ to E_G, there exists a cone $D \subseteq 2^{\mathbb{N}}$ such that θ maps D into a single E_G-class.*

Let $\equiv_1$ be the recursive isomorphism relation on $2^{\mathbb{N}}$, defined by $x \equiv_1 y$ iff there exist a recursive permutation π of $\mathbb{N}$ such that $\pi(x) = y$. For each $x \in 2^{\mathbb{N}}$, let x' be the Turing jump of x. Then it is well-known that the map $x \mapsto x'$ is a Borel reduction from $\equiv_T$ to $\equiv_1$. (For example, see Rogers [10, Theorem 13.1].) Clearly the following result is an immediate consequence of Theorem 1.1.

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Corollary 1.2. $\equiv_T \not\leq_c \equiv_1$.

Let $\mathbb{F}_2$ be the free group on two generators and let E_∞ be the orbit equivalence relation arising from the shift action of $\mathbb{F}_2$ on $2^{\mathbb{F}_2}$. Then E_∞ is a universal countable Borel equivalence relation and hence $\equiv_T \leq_B E_\infty$. Of course, the conclusion of Theorem 1.1 continues to hold if $\mathbb{N}$ is replaced by any other countably infinite set.

Corollary 1.3. $\equiv_T \not\leq_c E_\infty$.

Finally we consider the isomorphism relation $\cong$ on the standard Borel space $\mathcal{G}$ of finitely generated groups. By Thomas-Velickovic [14], $\cong$ is also a universal countable Borel equivalence relation and hence $\equiv_T \leq_B \cong$. Following Champetier [1], the space $\mathcal{G}$ can be defined as follows. Let $\mathbb{F}_\infty$ be the free group on countably many generators $X = \{x_i \mid i \in \mathbb{N}\}$. Suppose that G is a finitely generated group and that $(g_0, \dots, g_n)$ is a finite sequence of generators. Then by considering the homomorphism $\pi : \mathbb{F}_\infty \to G$ defined by

$$\pi(x_i) = \begin{cases} g_i & \text{if } 0 \leq i \leq n \\ 1 & \text{otherwise,} \end{cases}$$

we see that G can be realized as a quotient $\mathbb{F}_\infty/N$, where N is a normal subgroup which contains all but finitely many elements of the basis X. (Of course, choosing a different generating sequence usually results in a different realization.) Thus we can identify $\mathcal{G}$ with the set of all such normal subgroups N of $\mathbb{F}_\infty$. With this identification, $\mathcal{G}$ is a Borel subset of the standard Borel space $\mathcal{P}(\mathbb{F}_\infty)$ and hence $\mathcal{G}$ is a standard Borel space. The isomorphism relation $\cong$ on $\mathcal{G}$ is the orbit equivalence relation of the action of a suitable countable subgroup of $\mathrm{Aut}(\mathbb{F}_\infty)$. More precisely, let $\mathrm{Aut}_f(\mathbb{F}_\infty)$ be the subgroup of $\mathrm{Aut}(\mathbb{F}_\infty)$ generated by the elementary Nielsen transformations

$$\{\alpha_i \mid i \in \mathbb{N}\} \cup \{\beta_{ij} \mid i \neq j \in \mathbb{N}\},$$

where α_i is the automorphism sending x_i to x_i^{-1} and leaving $X \smallsetminus \{x_i\}$ fixed; and β_{ij} is the automorphism sending x_i to $x_i x_j$ and leaving $X \smallsetminus \{x_i\}$ fixed. Then the natural action of $\mathrm{Aut}_f(\mathbb{F}_\infty)$ on $\mathbb{F}_\infty$ induces a corresponding action on the space $\mathcal{G}$ of normal subgroups of $\mathbb{F}_\infty$ which contain all but finitely many elements of the basis X; and if N, $M \in \mathcal{G}$ are two such normal subgroups, then $\mathbb{F}_\infty/N \cong \mathbb{F}_\infty/M$ iff there exists $\varphi \in \mathrm{Aut}_f(\mathbb{F}_\infty)$ such that $\varphi[N] = M$. (For example, see Champetier [1] and Lyndon-Schupp [9].) Hence, applying Theorem 1.1 once more, we obtain the following result.

Corollary 1.4. $\equiv_T \not\leq_c \cong$.

Remark 1.5. If G is a countable group acting continuously on a 0-dimensional Polish space X and E_G^X is the corresponding orbit equivalence relation, then the proof of Dougherty-Jackson-Kechris [3, Proposition 1.8] shows that $E_G^X \leq_c E_\infty$. In particular, it follows that $\equiv_1 \leq_c E_\infty$ and that $\cong \leq_c E_\infty$. Also the proof of Thomas-Velickovic [14, Theorem 3] shows that $E_\infty \leq_c \cong$. These seem to be the only cases where it is currently known that a continuous reduction exists between distinct $E \neq F$ amongst the equivalence relations $\equiv_T$, $\equiv_1$, E_∞ and $\cong$ considered in this section. In fact, it is not even known whether there exist Borel reductions from E_∞ to $\equiv_T$ or $\equiv_1$. (Of course, these questions are equivalent to asking whether $\equiv_T$ or $\equiv_1$ is countable universal. For a discussion of these very interesting questions, see Dougherty-Kechris [4].) It is also not known whether there exists a Borel reduction from $\equiv_1$ to $\equiv_T$.

The remainder of this paper is organized as follows. In Section 2, we shall recall some basic notions from the theory of countable Borel equivalence relations and recursion theory. In Section 3, we shall prove Theorem 1.1. In Section 4, we shall prove that the recursive isomorphism relation $\equiv_1$ is not a normal subrelation of the Turing equivalence relation $\equiv_T$. Finally, in Section 5, we shall mention an open problem which is related to Martin's Conjecture on degree invariant Borel maps.

2. Preliminaries

In this section, we shall recall some basic notions from the theory of Borel equivalence relations and recursion theory.

2.1. **Borel equivalence relations.** In this paper, we shall only be concerned with *countable Borel equivalence relations*; i.e. those Borel equivalence relations E such that every E-class is countable. By Feldman-Moore [5], if E is a countable Borel equivalence relation on the standard Borel space X, then there exists a Borel action of a countable group G on X such that $E = E_G^X$, where E_G^X is the orbit equivalence relation defined by

$$x \, E_G^X \, y \quad \Longleftrightarrow \quad (\, \exists g \in G \,) \; g \cdot x = y.$$

It follows easily that if $A \subseteq X$ is a Borel subset, then the corresponding *E-saturation* $[\, A \,]_E = \{\, x \in X \mid (\, \exists a \in A \,) \; a \, E \, x \,\}$ is also a Borel subset.

Suppose that E, E' are countable Borel equivalence relations on the standard Borel spaces X, X'. Then the Borel map $\theta : X \to X'$ is a *Borel homomorphism* from E to E' if $x E y \Rightarrow \theta(x) E' \theta(y)$. If the Borel homomorphism $\theta : X \to X'$ from E to E' is countable-to -one, then we say that θ is a *weak Borel reduction* and we write $E \leq_B^w E'$. In this case, since E' is a countable Borel equivalence relation, it follows that the preimage $\theta^{-1}(\, [x']_{E'} \,)$ of every E'-class $[x']_{E'}$ is countable. A countable Borel equivalence relation E is said to be *weakly universal* iff $F \leq_B^w E$ for every countable Borel equivalence relation F. For example, Kechris [12, Corollary 4.9] has shown that the Turing equivalence relation $\equiv_T$ is weakly universal; and since $\equiv_T \, \leq_B \, \equiv_1$, it follows that $\equiv_1$ is also weakly universal. (The material in Thomas [12, Section 4] is entirely due to Kechris and Miller.)

The countable Borel equivalence relation E on the standard Borel space X is said to be *smooth* iff $E \leq_B \Delta(Z)$ for some standard Borel space Z, where $\Delta(Z)$ is the identity relation on Z. Equivalently, E is smooth iff there exists a Borel E-tranversal $T \subseteq X$; i.e. a Borel subset T which meets every E-class $[\, x \,]_E$ in exactly one point. We shall make use of the following easy observation in Section 4.

Lemma 2.1. *Suppose that E, E' are countable Borel equivalence relations on the standard Borel spaces X, X' respectively and that $E \leq_B^w E'$. If E' is smooth, then E is also smooth.*

Proof. Suppose that E' is smooth. Let $\theta : X \to X'$ be a weak Borel reduction from E to E' and let $F = \theta^{-1}(E')$. Then F is a countable Borel equivalence relation on X such that $F \supseteq E$ and $F \sim_B E'$. In particular, it follows that F is also smooth. Let $T \subseteq X$ be a Borel F-transversal. By Feldman-Moore [5], there exists a Borel action of a countable group $\Gamma = \{\, \gamma_n \mid n \in \mathbb{N} \,\}$ on X such that $F = E_\Gamma^X$. Hence we can define a Borel reduction $\psi : X \to X$ from E to $\Delta(X)$ by $\psi(x) = \gamma_n \cdot t$, where $t \in T \cap [\, x \,]_F$ and n is minimal such that $\gamma_n \cdot t \; E \; x$. □

2.2. **Recursion theory.** If $T \subseteq 2^{<\mathbb{N}}$ is a tree, then $[T] \subseteq 2^{\mathbb{N}}$ denotes the set of infinite branches of T. The tree T is said to be *pointed* iff T is perfect and $T \leq_T x$ for all $x \in [T]$. It is easily seen that if T is pointed and $T \leq_T z \in 2^{\mathbb{N}}$, then there exists a branch $x \in [T]$ such that $x \equiv_T z$. Thus $[T]$ is a complete Borel $\equiv_T$-section for the cone $C = \{ z \in 2^{\mathbb{N}} \mid T \leq_T z \}$. Conversely, suppose that $A \subseteq 2^{\mathbb{N}}$ is a $\leq_T$-cofinal Borel subset. Then a remarkable theorem of Martin says that there exists a pointed tree T such that $[T] \subseteq A$. (A proof of this theorem can be found in Kechris [7].) In particular, it follows that if $A \subseteq 2^{\mathbb{N}}$ is a $\leq_T$-cofinal $\equiv_T$-invariant Borel subset, then A contains a cone.

If $s, t \in 2^{<\mathbb{N}}$, then their concatenation is denoted by $s * t$. If $x, y \in 2^{\mathbb{N}}$, then their disjoint sum $x \oplus y \in 2^{\mathbb{N}}$ is defined by

$$(x \oplus y)(n) = \begin{cases} x(\frac{n}{2}), & \text{if } n \text{ is even;} \\ y(\frac{n-1}{2}), & \text{if } n \text{ is odd.} \end{cases}$$

Throughout this paper, φ_n denotes the nth partial recursive function in some standard enumeration and $\mathbf{0}' = \{ n \mid \varphi_n(n) \downarrow \}$.

3. The proof of Theorem 1.1

In this section, we shall present the proof of Theorem 1.1. For each countable subgroup H of $\mathrm{Sym}(\mathbb{N})$, let E_H be the orbit equivalence relation of the action of H on $2^{\mathbb{N}}$. Notice that there exists a cone $C \subseteq 2^{\mathbb{N}}$ such that $(E_H \restriction C) \subseteq (\equiv_T \restriction C)$.

Definition 3.1. If $E \subseteq F$ are countable Borel equivalence relations on the standard Borel space X, then F is *smooth over* E iff there exists a Borel homomorphism $\theta : X \to X$ from F to E such that $\theta(x) \, F \, x$ for all $x \in X$. (Of course, this implies that θ is actually a Borel reduction from F to E.)

Theorem 1.1 is a straightforward consequence of the following result, together with a deep result of Slaman-Steel [11].

Theorem 3.2. *Suppose that H is a countable subgroup of* $\mathrm{Sym}(\mathbb{N})$ *and that $D \subseteq 2^{\mathbb{N}}$ is a cone such that $(E_H \restriction D) \subseteq (\equiv_T \restriction D)$. Then $\equiv_T \restriction D$ is not smooth over $E_H \restriction D$.*

Proof of Theorem 1.1. Let G be a countable subgroup of $\mathrm{Sym}(\mathbb{N})$ and suppose that $\theta : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ is a continuous homomorphism from $\equiv_T$ to E_G. Let $r \in 2^{\mathbb{N}}$ be a real such that $g \leq_T r$ for all $g \in G$. For each $g \in G$, let $\tilde{g} \in \mathrm{Sym}(\mathbb{N})$ be the permutation defined by

$$\tilde{g}(n) = \begin{cases} 2g(n/2), & \text{if } n \text{ is even;} \\ n, & \text{if } n \text{ is odd.} \end{cases}$$

Let $H = \{ \tilde{g} \mid g \in G \}$ and $C = \{ z \in 2^{\mathbb{N}} \mid r \leq_T z \}$. Then $(E_H \restriction C) \subseteq (\equiv_T \restriction C)$. Let $\pi : 2^{\mathbb{N}} \to C$ be the continuous reduction from E_G to $E_H \restriction C$ defined by $\pi(x) = x \oplus r$ and let $\psi = \pi \circ \theta$. Then ψ is a continuous homomorphism from $\equiv_T$ to $(E_H \restriction C)$. Hence there exists a cone $D \subseteq 2^{\mathbb{N}}$ such that $\psi(x) \leq_T x$ for all $x \in D$. Applying Martin's Theorem, there exists a cone $D' \subseteq D$ such that either:

(i) $\psi(x) \equiv_T x$ for all $x \in D'$; or

(ii) $\psi(x) <_T x$ for all $x \in D'$.

First suppose that (i) holds. Then $D' \subseteq C$ and $\equiv_T \restriction D'$ is smooth over $E_H \restriction D'$, which contradicts Theorem 3.2. Thus (ii) holds. Since $(E_H \restriction C) \subseteq (\equiv_T \restriction C)$, we can also regard ψ as a Borel homomorphism from $\equiv_T$ to $\equiv_T$. Hence by Slaman-Steel [11, Theorem 2], there exists a cone $D'' \subseteq D'$ such that ψ maps D'' to a single $\equiv_T$-class; say, $[\, y\,]_{\equiv_T}$. Let $A = \pi^{-1}([\, y\,]_{\equiv_T})$. Then A is countable and hence there exists an element $a \in A$ such that $\theta^{-1}(a)$ is $\leq_T$-cofinal. Applying Martin's Theorem once more, it follows that there exists a cone $D''' \subseteq D''$ such that θ maps D''' to $[\, a\,]_{E_G}$. □

Thus it only remains to prove Theorem 3.2. Suppose that H is a countable subgroup of $\mathrm{Sym}(\mathbb{N})$ and that $D \subseteq 2^{\mathbb{N}}$ is a cone such that $(E_H \restriction D) \subseteq (\equiv_T \restriction D)$ and $\equiv_T \restriction D$ is smooth over $E_H \restriction D$. Let $\theta : D \to D$ be a Borel homomorphism from $\equiv_T \restriction D$ to $E_H \restriction D$ such that $\theta(x) \equiv_T x$ for all $x \in D$. Since θ is countable-to-one, it follows that $\theta(D)$ is a Borel subset of $2^{\mathbb{N}}$. Clearly $\theta(D)$ is $\leq_T$-cofinal and hence there exists a pointed tree T such that $[\, T\,] \subseteq \theta(D)$. In particular, it follows that if $x, y \in [\, T\,]$, then

$$x \equiv_T y \iff x \, E_H \, y.$$

Let $H = \{ h_n \mid n \in \mathbb{N} \}$ and let $s \in 2^{\mathbb{N}}$ code the sequence $(\, h_n \mid n \in \mathbb{N}\,) \in (\mathbb{N}^{\mathbb{N}})^{\mathbb{N}}$. Then after replacing T by a suitable pointed subtree if necessary, we can suppose that $s \leq_T T$. Let $x \in [\, T\,]$ be the leftmost branch of T. Then $x \equiv_T T$. Define an increasing sequence of nodes $y_n \in T$ as follows:

- $y_0 = \emptyset$.
- Suppose that y_n has been defined and that $y_n \subseteq y_n^+ \in T$ is the next branching node. Let $|y_n^+| = \ell_n$. If $h_n(\ell_n) \notin x$, let $y_{n+1} = y_n^+ * 1$. Otherwise, let $y_{n+1} = y_n^+ * 0$.

Let $y = \lim y_n \in [T]$. Then $T \leq_T y \leq_T T \oplus x \oplus s \equiv_T T$ and so $y \equiv_T x$. But by construction, we have that $(x, y) \notin E_H$, which is a contradiction. This completes the proof of Theorem 3.2.

4. Normal Subrelations

By Theorem 3.2, the Turing equivalence relation $\equiv_T$ is not smooth over the recursive isomorphism relation $\equiv_1$. In this section, we shall continue our study of the relationship between $\equiv_T$ and $\equiv_1$ from the perspective of the theory of countable Borel equivalence relations.

Definition 4.1. If $E \subseteq F$ are countable Borel equivalence relations on the standard Borel space X, then $\mathrm{End}_F(E)$ denotes the set of all Borel maps ψ from a Borel subset $\mathrm{dom}\,\psi \subseteq X$ to X such that for all x, $y \in \mathrm{dom}\,\psi$,

(a) $\psi(x)\, F\, x$; and
(b) $\psi(x)\, E\, \psi(y)$ iff $x\, E\, y$.

Definition 4.2. If $E \subseteq F$ are countable Borel equivalence relations on the standard Borel space X, then E is said to be a *normal subrelation* of F, written $E \trianglelefteq F$, iff there exists a countable family $\{\, \psi_n \mid n \in \mathbb{N} \,\} \subseteq \mathrm{End}_F(E)$ such that

$$x\, F\, y \iff (\,\exists n\,)\ \psi_n(x) = y.$$

Remark 4.3. In this case, we can suppose that $\mathrm{dom}\,\psi_n$ is E-invariant for each $n \in \mathbb{N}$. To see this, let $D_n = [\,\mathrm{dom}\,\psi_n\,]_E$ be the E-saturation of $\mathrm{dom}\,\psi_n$. Then there exists a Borel map $c_n : D_n \to \mathrm{dom}\,\psi_n$ such that:

(i) $c_n(x) = x$ for all $x \in \mathrm{dom}\,\psi_n$; and
(ii) $c_n(x) E x$ for all $x \in D_n$.

Thus we can extend ψ_n to the Borel map $\psi_n^+ = \psi_n \circ c_n \in \mathrm{End}_F(E)$ such that $D_n = \mathrm{dom}\,\psi_n^+$ is E-invariant.

Example 4.4. If $E \subseteq F$ are countable Borel equivalence relations on the standard Borel space X and E is smooth, then $E \trianglelefteq F$. To see this, let T be a Borel E-transversal and let $c : X \to T$ be the Borel map such that $c(x) \mathrel{E} x$ for all $x \in X$. Let $\Gamma = \{ \gamma_n \mid n \in \mathbb{N} \}$ be a countable group such that $F \restriction T = E_\Gamma^T$ for a suitable Borel action of Γ on T; and let $\Delta = \{ \delta_m \mid m \in \mathbb{N} \}$ be a countable group such that $E = E_\Delta^X$ for a suitable Borel action of Δ on X. Then the family $\{ \delta_m \circ \gamma_n \circ c \mid n, m \in \mathbb{N} \}$ witnesses that $E \trianglelefteq F$.

The remainder of this section is devoted to the proof of the following result.

Theorem 4.5. *$\equiv_1$ is not a normal subrelation of $\equiv_T$.*

Our proof is based upon the following observations.

Lemma 4.6. *Suppose that $E \subseteq F$ are countable Borel equivalence relations on the standard Borel space X and that $E \trianglelefteq F$. If $Y \subseteq X$ is a complete Borel F-section, then $E \leq_B^w (E \restriction Y)$.*

Proof. Let $\{ \psi_n \mid n \in \mathbb{N} \} \subseteq \mathrm{End}_F(E)$ witness that $E \trianglelefteq F$. Then we can suppose that $\mathrm{dom}\,\psi_n$ is E-invariant for each $n \in \mathbb{N}$. Let $Z = [\, Y \,]_E$ be the E-saturation of Y and let $c : Z \to Y$ be a Borel map such that $c(z) \mathrel{E} z$ and $c(z) \in Y$ for each $z \in Z$. Consider the Borel map $\theta : X \to Y$ defined by

$$\theta(x) = (\, c \circ \psi_n \,)(x),$$

where n is least such that $x \in \mathrm{dom}\,\psi_n$ and $\psi_n(x) \in Z$. Then θ is a Borel homomorphism from E to $E \restriction Y$. Since $\theta(x) \mathrel{F} x$ for all $x \in X$, it follows that θ is countable-to-one and hence is a weak Borel reduction. □

Lemma 4.7. *Let E be a countable Borel equivalence relation such that $E \subseteq\, \equiv_T$. Suppose that there exists a pointed tree T such that:*

(a) *$E \restriction [\, T \,]$ is the identity relation; and*

(b) *$E \restriction C$ is not smooth, where $C = \{ x \in 2^{\mathbb{N}} \mid (\, \exists y \in [\, T \,]\,)\; y \equiv_T x \}$.*

Then E is not a normal subrelation of $\equiv_T$.

Proof. Suppose that E is a normal subrelation of $\equiv_T$. Then it follows easily that $E \restriction C$ is a normal subrelation of $\equiv_T \restriction C$ and hence $(E \restriction C) \leq_B^w (E \restriction [\, T \,])$. But then Lemma 2.1 implies that $E \restriction C$ is smooth, which is a contradiction. □

Before proving Theorem 4.5, we shall illustrate the use of Lemma 4.7 by means of the following simple application. Let E_0 be the *Vitali equivalence relation* defined on $2^{\mathbb{N}}$ by $x \, E_0 \, y$ iff $x(n) = y(n)$ for all but finitely many n. Then clearly $E_0 \subseteq \, \equiv_T$.

Proposition 4.8. *E_0 is not a normal subrelation of $\equiv_T$.*

Proof. For each $s \in 2^{<\mathbb{N}}$, let $u_s \in 2^{<\mathbb{N}}$ be the binary sequence defined inductively by $u_\emptyset = \emptyset$ and $u_{s*i} = u_s * i * u_s$. Then $T = \{ \, t \in 2^{<\mathbb{N}} \mid (\exists s) \, t \subseteq u_s \, \}$ is a perfect recursive tree such that $E_0 \restriction [\, T \,]$ is the identity relation. Since E_0 is not smooth and $[\, T \,]$ is a complete Borel $\equiv_T$-section, the result follows from Lemma 4.7. □

Theorem 4.5 is an easy consequence of the following result.

Theorem 4.9. *There exists a pointed tree T such that:*

(i) *$T \equiv_T \mathbf{0}'$; and*

(ii) *$\equiv_1 \restriction [\, T \,]$ is the identity relation.*

Proof of Theorem 4.5. Let $C = \{ \, x \in 2^{\mathbb{N}} \mid \mathbf{0}' \leq_T x \, \}$. Then the map $z \mapsto z'$ is a Borel reduction from $\equiv_T$ to $\equiv_1 \restriction C$ and so $\equiv_1 \restriction C$ is weakly universal. Hence the result follows by Lemma 4.7 and Theorem 4.9. □

Thus it only remains to prove Theorem 4.9. Recall that φ_n denotes the nth partial recursive function in some standard enumeration. For each $s \in 2^{<\mathbb{N}}$, let $a_s \in 2^{<\mathbb{N}}$ be the binary sequence defined inductively as follows. First let $a_\emptyset = \emptyset$. Now suppose that $a_s \in 2^{\ell_n}$ has been defined for each $s \in 2^n$ and that $a_s(2i) = \mathbf{0}'(i)$ for each $2i < \ell_n$.

(i) If it exists, let $p = 2j$ be the least even integer such that $\varphi_n(p) \downarrow$ and $\ell_n \leq \varphi_n(p)$ is odd. For each $s \in 2^n$, define $b_s = a_s * \alpha_s \in 2^{\varphi_n(p)+2}$, where for each $\ell_n \leq r \leq \varphi_n(p) + 1$,

$$\alpha_s(r) = \begin{cases} \mathbf{0}'(r/2) & \text{if } r \text{ is even;} \\ 1 - \mathbf{0}'(j) & \text{if } r = \varphi_n(p); \\ 1 & \text{otherwise.} \end{cases}$$

If no such p exists, then we define $b_s = a_s$.

(ii) Next if it exists, let $q \geq |b_s|$ be the least odd integer such that $\varphi_n(q) \downarrow$ and $\varphi_n(q) = 2k$ is even. As above, extend each b_s to a sequence $c_s = b_s * \beta_s$ of odd length such that $\beta_s(q) = 1 - \mathbf{0}'(k)$. If no such q exists, then we define $c_s = b_s$.

(iii) Next if it exists, let $m \geq |c_s|$ be the least odd integer such that $\varphi_n(m) \downarrow$ is odd and $|c_s| \leq \varphi_n(m) \neq m$. As above, extend each c_s to a sequence $d_s = c_s * \gamma_s$ of odd length such that $\gamma_s(m) = 1$ and $\gamma_s(\varphi_n(m)) = 0$. If no such m exists, then we define $d_s = c_s$.

(iv) Finally for each $i \in \{0, 1\}$, let $a_{s*i} = d_s * i * \delta_s$, where the odd values of δ_s mimic those of a_s. (We have included δ_s to ensure that $E_0 \upharpoonright [T]$ is the identity relation.)

Clearly $T = \{ t \in 2^{<\mathbb{N}} \mid (\exists s)\, t \subseteq a_s \}$ is a pointed tree such that $T \equiv_T \mathbf{0}'$. To see that $\equiv_1 \upharpoonright [T]$ is the identity relation, suppose that $x \in [T]$ and that π is a recursive permutation such that $\pi(x) \in [T]$. Since there are infinitely many n such that $\varphi_n = \pi$, clauses (i) and (ii) ensure that the symmetric difference $\pi(\mathbb{E}) \bigtriangleup \mathbb{E}$ is finite, where $\mathbb{E}$ is the set of even natural numbers. Similarly, clause (iii) ensures that the set $\{ k \in \mathbb{N} \mid k \text{ is odd and } \pi(k) \neq k \}$ is finite. It follows easily that $\pi(x)\, E_0\, x$. Since $E_0 \upharpoonright [T]$ is the identity relation, it follows that $\pi(x) = x$. This completes the proof of Theorem 4.9.

5. Concluding remarks

In this section, we shall sketch an alternative approach to Theorem 4.5, which relies on a consequence of Martin's Conjecture on degree invariant Borel maps. Here, by Martin's Conjecture, we mean the following special case of a more general conjecture (also known as the 5th Victoria Delfino Problem) which was formulated by Martin in Kechris-Moschovakis [8].

Martin's Conjecture. *If $f : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ is a Borel homomorphism from $\equiv_T$ to $\equiv_T$, then exactly one of the following conditions holds:*

(i) *There exists a cone $C \subseteq 2^{\mathbb{N}}$ such that f maps C into a single $\equiv_T$-class.*

(ii) *There exists a cone $C \subseteq 2^{\mathbb{N}}$ such that $x \leq_T f(x)$ for all $x \in C$.*

Martin's Conjecture has many interesting consequences for the class of weakly universal countable Borel equivalence relations. For example, it implies that $\equiv_T$ is not countable universal and it implies the existence of uncountably many weakly universal countable Borel equivalence relations up to Borel bireducibility. (It is currently not known whether there are any weakly universal relations which are not countable universal. For a fuller discussion, see Dougherty-Kechris [4] and Thomas [13].)

My original approach to Theorem 4.5 depended upon an appeal to Martin's Conjecture. More specifically, let $M = \{ z \in 2^{\mathbb{N}} \mid [z]_{\equiv_T} \text{ is a minimal degree} \}$ and let $\theta : M \to 2^{\mathbb{N}}$ be the Borel map defined by $\theta(z) = z'$. Then by the Cooper Jump Inversion Theorem [2], for every $\mathbf{0}' \leq_T x \in 2^{\mathbb{N}}$, there exists $z \in M$ such that $\theta(z) \equiv_T x$. Thus, letting $C = \{ x \in 2^{\mathbb{N}} \mid \mathbf{0}' \leq_T x \}$ and $Y = \theta(M)$, it follows that Y is a complete Borel section for $\equiv_T \restriction C$. Hence, by Lemma 4.6, in order to show that $\equiv_1$ is not a normal subrelation of $\equiv_T$, it is enough to prove that $(\equiv_1 \restriction C) \not\leq^w_B (\equiv_1 \restriction Y)$. As we noted in Section 4, $(\equiv_1 \restriction C)$ is weakly universal. Hence, since θ witnesses that $(\equiv_T \restriction M) \sim_B (\equiv_1 \restriction Y)$, it is enough to prove the following conjecture.

Conjecture 5.1. $\equiv_T \restriction M$ is not weakly universal.

As pointed out in Thomas [13, Corollary 2.4], Martin's Conjecture implies that if $A \subseteq 2^{\mathbb{N}}$ is a $\equiv_T$-invariant Borel subset, then $\equiv_T \restriction A$ is weakly universal iff A contains a cone. In particular, Conjecture 5.1 follows from Martin's Conjecture. Unfortunately, there are currently no naturally occurring classes $D \subseteq 2^{\mathbb{N}}$ of degrees for which it is known that $\equiv_T \restriction D$ is not weakly universal.

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