TOPOLOGICAL FULL GROUPS OF MINIMAL SUBSHIFTS AND THE CLASSIFICATION PROBLEM FOR FINITELY GENERATED COMPLETE GROUPS

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ABSTRACT. Using results on the structure of the topological full groups of minimal subshifts, we prove that the isomorphism relation on the space of finitely generated complete groups is not smooth.

1. INTRODUCTION

If G is a group and $g \in G$ is any element, then the corresponding *inner automorphism* $i_g \in \mathrm{Aut}(G)$ is defined by

$$i_g(x) = g\,x\,g^{-1}, \qquad x \in G.$$

A group G is said to be *complete* if G is centerless and every automorphism of G is inner. Let $\mathcal{FG}$ be the space of finitely generated groups and let

$$\mathcal{FG}_{cmp} = \{\, G \in \mathcal{FG} \mid G \text{ is complete }\}.$$

(Here $\mathcal{FG}$ denotes the Polish space of marked finitely generated groups, which was introduced by Grigorchuk [10]; i.e. the elements of $\mathcal{FG}$ are the isomorphism types of *marked groups* $(G, \bar{c})$, where G is a finitely generated group and $\bar{c}$ is a finite sequence of generators.) Then it is easily checked that $\mathcal{FG}_{cmp}$ is a Borel subset of $\mathcal{FG}$ and hence $\mathcal{FG}_{cmp}$ is a standard Borel space. The main result of this paper constitutes the first step in the project of analyzing the Borel complexity of the isomorphism relation on $\mathcal{FG}_{cmp}$.

Theorem 1.1. *The isomorphism relation on the space $\mathcal{FG}_{cmp}$ of finitely generated complete groups is not smooth.*

The proof of Theorem 1.1 makes use of results on the structure of the topological full groups of minimal subshifts. More precisely, if (X, T) is a minimal subshift and

$[[\,T\,]]$ is the corresponding topological full group, then the commutator subgroup $[[\,T\,]]'$ is an infinite finitely generated simple group and hence $\mathrm{Aut}([[\,T\,]]')$ is a (not necessarily finitely generated) complete group. Furthermore, if $(\,Y,S\,)$ is another minimal subshift, then $\mathrm{Aut}([[\,T\,]]') \cong \mathrm{Aut}([[\,S\,]]')$ if and only if $(\,X,T\,)$ and $(\,Y,S\,)$ are flip conjugate. (A fuller discussion, including the relevant definitions, will be presented in Section 3.) Hence, in order to prove Theorem 1.1, it is enough to find a standard Borel space $\mathcal{M}$ of minimal subshifts such that the following conditions are satisfied:

- $\mathrm{Aut}([[\,T\,]]')$ is finitely generated for each $(\,X,T\,) \in \mathcal{M}$.
- The flip conjugacy relation on $\mathcal{M}$ is nonsmooth.

In [3], Clemens showed that the topological conjugacy relation for *arbitrary* subshifts is a universal countable Borel equivalence relation. Unfortunately, it is currently not known whether or not the topological conjugacy relation for *minimal* subshifts is strictly more complex than the Vitali equivalence relation E_0. However, it seems reasonable to conjecture that the following strengthening of Theorem 1.1 should be true. (Of course, it would be far more interesting if Conjecture 1.2 turned out to be false.)

Conjecture 1.2. The isomorphism relation on the space $\mathcal{FG}_{cmp}$ of finitely generated complete groups is countable universal.

This paper is organized as follows. In Section 2, we will recall some basic notions and results from the theory of countable Borel equivalence relations, including the definition of the space $\mathcal{FG}$ of (marked) finitely generated groups. In Section 3, we will discuss the basic strategy that will be employed in the proof of Theorem 1.1 and we will recall some results concerning the structure of the topological full groups $[[\,T\,]]$ of Cantor minimal systems. In Sections 4 and 5, we will discuss the structure of $\mathrm{Aut}([[\,T\,]]')$; and, in particular, we will present some criteria which are sufficient to ensure that $\mathrm{Aut}([[\,T\,]]')$ is finitely generated. Finally, in Section 6, we will present the proof of Theorem 1.1.

2. Preliminaries

In this section, we will recall some basic notions and results from the theory of countable Borel equivalence relations, and we will present a brief discussion of the Polish space $\mathcal{FG}$ of marked finitely generated groups.

2.1. **Countable Borel equivalence relations.** In this subsection, we will recall some basic notions and results from the theory of countable Borel equivalence relations. (A detailed development of the theory can be found in Dougherty-Jackson-Kechris [4] and Jackson-Kechris-Louveau [14].)

If E, F are Borel equivalence relations on the standard Borel spaces X, Y, then a Borel map $f : X \to Y$ is a *homomorphism* from E to F if for all x, $y \in X$,

$$x \mathrel{E} y \quad \Longrightarrow \quad f(x) \mathrel{F} f(y).$$

If f satisfies the stronger property that for all x, $y \in X$,

$$x \mathrel{E} y \quad \Longleftrightarrow \quad f(x) \mathrel{F} f(y),$$

then f is said to be a *Borel reduction* and we write $E \leq_B F$. If both $E \leq_B F$ and $F \leq_B E$, then E and F are said to be *Borel bireducible* and we write $E \sim_B F$. Finally we write $E <_B F$ if both $E \leq_B F$ and $F \nleq_B E$.

In this paper, we will only be concerned with *countable Borel equivalence relations*; i.e. Borel equivalence relations E such that every E-equivalence class is countable. With respect to Borel reducibility, the least complex countable Borel equivalence relations are those which are *smooth*; i.e. those countable Borel equivalence relations E on a standard Borel space X such that E is Borel reducible to the identity relation Id_Y on some (equivalently every) uncountable standard Borel space Y. Next in complexity come those countable Borel equivalence relations E which are Borel bireducible with the *Vitali equivalence relation E_0*, which is defined on the space $2^{\mathbb{N}}$ of infinite binary sequences by

$$x \mathrel{E_0} y \quad \Longleftrightarrow \quad x(n) = y(n) \text{ for all but finitely many } n.$$

More precisely, by Harrington-Kechris-Louveau [12], if E is any (not necessarily countable) Borel equivalence relation, then E is nonsmooth if and only if $E_0 \leq_B E$. It turns out that there is also a most complex countable Borel equivalence relation E_∞, which is *universal* in the sense that $F \leq_B E_\infty$ for every countable Borel

equivalence relation F. (Clearly this universality property uniquely determines E_∞ up to Borel bireducibility.) Furthermore, E_∞ is strictly more complex than E_0. The universal countable Borel relation E_∞ has a number of natural realizations in many areas of mathematics, including algebra, topology and recursion theory. In particular, by Thomas-Velickovic [20], the isomorphism relation $\cong$ on the space $\mathcal{FG}$ of finitely generated groups is a universal countable Borel equivalence relation.

Finally, recall that if E, F are countable Borel equivalence relations on the standard Borel spaces X, Y, then E is said to be *weakly Borel reducible* to F if there exists a countable-to-one Borel homomorphism $f : X \to Y$ from E to F. In this case, we write $E \leq_B^w F$. In the proof of Theorem 1.1, we will make use of the following observation. (For example, see Thomas [19, Proposition 2.1].)

Proposition 2.1. *Suppose that E, F are countable Borel equivalence relations and that $E \leq_B^w F$. If E is nonsmooth, then F is also nonsmooth.*

2.2. **The space of marked finitely generated groups.** In this subsection, we will present a brief discussion of the Polish space $\mathcal{FG}$ of (marked) finitely generated groups, which is defined as follows. A marked group $(G, \bar{s})$ consists of a finitely generated group with a distinguished sequence $\bar{s} = (s_1, \cdots, s_m)$ of generators. (Here the sequence $\bar{s}$ is allowed to contain repetitions and we also allow the possibility that the sequence contains the identity element.) Two marked groups $(G, (s_1, \cdots, s_m))$ and $(H, (t_1, \cdots, t_n))$ are said to be *isomorphic* if $m = n$ and the map $s_i \mapsto t_i$ extends to a group isomorphism between G and H.

Definition 2.2. For each $m \geq 2$, let $\mathcal{FG}_m$ be the set of *isomorphism types* of marked groups $(G, (s_1, \cdots, s_m))$ with m distinguished generators.

Let $\mathbb{F}_m$ be the free group on the generators $\{x_1, \cdots, x_m\}$. Then for each marked group $(G, (s_1, \cdots, s_m))$, we can define an associated epimorphism $\theta_{G,\bar{s}} : \mathbb{F}_m \to G$ by $\theta_{G,\bar{s}}(x_i) = s_i$. It is easily checked that two marked groups $(G, (s_1, \cdots, s_m))$ and $(H, (t_1, \cdots, t_m))$ are isomorphic if and only if $\ker \theta_{G,\bar{s}} = \ker \theta_{H,\bar{t}}$. Thus we can naturally identify $\mathcal{FG}_m$ with the compact space $\mathcal{N}_m$ of normal subgroups of $\mathbb{F}_m$; and hence, via this identification, we can regard $\mathcal{FG}_m$ as a compact space.

For each $m \geq 2$, there is a natural embedding of $\mathcal{N}_m$ into $\mathcal{N}_{m+1}$ defined by

$$N \mapsto \text{ the normal closure of } N \cup \{x_{m+1}\} \text{ in } \mathbb{F}_{m+1}.$$

Thus we can identify $\mathcal{N}_m$ with the clopen subset $\{\, N \in \mathcal{N}_{m+1} \mid x_{m+1} \in N \,\}$ of $\mathcal{N}_{m+1}$ and form the locally compact Polish space $\mathcal{N} = \bigcup \mathcal{N}_m$. Note that $\mathcal{N}$ can be identified with the space of normal subgroups N of the free group $\mathbb{F}_\infty$ on countably many generators such that N contains all but finitely many elements of the basis $B = \{x_i \mid i \in \mathbb{N}^+\}$. Similarly, we can form the locally compact Polish space $\mathcal{FG} = \bigcup \mathcal{FG}_m$ of finitely generated groups via the corresponding natural embedding

$$(G, (s_1, \cdots, s_m)) \mapsto (G, (s_1, \cdots, s_m, 1))$$

From now on, we will identify $\mathcal{FG}_m$ and $\mathcal{N}_m$ with the corresponding clopen subsets of $\mathcal{FG}$ and $\mathcal{N}$. If $\Gamma \in \mathcal{FG}$, then we will write $\Gamma = (G, (s_1, \cdots, s_m))$, where m is the least integer such that $\Gamma \in \mathcal{G}_m$. Following the usual convention, we will completely identify the Polish spaces $\mathcal{FG}$ and $\mathcal{N}$; and we will work with whichever space is most convenient in any given context.

In the remaining sections of this paper, the symbol $\cong$ will always denote the usual isomorphism relation on the space $\mathcal{FG}$ of finitely generated groups; i.e. two marked groups are $\cong$-equivalent if their underlying groups (obtained by forgetting about the distinguished sequences of generators) are isomorphic. Clearly $\cong$ is a countable Borel equivalence relation on $\mathcal{FG}$. (This is one of the many advantages of working with the space $\mathcal{FG}$ of marked finitely generated groups rather than with the space $\mathcal{FG}'$ associated with the logic action.)

Finally, we should mention that we will often slightly abuse notation and denote the elements of $\mathcal{FG}$ by G, H, etc. instead of the more accurate $(G, \bar{c})$, $(H, \bar{d})$, etc.

3. Full groups of Cantor minimal systems

In this section, we will discuss the basic strategy that will be employed in the proof of Theorem 1.1. First it is necessary to say a few words on the structure of the topological full groups of Cantor minimal systems. Let (X, T) be a *Cantor dynamical system*; i.e. X is a Cantor set and $T : X \to X$ is a homeomorphism. Then (X, T) is said to be a *Cantor minimal system* if X has no nonempty proper closed T-invariant subsets.

Definition 3.1. If (X, T) is a Cantor minimal system, then the *topological full group* $[[T]]$ is the group of all homeomorphisms $\pi : X \to X$ such that there exists

a partition $X = C_1 \sqcup \cdots \sqcup C_m$ into clopen subsets and $\ell_1, \cdots, \ell_m \in \mathbb{Z}$ such that $\pi \upharpoonright C_i = T^{\ell_i} \upharpoonright C_i$ for each $1 \leq i \leq m$.

The Cantor minimal systems (X, T) and (Y, S) are said to be *topologically conjugate* if there exists a homeomorphism $\pi : X \to Y$ such that $\pi \circ T = S \circ \pi$. If (X, T) is topologically conjugate to either (Y, S) or (Y, S^{-1}), then (X, T) and (Y, S) are said to be *flip conjugate*. The following theorem combines the work of Giordano-Putnam-Skau [8] and Bezuglyi-Medynets [1].

Theorem 3.2. *If (X, T), (Y, S) are Cantor minimal systems, then the following statements are equivalent.*

(i) *(X, T), (Y, S) are flip conjugate.*

(ii) *The topological full groups $[[T]]$, $[[S]]$ are isomorphic as abstract groups.*

(iii) *The commutator subgroups $[[T]]'$, $[[S]]'$ are isomorphic as abstract groups.*

If $n \geq 2$, then the shift transformation σ on the Cantor space $n^{\mathbb{Z}}$ is defined by $\sigma(x)_k = x_{k+1}$. An infinite subset $X \subseteq n^{\mathbb{Z}}$ is said to be a *subshift* if X is a closed σ-invariant subset. The subshift X is *minimal* if the corresponding Cantor dynamical system (X, T) is minimal, where $T = \sigma \upharpoonright X$. In this case, we also say that (X, T) is a minimal subshift. The following result is due to Matui [15, Theorem 5.4].

Theorem 3.3. *Let (X, T) be a Cantor minimal system.*

(a) *The commutator subgroup $[[T]]'$ is an infinite simple group.*

(b) *The commutator subgroup $[[T]]'$ is finitely generated if and only if (X, T) is topologically conjugate to a minimal subshift over a finite alphabet.*

Remark 3.4. In the statements of his results in [15], Matui always refers to $[[T]]'_0$ rather than $[[T]]'$. (The subgroup $[[T]]_0 \leqslant [[T]]$ will be defined in Section 4.) However, as Matui points out in the proof of [15, Corollary 5.5], the quotient group $[[T]]/[[T]]'_0$ is abelian and so it follows that $[[T]]'_0 = [[T]]'$.

The proof of Theorem 1.1 is based upon the following two easy corollaries of Theorems 3.2 and 3.3.

Corollary 3.5. *If (X, T) is a Cantor minimal system, then the automorphism group $\mathrm{Aut}([[T]]')$ is complete.*

Proof. By a classical result of Burnside, if S is a nonabelian simple group, then $\mathrm{Aut}(S)$ is complete. (For example, see Robinson [16, 13.5.10].) $\qquad\square$

Corollary 3.6. *If* (X, T), (Y, S) *are Cantor minimal systems, then the following are equivalent.*

(i) (X, T), (Y, S) *are flip conjugate.*

(iii) *The automorphism subgroups* $\mathrm{Aut}([[T]]')$, $\mathrm{Aut}([[S]]')$ *are isomorphic as abstract groups.*

Proof. It is well-known that if S is a simple nonabelian group, then S is the unique minimal nontrivial normal subgroup of $\mathrm{Aut}(S)$. (For example, once again, see Robinson [16, 13.5.10].) Hence if $\pi : \mathrm{Aut}([[T]]') \to \mathrm{Aut}([[S]]')$ is an isomorphism, then $\pi([[T]]') = [[S]]'$, and so the result follows from Theorem 3.2. $\qquad\square$

Let $\mathcal{M}_2$ be the set of minimal subshifts $X \subseteq 2^{\mathbb{Z}}$. Then $\mathcal{M}_2$ is a Borel subset of the standard Borel space $K(2^{\mathbb{Z}})$ of closed subspaces of $2^{\mathbb{Z}}$ and thus $\mathcal{M}_2$ is a standard Borel space. By Clemens [3, Lemma 9], the topological conjugacy relation E_{tc} is a countable Borel equivalence relation on $\mathcal{M}_2$; and, of course, this implies that the flip conjugacy relation E_{fp} is also a countable Borel equivalence relation. Furthermore, by Thomas [19, Section 4], the relations E_{tc} and E_{fp} are both nonsmooth. This suggests that it should be possible to prove the nonsmoothness of the isomorphism relation on the space $\mathcal{FG}_{cmp}$ of finitely generated complete groups by a consideration of the groups $\mathrm{Aut}([[T_X]]')$ for suitably chosen $X \in \mathcal{M}_2$, where $T_X = \sigma \restriction X$. This will indeed turn out to be the case. However, there is a complication; namely, there exist examples of minimal subshifts $X \in \mathcal{M}_2$ such that the complete group $\mathrm{Aut}([[T_X]]')$ is not finitely generated. (For example, see Salo [17].)

4. The structure of $\mathrm{Aut}([[T]]')$

Let (X, T) be a Cantor minimal system. In this section, we will discuss the structure of $\mathrm{Aut}([[T]]')$; and, in particular, we will present some criteria which are sufficient to ensure that $\mathrm{Aut}([[T]]')$ is finitely generated.

Let $H = \mathrm{Homeo}(X)$ be the group of all homeomorphisms of the Cantor set X. Then the following result is implicitly contained in Bezuglyi-Medynets [1].

Theorem 4.1. *If (X, T) is a Cantor minimal system, then*

$$\mathrm{Aut}([[T]]') \cong N_H([[T]]).$$

Proof. By Bezuglyi-Medynets [1, Theorem 5.8], if Γ is either $[[T]]$ or $[[T]]'$ and $\pi \in \mathrm{Aut}(\Gamma)$, then there exists $h \in H$ such that $\pi(g) = h\, g\, h^{-1}$ for all $g \in \Gamma$. Thus, in both cases, we can identify $\mathrm{Aut}(\Gamma)$ with $N_H(\Gamma)$. Since $[[T]]'$ is a characteristic subgroup of $[[T]]$, it follows that $N_H([[T]]) \leqslant N_H([[T]]')$. On the other hand, applying Bezuglyi-Medynets [1, Lemma 5.12], it follows that $N_H([[T]]') \leqslant N_H([[T]])$. $\square$

Let $C(X, \mathbb{Z})$ be the space of continuous maps $f : X \to \mathbb{Z}$ and let μ be any T-invariant probability measure on X. For each $g \in [[T]]$, let $n_g \in C(X, \mathbb{Z})$ be the continuous map defined by $g(x) = T^{n_g}(x)$ for each $x \in X$. Then the *index map* is defined by

$$I(g) = \int n_g \, d\mu, \qquad g \in [[T]].$$

By Giordano-Putnam-Skau [8, Section 5], the index map is a group homomorphism $I : [[T]] \to \mathbb{Z}$. Furthermore, the index map I does not depend on the choice of the T-invariant probability measure μ.

Definition 4.2. The kernel of the index map I is denoted by $[[T]]_0$.

Of course, we have that $[[T]]' \leqslant [[T]]_0$.

Definition 4.3. If (X, T) is a Cantor minimal system, then:

- $C(T) = \{\, h \in \mathrm{Homeo}(X) \mid hTh^{-1} = T \,\}$.
- $C^\varepsilon(T) = \{\, h \in \mathrm{Homeo}(X) \mid hTh^{-1} = T \text{ or } hTh^{-1} = T^{-1} \,\}$.

The following result is due to Giordano-Putnam-Skau [8, Corollary 5.12].

Theorem 4.4. *If (X, T) is a Cantor minimal system, then*

$$N_H([[T]]) \cong [[T]]_0 \rtimes C^\varepsilon(T).$$

In particular, if $[[T]]_0$ and $C^\varepsilon(T)$ are both finitely generated, then $\mathrm{Aut}([[T]]')$ is also finitely generated.

Definition 4.5. If (X, T) is a Cantor minimal system, then the corresponding K^0-group is defined to be

$$K^0(X, T) = C(X, \mathbb{Z})/B_T,$$

where $B_T = \{\, f - f \circ T^{-1} \mid f \in C(X, \mathbb{Z}) \,\}$.

The following result is also due to Matui [15, Theorem 4.8].

Theorem 4.6. *If (X, T) is a Cantor minimal system, then*

$$[[T]]_0 / [[T]]' \cong K^0(X, T) / 2K^0(X, T).$$

In particular, if (X, T) is topologically conjugate to a minimal subshift over a finite alphabet and $K^0(X, T)/2K^0(X, T)$ is a finite group, then $[[T]]_0$ is finitely generated.

5. The structure of $K^0(X, T)/2K^0(X, T)$

In this section, we will explain how to compute the group $K^0(X, T)/2K^0(X, T)$; and, in the case when $K^0(X, T)/2K^0(X, T)$ is a finite group, we will explain how to find a finite subset $S \subseteq [[T]]_0$ such that $[[T]]' \cup S$ generates $[[T]]_0$. First it is necessary to describe how to associate a Bratteli diagram with any Cantor minimal system (X, T).

Definition 5.1. A *Bratteli diagram* (V, E) consists of a set $V(B) = \bigsqcup_{n \in \mathbb{N}} V_n$ of *vertices*, where each V_n is a finite nonempty set, and a set $E(B) = \bigsqcup_{n \in \mathbb{N}^+} E_i$ of *edges*, where each E_n is a finite nonempty set, with the following properties:

(i) $V_0 = \{\, v_0 \,\}$ consists of a single vertex.

(ii) There exist a *source map* $\mathbf{s} : E \to V$ and a *range map* $\mathbf{r} : E \to V$ such that
   - $\mathbf{r}(E_n) \subseteq V_n$ and $\mathbf{s}(E_n) \subseteq V_{n-1}$;
   - $\mathbf{s}^{-1}(v) \neq \emptyset$ for all $v \in V$;
   - $\mathbf{r}(v) \neq \emptyset$ for all $v \in V \smallsetminus V_0$.

If $\mathbf{s}(e) = u$ and $\mathbf{r}(e) = v$, then we say that the edge e connects the vertices u and v.

If $V_n = \{\, u_1, \cdots, u_k \,\}$ and $V_{n+1} = \{\, v_1, \cdots, v_m \,\}$, then we define the corresponding *nth incidence matrix* to be the $m \times k$ matrix $M_n = (\, m_{ji} \,)$, where m_{ji} is the number of edges connecting u_i and v_j. For each $n \in \mathbb{N}$, let $\mathbb{Z}^{V_n}$ be the free group on the set V_n. Then the abelian group $K_0(V, E)$ is the inductive limit of

$$\mathbb{Z}^{V_0} \xrightarrow{\varphi_0} \mathbb{Z}^{V_1} \xrightarrow{\varphi_1} \mathbb{Z}^{V_2} \xrightarrow{\varphi_2} \dots \xrightarrow{\varphi_{n-1}} \mathbb{Z}^{V_n} \xrightarrow{\varphi_n} \mathbb{Z}^{V_{n+1}} \xrightarrow{\varphi_{n+1}} \dots$$

where the homomorphism φ_n is given by matrix multiplication by the nth incidence matrix M_n. (In fact, $K_0(V, E)$ comes equipped with the richer structure of an ordered group with a distinguished order unit. However, we will not make use of this richer structure in this paper.)

From now on, let (X, T) be a fixed Cantor minimal system. Then, following Herman-Putnam-Skau [13], we can associate a corresponding Bratteli diagram as follows.

Definition 5.2. If $\{ A_i \in | \ i \in I \}$ are clopen subsets of X and $\{ h_i \mid i \in I \}$ are positive integers such that

$$\mathcal{P} = \{ T^\ell(A_i) \mid 0 \le \ell < h_i, i \in I \}$$

is a partition of X, then $\mathcal{P}$ is called a *Kakutani-Rokhlin partition*. The clopen set $B(\mathcal{P}) = \bigsqcup_{i \in I} A_i$ is called the *base* of $\mathcal{P}$ and the clopen set $H(\mathcal{P}) = \bigsqcup_{i \in I} T^{h_i - 1}(A_i)$ is called the *top* of $\mathcal{P}$.

Warning 5.3. Here we follow the conventions of Gjerde-Johansen [9], Grigorchuk-Medynets [11] and Skau [18]. On the other hand, in Giordano-Putnam-Skau [7] and Herman-Putnam-Skau [13], the roles of the bases and tops are reversed. For this reason, depending on the conventions adopted by the authors, it is occasionally necessary to slightly re-word the results in the literature.

Fix any point $x_0 \in X$. Then, by Herman-Putnam-Skau [13, Theorem 4.2], there exists a sequence $\mathcal{P} = (\mathcal{P}_n)_{n \ge 0}$ of Kakutani-Rokhlin partitions

$$\mathcal{P}_n = \{ T^\ell(A_i^{(n)}) \mid 0 \le \ell < h_i, i \in I_n \}$$

satisfying the following conditions:

(a) $\mathcal{P}_0 = \{ A_0^0 \} = \{ X \}$;

(b) $\mathcal{P}_{n+1}$ strictly refines $\mathcal{P}_n$ for each $n \ge 1$;

(c) $\bigcup_{n \ge 0} \mathcal{P}_n$ generates the topology of X;

(d) $\bigcap_{n \ge 0} B(\mathcal{P}_n) = \{ x_0 \}$.

In addition, after deleting finitely many partitions, $\mathcal{P}_1, \cdots, \mathcal{P}_r$ if necessary, we can suppose that $\mathcal{P} = (\mathcal{P}_n)_{n \ge 0}$ also satisfies the following condition:

(e) For all $n \ge 1$ and $i \in I_n$, the integer $h_i \ge 5$.

Given such a sequence $\mathcal{P} = (\mathcal{P}_n)_{n \geq 0}$ of Kakutani-Rokhlin partitions, we can define a corresponding Bratteli diagram $(V_\mathcal{P}, E_\mathcal{P})$ with vertex set $V_\mathcal{P} = \bigsqcup_{n \geq 0} V_n$ such that:

- $V_n = \{A_i^{(n)} \mid i \in I_n\}$.
- If $A_i^{(n)} \in V_n$ and $A_j^{(n+1)} \in V_{n+1}$, then the number of edges from $A_i^{(n)}$ to $A_j^{(n+1)}$ is equal to the number of elements $A \in \{T^\ell(A_j^{(n+1)}) \mid 0 \leq \ell < h_j\}$ such that $A \subseteq A_i^{(n)}$.

The following result is due to Herman-Putnam-Skau [13, Theorem 5.4].

Theorem 5.4. *With the above notation, $K^0(X, T) \cong K_0(V_\mathcal{P}, E_\mathcal{P})$.*

In the remainder of this section, we will collect together some results which, in the case when $K^0(X, T)/2K^0(X, T)$ is a finite group, will enable us to find a finite subset $S \subseteq [[T]]_0$ such that $[[T]]' \cup S$ generates $[[T]]_0$. We will continue to work with a fixed sequence $\mathcal{P} = (\mathcal{P}_n)_{n \geq 0}$ of Kakutani-Rokhlin partitions as above.

Definition 5.5. Let $n \geq 1$. Then an element $g \in [[T]]$ is an *n-permutation* if for each $i \in I_n$ and $0 \leq \ell < h_i$, there exists an integer $k_{i\ell} \in \mathbb{Z}$ such that:

(a) $g \upharpoonright T^\ell(A_i^{(n)}) = T^{k_{i\ell}} \upharpoonright T^\ell(A_i^{(n)})$;

(b) $0 \leq k_{i\ell} + \ell < h_i$.

Remark 5.6. It follows that for each $i \in I_n$, there exists a permutation π_i of $\{0, 1, \cdots, h_i - 1\}$ such that $k_{i\ell} + \ell = \pi_i(\ell)$ and hence $g(T^\ell(A_i^{(n)})) = T^{\pi_i(\ell)}(A_i^{(n)})$. Thus g permutes the atoms of the partition $\mathcal{P}_n$. It also follows that $\sum_{0 \leq \ell < h_i} k_{i\ell} = 0$ for each $i \in I_n$ and this easily implies that $g \in [[T]]_0$.

For each $n \geq 1$, let G_n be the group of n-permutations. Then $G_n \leqslant [[T]]_0$ and it is easily seen that $G_n \leqslant G_{n+1}$. More precisely, for each $n \geq 1$ and $i \in I_n$, let

$$\Delta_i = \{T^\ell(A_i^{(n)}) \mid 0 \leq \ell < h_i\}.$$

Then $G_n = \prod_{i \in I_n} \mathrm{Sym}(\Delta_i)$; and, letting $M_n = (m_{ji})$ be the nth incidence matrix of the Bratteli diagram $(V_\mathcal{P}, E_\mathcal{P})$, the inclusion $G_n \leqslant G_{n+1}$ is the *block diagonal* embedding $\prod_{i \in I_n} \mathrm{Sym}(\Delta_i) \hookrightarrow \prod_{j \in I_{n+1}} \mathrm{Sym}(\Delta_j)$ such that:

- for each $j \in I_{n+1}$ and G_n-orbit $\Sigma \subseteq \Delta_j$, there exists a unique $i \in I_n$ such that $\mathrm{Sym}(\Delta_i)$ acts naturally on Σ and $\mathrm{Sym}(\Delta_{i'})$ acts trivially on Σ for each $i' \in I_n \smallsetminus \{i\}$;

- if $i \in I_n$ and $j \in I_{n+1}$, then $\mathrm{Sym}(\Delta_i)$ has m_{ji} natural orbits on Δ_j.

Definition 5.7. $G_{\mathcal{P}} = \bigcup_{n \geq 1} G_n$ is the *group of permutations* associated with the sequence $\mathcal{P} = (\mathcal{P}_n)_{n \geq 0}$ of Kakutani-Rokhlin partitions.

Next, for each $n \geq 1$, let $H_n = \prod_{i \in I_n} \mathrm{Alt}(\Delta_i) \trianglelefteq G_n$. Since $h_i = |\Delta_i| \geq 5$ for all $n \geq 1$ and $i \in I_n$, it follows that $H_n = G_n'$ and hence $G_n/G_n' \cong \mathbb{Z}_2^{I_n} \cong \mathbb{Z}_2^{V_n}$. (Here $\mathbb{Z}_2$ is the cyclic group of order 2.) Furthermore, the natural homomorphism $G_n/G_n' \to G_{n+1}/G_{n+1}'$ corresponds to the homomorphism

$$\mathbb{Z}_2^{V_n} = \mathbb{Z}^{V_n}/2\mathbb{Z}^{V_n} \to \mathbb{Z}^{V_{n+1}}/2\mathbb{Z}^{V_{n+1}} = \mathbb{Z}_2^{V_{n+1}}$$

induced by the homomorphism $\mathbb{Z}^{V_n} \xrightarrow{\varphi_n} \mathbb{Z}^{V_{n+1}}$, where φ_n is given by matrix multiplication by the nth incidence matrix M_n. Hence the quotient $G_{\mathcal{P}}/G_{\mathcal{P}}'$ is equal to the inductive limit

$$\mathbb{Z}_2^{V_1} \to \mathbb{Z}_2^{V_2} \to \cdots \to \mathbb{Z}_2^{V_n} \to \mathbb{Z}_2^{V_{n+1}} \to \cdots$$

and it follows that

$$(5.8) \qquad G_{\mathcal{P}}/G_{\mathcal{P}}' \cong K_0(V_{\mathcal{P}}, E_{\mathcal{P}})/2K_0(V_{\mathcal{P}}, E_{\mathcal{P}}) \cong K^0(X, T)/2K^0(X, T);$$

and thus, applying Theorem 4.6, we obtain that $G_{\mathcal{P}}/G_{\mathcal{P}}' \cong [[T]]_0/[[T]]'$. In fact, as we will next explain, the isomorphism between $G_{\mathcal{P}}/G_{\mathcal{P}}'$ and $[[T]]_0/[[T]]'$ is completely canonical. Recall that $G_{\mathcal{P}} \leqslant [[T]]_0$ and that $[[T]]' = [[T]]_0'$.

Proposition 5.9. $G_{\mathcal{P}} \cap [[T]]' = G_{\mathcal{P}}'$ and the canonical induced injection

$$G_{\mathcal{P}}/G_{\mathcal{P}}' \hookrightarrow [[T]]_0/[[T]]'$$

is an isomorphism.

Proof sketch. Let $\mathrm{Orb}^+(x_0) = \{\, T^r(x_0) \mid r \geq 0 \,\}$ be the forward orbit of x_0 and let

$$[[T]]_{x_0} = \{\, g \in [[T]] \mid g(\mathrm{Orb}^+(x_0)) = \mathrm{Orb}^+(x_0) \,\}.$$

Then, by Grigorchuk-Medynets [11, Section 5], we have that $G_{\mathcal{P}} = [[T]]_{x_0}$. Let

$$\mathrm{sgn} : [[T]]_{x_0} = G_{\mathcal{P}} \to K^0(X, T)/2K^0(X, T)$$

be the composition of the quotient map $G_{\mathcal{P}} \to G_{\mathcal{P}}/G'_{\mathcal{P}}$ and the isomorphism (5.8). Then, by Matui [15, Section 4], the surjective homomorphism sgn extends to a surjective homomorphism

$$\mathrm{sgn} : [[\,T\,]]_0 \to K^0(X,T)/2K^0(X,T)$$

such that $\ker \mathrm{sgn} = [[\,T\,]]'_0 = [[\,T\,]]'$. The result follows.   □

Let $g \mapsto \bar{g}$ be the quotient map $G_{\mathcal{P}} \to G_{\mathcal{P}}/G'_{\mathcal{P}}$.

Corollary 5.10. *If $S \subseteq G_{\mathcal{P}}$ is a subset such that $\{\,\bar{s} \mid s \in S\,\}$ generates $G_{\mathcal{P}}/G'_{\mathcal{P}}$, then $[[\,T\,]]' \cup S$ generates $[[\,T\,]]_0$.*

6. THE PROOF OF THEOREM 1.1

The basic idea of the proof of Theorem 1.1 is easily explained. Let $\mathrm{Ec}(2^{\mathbb{N}})$ be the set of eventually constant sequences $z \in 2^{\mathbb{N}}$ and let $\mathrm{Nec}(2^{\mathbb{N}}) = 2^{\mathbb{N}} \smallsetminus \mathrm{Ec}(2^{\mathbb{N}})$. Also let $T = \{\, 2^{m+1} - 1 \mid m \in \mathbb{N}^+ \,\}$ and let

$$\mathrm{Thin}(2^{\mathbb{N}}) = \{\, z \in \mathrm{Nec}(2^{\mathbb{N}}) \mid z(n) = 0 \text{ for all } n \in \mathbb{N} \smallsetminus T \,\}.$$

Then it is easily checked that $E_0 \upharpoonright \mathrm{Thin}(2^{\mathbb{N}})$ is Borel bireducible with E_0 and hence $E_0 \upharpoonright \mathrm{Thin}(2^{\mathbb{N}})$ is nonsmooth. Finally, recall that $\mathcal{M}_2$ is the standard Borel space of minimal subshifts of $2^{\mathbb{Z}}$. Then Theorem 1.1 is a consequence of the following result.

Lemma 6.1. *There exist Borel maps $z \overset{\varphi}{\mapsto} X_z$ from $\mathrm{Thin}(2^{\mathbb{N}})$ to $\mathcal{M}_2$ and $z \overset{\psi}{\mapsto} G_z$ from $\mathrm{Thin}(2^{\mathbb{N}})$ to $\mathcal{FG}$ such that, letting $T_z = \sigma \upharpoonright X_z$, the following conditions are satisfied:*

(i) *$z \overset{\varphi}{\mapsto} X_z$ is an injective Borel homomorphism from $E_0 \upharpoonright \mathrm{Thin}(2^{\mathbb{N}})$ to the topological conjugacy relation E_{tc} on $\mathcal{M}_2$;*

(ii) *$G_z \cong \mathrm{Aut}([[\,T_z\,]]')$ for each $z \in \mathrm{Thin}(2^{\mathbb{N}})$.*

Proof of Theorem 1.1. By Proposition 2.1, since $E_0 \upharpoonright \mathrm{Thin}(2^{\mathbb{N}})$ is nonsmooth, it is enough to show that the map $z \overset{\psi}{\mapsto} G_z$ is a weak Borel reduction from $E_0 \upharpoonright \mathrm{Thin}(2^{\mathbb{N}})$ to the isomorphism relation on the space $\mathcal{FG}_{cmp}$ of finitely generated complete groups. To see this, first recall that by Corollary 3.5, if (X,T) is any Cantor minimal system, then $\mathrm{Aut}([[\,T\,]]')$ is a complete group. Hence, since $G_z \in \mathcal{FG}$ and $G_z \cong \mathrm{Aut}([[\,T_z\,]]')$, it follows that $G_z \in \mathcal{FG}_{cmp}$ for each $z \in \mathrm{Thin}(2^{\mathbb{N}})$.

Next note that if y, $z \in \mathrm{Thin}(2^{\mathbb{N}})$ and $y \, E_0 \, z$, then the Cantor minimal systems (X_y, T_y) and (X_z, T_z) are topologically conjugate; and hence, by Corollary 3.6,

$$G_y \cong \mathrm{Aut}([[\,T_y\,]]') \cong \mathrm{Aut}([[\,T_z\,]]') \cong G_z.$$

Thus, the map $z \overset{\psi}{\mapsto} G_z$ is a Borel homorphism from $E_0 \restriction \mathrm{Thin}(2^{\mathbb{N}})$ to the isomorphism relation on $\mathcal{FG}_{cmp}$.

Finally, again by Corollary 3.6, if y, $z \in \mathrm{Thin}(2^{\mathbb{N}})$ and $G_y = G_z$, then (X_y, T_y) and (X_z, T_z) are flip conjugate. Using the facts that flip conjugacy is a countable Borel equivalence relation on $\mathcal{M}_2$ and that the map $z \mapsto X_z$ is injective, it follows that ψ is countable-to-one. Thus ψ is a weak Borel reduction from $E_0 \restriction \mathrm{Thin}(2^{\mathbb{N}})$ to the isomorphism relation on $\mathcal{FG}_{cmp}$ $\hfill \square$

The remainder of this section is devoted to the proof of Lemma 6.1. In fact, we will show that the construction in Thomas [19], which associates a Toeplitz flow $X_z \in \mathcal{M}_2$ to each $z \in \mathrm{Thin}(2^{\mathbb{N}})$, satisfies our requirements. Throughout this section, if $Y \subseteq 2^{\mathbb{Z}}$, then its closure will be denoted by $\overline{Y}$.

Definition 6.2. An element $x \in 2^{\mathbb{Z}}$ is said to be a *Toeplitz sequence* if for all $a \in \mathbb{Z}$, there exists $b \in \mathbb{N}^+$ such that $x(a + kb) = x(a)$ for all $k \in \mathbb{Z}$.

It is well-known that if x is a nonperiodic Toeplitz sequence, then the *Toeplitz flow* $X = \overline{\{\, \sigma^n(x) \mid n \in \mathbb{Z} \,\}}$ is a minimal subshift of $2^{\mathbb{Z}}$. (For the basic theory of Toeplitz flows, see Downarowicz [5].) For each $z \in 2^{\mathbb{N}}$, let $\tilde{z} \in 2^{\mathbb{Z}}$ be the Toeplitz sequence defined as follows.

- For each $m \geq 1$, let $B_m = [\,0, 2^m - 1\,]$ and suppose inductively that we have defined the value $\tilde{z}(\ell)$ for all integers $\ell \in B_m \smallsetminus \{\, a_m, b_m \,\}$, where $0 \leq a_m < b_m \leq 2^m - 1$. Let $c_m = a_m$ if m is odd and $c_m = b_m$ if m is even. Then for all $k \in \mathbb{Z}$, we define

$$\tilde{z}(c_m + k\,2^m) = z(m - 1).$$

For example, at the beginning of stage 3 of the construction, $\tilde{z} \restriction B_3$ is given by

$$z(0) \, * \, z(0) \, z(1) \, z(0) \, * \, z(0) \, z(1)$$

where the $*$ indicates that the value has not yet been defined. We then define $\tilde{z}(1 + 8k) = z(2)$ for all $k \in \mathbb{Z}$ and hence obtain that for all $k \in \mathbb{Z}$,

$$\tilde{z} \restriction [\, 8k, 8(k+1)\,) = z(0)\, z(2)\, z(0)\, z(1)\, z(0)\, *\, z(0)\, z(1)$$

Remark 6.3. For later use, note that $\lim_{m \to \infty} \min\{\, a_m, 2^m - 1 - b_m\,\} = \infty$.

It is easily checked that if $z \in 2^{\mathbb{N}}$, then $\tilde{z}$ is a periodic sequence if and only if $z \in \mathrm{Ec}(2^{\mathbb{N}})$ is an eventually constant sequence. For each sequence $z \in \mathrm{Nec}(2^{\mathbb{N}})$, let $X_z = \overline{\{\, \sigma^n(\tilde{z}) \mid n \in \mathbb{Z}\,\}}$ be the corresponding Toeplitz flow and let $T_z = \sigma \restriction X_z$. The following results were proved in Thomas [19, Section 4].

Proposition 6.4. *If y, $z \in \mathrm{Nec}(2^{\mathbb{N}})$ and $y\, E_0\, z$, then the Toeplitz flows $(\, X_y, T_y\,)$ and $(\, X_z, T_z\,)$ are topologically conjugate.*

Proposition 6.5. *The Borel map $z \mapsto X_z$ is injective on $\mathrm{Thin}(2^{\mathbb{N}})$.*

Combining these results, we see that the map $z \mapsto X_z$ is an injective Borel homomorphism from $E_0 \restriction \mathrm{Thin}(2^{\mathbb{N}})$ to the topological conjugacy relation E_{tc} on $\mathcal{M}_2$. (It is not clear whether the map $z \mapsto X_z$ is countable-to-one on the whole of $\mathrm{Nec}(2^{\mathbb{N}})$.)

Until further notice, we will fix some sequence $z \in \mathrm{Nec}(2^{\mathbb{N}})$. We will begin our analysis of the structure of $\mathrm{Aut}([[\,T_z\,]]')\cong [[\,T_z\,]]_0 \rtimes C^{\varepsilon}(T_z)$ by computing $C^{\varepsilon}(T_z) = \{\, h \in \mathrm{Homeo}(X_z) \mid hT_zh^{-1} = T_z \text{ or } hT_zh^{-1} = T_z^{-1}\,\}$.

Lemma 6.6. $C(T_z) = \{\, T_z^{\ell} \mid \ell \in \mathbb{Z}\,\}$.

Proof. Recall that at the end of stage m of the construction of $\tilde{z}$, we have that $\tilde{z} \restriction B_m = \bar{c} * \bar{d}$ and $\tilde{z} \restriction B_{m+1} = \bar{c} * \bar{d}\,\bar{c} * \bar{d}$ for some finite binary sequences $\bar{c}, \bar{d}$. Then during stage $m + 1$, one of the $*$ in $\tilde{z} \restriction B_{m+1}$ is replaced by the value $z(m)$. Hence each of the subsequences $(\tilde{z} \restriction B_{m+1}) \restriction [0, 2^m - 1]$ and $(\tilde{z} \restriction B_{m+1}) \restriction [2^m, 2^{m+1} - 1]$ either remains unchanged as $\bar{c} * \bar{d}$ or else is completely filled in as $\bar{c}\, z(m)\, \bar{d}$. Thus $\tilde{z}$ satisfies condition $(*)$ of Bułatek-Kwiatkowski [2, Theorem 1] and so the result follows by Bułatek-Kwiatkowski [2, Theorem 1]. (Toeplitz sequences satisfying condition $(*)$ are sometimes called *generalized Oxtoby sequences*. For example, see Downarowicz-Kwiatkowski-Lacroix [6].) $\qquad\square$

Definition 6.7. Let $\tau : 2^{\mathbb{Z}} \to 2^{\mathbb{Z}}$ be the *flip map* defined by $(x_{\ell}) \overset{\tau}{\mapsto} (x_{-\ell})$.

Clearly $\tau\sigma\tau^{-1} = \sigma^{-1}$ and so the following result implies that $\tau T_z \tau^{-1} = T_z^{-1}$.

Lemma 6.8. X_z *is τ-invariant.*

Proof. At the end of stage 3 of the construction, for all $k \in \mathbb{Z}$,

$$\tilde{z} \restriction [\, 8k, 8k + 7 \,] = z(0)\, z(2)\, z(0)\, z(1)\, z(0) \, * \, z(0)\, z(1),$$

and hence $\tilde{z} \restriction [\, 8k - 1, 8k + 11 \,]$ is given by

$$z(1)\, z(0)\, z(2)\, z(0)\, z(1)\, z(0) \, * \, z(0)\, z(1) z(0)\, z(2)\, z(0)\, z(1).$$

Furthermore, since z is not eventually constantly 1, for some $k \in \mathbb{Z}$, we must at some later stage replace the symbol $*$ by 0. Thus $\tilde{z}$ contains the following subsequence:

$$z(1)\, z(0)\, z(2)\, z(0)\, z(1)\, z(0)\, 0\, z(0)\, z(1) z(0)\, z(2)\, z(0)\, z(1).$$

Continuing in this fashion, we see that $s \in X_z$, where $s = (\, s(\ell)\,) \in 2^{\mathbb{Z}}$ is the sequence such that $s(0) = 0$ and

$$s(2^n + k2^{n+1}) = s(-(2^n + k2^{n+1})) = z(n)$$

for each n, $k \geq 0$. Clearly $\tau(s) = s$. Hence, for each $\ell \in \mathbb{Z}$, we have that

$$\sigma^\ell(s) = (\tau\sigma^{-\ell}\tau^{-1})(s) = (\tau\sigma^{-\ell})(s) \in \tau(X_z),$$

and it follows that $X_z = \tau(X_z)$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Thus we obtain the following explicit pair of generators for $C^\varepsilon(T_z)$.

Lemma 6.9. $C^\varepsilon(T_z) = \langle\, T_z, \tau_z \,\rangle$, *where $\tau_z = \tau \restriction X_z$.*

Proof. We have already seen that $\langle\, T_z, \tau_z \,\rangle \leqslant C^\varepsilon(T_z)$. On the other hand, notice that if $S \in C^\varepsilon(T_z) \smallsetminus C(T_z)$, then $S\, T_z\, S^{-1} = T_z^{-1} = \tau_z T_z \tau_z^{-1}$ and hence we have that $\tau_z^{-1}S \in C(T_z) = \{\, T_z^\ell \mid \ell \in \mathbb{Z} \,\}$. $\qquad\qquad\qquad\qquad\square$

Next we will compute $K^0(X_z, T_z)/2K^0(X_z, T_z)$. First we need to recall some of the basic notions from the theory of Toeplitz flows.

Definition 6.10. Suppose that $x \in 2^{\mathbb{Z}}$ is a Toeplitz sequence.

(i) For each $a \in \mathbb{Z}$, the corresponding *minimal period* $\mathrm{per}_x(a)$ is the least integer $b \geq 1$ such that $x(a + kb) = x(a)$ for all $k \in \mathbb{Z}$.

(ii) The set of *essential periods* of x is $\{\, \mathrm{per}_x(a) \mid a \in \mathbb{Z} \,\}$.

The following result was proved in Thomas [19, Lemma 4.5]. For the sake of completeness, we include the proof here.

Lemma 6.11. *If $z \in \mathrm{Nec}(2^{\mathbb{N}})$, then the set of essential periods of $\tilde{z}$ is given by* $\{\, 2^m \mid m \in \mathbb{N}^+ \,\}$

Proof. With the above notation, it is enough to show that each $c_m \in B_m$ has minimal period 2^m. This is clear when $m = 1$. So suppose that $m > 1$. Then at the beginning of stage m of the construction, $\tilde{z} \restriction B_m$ has the form $\bar{a} * \bar{b}\,\bar{a} * \bar{b}$, where $\bar{a} * \bar{b}$ has length 2^{m-1}; and at the end of stage m, we know that $\tilde{z}$ has the form

$$\cdots \bar{a} * \bar{b}\,\bar{a}\,z(m-1)\,\bar{b}\,\bar{a} * \bar{b}\,\bar{a}\,z(m-1)\,\bar{b}\,\bar{a} * \bar{b}\,\bar{a}\,z(m-1)\,\bar{b} \cdots$$

Clearly 2^m is a period of c_m. Also, since z is not eventually constant, we must eventually replace some $*$ by a value $z(\ell) \neq z(m-1)$ and so 2^{m-1} is not a period of c_m. Thus c_m has minimal period 2^m. $\qquad\square$

Definition 6.12. For each $z \in \mathrm{Nec}(2^{\mathbb{N}})$, and $m \in \mathbb{N}^+$, let $W_m(\tilde{z})$ be the set of subsequences of $\tilde{z}$ of the form $\tilde{z} \restriction [\, k\,2^m, (k+1)2^m \,)$ for some $k \in \mathbb{Z}$.

The following result was proved in Thomas [19, Lemma 4.7].

Lemma 6.13. *If $z \in \mathrm{Nec}(2^{\mathbb{N}})$, then $|W_m(\tilde{z})| = 2$ for all $m \in \mathbb{N}^+$.*

Proof. If at the end of stage m of the construction, $\tilde{z} \restriction B_m$ has the form $\bar{c} * \bar{d}$, then $W_m(\tilde{z}) = \{\, \bar{c}\,0\,\bar{d}, \bar{c}\,1\,\bar{d} \,\}$. $\qquad\square$

We are now ready to begin our construction of a suitable sequence $\mathcal{P} = (\,\mathcal{P}_n\,)_{n \geq 0}$ of Kakutani-Rokhlin partitions of X_z. First, for each $m \in \mathbb{N}^+$ and $0 \leq k < 2^m$, let $A_k^m = \{\, T_z^n(\tilde{z}) \mid n \equiv k \mod 2^m \,\}$. Then, by Williams [21, Lemma 2.3], we have that:

(i) $\{\, \overline{A_k^m} \mid 0 \leq k < 2^m \,\}$ is a partition of X_z into clopen subsets.

(ii) If $m < n$ and $k \equiv \ell \mod 2^m$, then $\overline{A_k^m} \supsetneq \overline{A_\ell^n}$.

(iii) $T_z(\overline{A_k^m}) = \overline{A_{k+1}^m}$ if $k < 2^m - 1$ and $T_z(\overline{A_{2^m-1}^m}) = \overline{A_0^m}$.

In addition, the following condition is also satisfied:

(v) $\bigcap_{m=1}^{\infty} \overline{A_0^m} = \{\, \tilde{z} \,\}$.

To see this, let $m \geq 1$ and suppose that at the end of stage m of the construction, $\tilde{z} \upharpoonright B_m$ has the form $\bar{c} * \bar{d}$. Then for each $n \equiv 0 \mod 2^m$, there exist $i_n, j_n \in \{0, 1\}$ such that

$$T_z^n(\tilde{z}) \upharpoonright [-2^m, 2^m - 1] = \bar{c}\, i_n\, \bar{d}\, \bar{c}\, j_n\, \bar{d}.$$

Let $t_m = \min\{a_m, 2^m - 1 - b_m\}$. Then for each $n \equiv k \mod 2^m$,

$$\tilde{z} \upharpoonright [-t_m, t_m - 1] = T_z^n(\tilde{z}) \upharpoonright [-t_m, t_m - 1].$$

By Remark 6.3, $\lim_{m \to \infty} t_m = \infty$ and hence (v) holds.

For each $m \in \mathbb{N}$, we can now define a corresponding Kakutani-Rohlin partition $\mathcal{P}_m$ of X_z as follows. First let $\mathcal{P}_0 = \{X_z\}$ be the trivial Kakutani-Rohlin partition. Also define $W_0(\tilde{z}) = \{\emptyset\}$ and $B_\emptyset^0 = X_z$. Next suppose that $m \in \mathbb{N}^+$. For each $w \in W_m(\tilde{z})$, let

$$B_w^m = \{x \in \overline{A_0^m} \mid x \upharpoonright [0, 2^m - 1] = w\}.$$

Then $\{B_w^m \mid w \in W_m(\tilde{z})\}$ is a partition of $\overline{A_0^m}$ into 2 clopen subsets. It follows that

$$\mathcal{P}_m = \{T_z^k(B_w^m) \mid w \in W_m(\tilde{z}), 0 \leq k < 2^m\}$$

is a Kakutani-Rohlin partition with base $\overline{A_0^m} = \bigsqcup \{B_w^m \mid w \in W_m(\tilde{z})\}$. Using (iii), it is easily checked that

$$\overline{A_0^m} = \overline{A_0^{m+1}} \sqcup T^{2^m}(\overline{A_0^{m+1}});$$

and it follows that $\mathcal{P}_{m+1}$ is a refinement of $\mathcal{P}_m$. We have already shown that $\bigcap_{m=1}^{\infty} \overline{A_0^m} = \{\tilde{z}\}$. Finally, we must check that $\bigcup_{m \geq 0} \mathcal{P}_m$ generates the topology; i.e. we must show that if $x \in \bigcap_{m \geq 1} T_z^{k_m}(B_{w_m}^m)$, then $\bigcap_{m \geq 1} T_z^{k_m}(B_{w_m}^m) = \{x\}$. (Here we adapt an argument from the proof of Gjerde-Johansen [9, Theorem 8].) There are three cases to consider.

Case 1: Suppose that there exists an integer $k \geq 0$ and an infinite subset $M \subseteq \mathbb{N}^+$ such that $k_m = k$ for all $m \in M$. Then

$$\bigcap_{m \geq 1} T_z^{k_m}(B_{w_m}^m) = \bigcap_{m \in M} T_z^{k_m}(B_{w_m}^m) \subseteq T_z^k \Big(\bigcap_{m \in M} \overline{A_0^m} \Big) = \{T_z^k(\tilde{z})\}.$$

Case 2: Next suppose that there exists an integer $k \geq 0$ and an infinite subset $M \subseteq \mathbb{N}^+$ such that $2^m - k_m = k$ for all $m \in M$. Then

$$\bigcap_{m \geq 1} T_z^{k_m}(B_{w_m}^m) = \bigcap_{m \in M} T_z^{2^m - k}(B_{w_m}^m) \subseteq T_z^{-k} \Big(\bigcap_{m \in M} \overline{A_0^m} \Big) = \{T_z^{-k}(\tilde{z})\}.$$

Case 3: Finally suppose that $\lim_{m \to \infty} k_m = \lim_{m \to \infty} 2^m - k_m = \infty$. Then, since $x \in T_z^{k_m}(B_{w_m}^m)$ uniquely determines $x \restriction [-k_m, 2^m - k_m)$, it follows that $\bigcap_{m \geq 1} T_z^{k_m}(B_{w_m}^m) = \{\, x \,\}$.

As in Section 5, we can now define a corresponding Bratteli diagram $(V_{\mathcal{P}}, E_{\mathcal{P}})$ with vertex set $V_{\mathcal{P}} = \bigsqcup_{m \geq 0} V_m$ such that:

- $V_m = \{\, B_w^m \mid w \in W_m(\tilde{z}) \,\}$.
- If $w \in W_m(\tilde{z})$ and $u \in W_{m+1}(\tilde{z})$, then the number of edges from B_w^m to B_u^{m+1} is equal to the number of elements $B \in \{\, B_u^{m+1}, T_z^{2^m}(B_u^{m+1}) \,\}$ such that $B \subseteq B_w^m$.

Lemma 6.14. *If $m > 0$, then with respect to suitable orderings of the vertex sets V_m and V_{m+1}, the corresponding incidence matrix is given by*

$$M_m = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}$$

Proof. Suppose that at the end of stage m of the construction, $\tilde{z} \restriction B_m$ has the form $\bar{c} * \bar{d}$. Then $W_m(\tilde{z}) = \{\, \bar{c}\,0\,\bar{d}, \bar{c}\,1\,\bar{d} \,\}$; and, at the beginning of stage $m + 1$, $\tilde{z} \restriction B_{m+1}$ has the form $\bar{c} * \bar{d}\,\bar{c} * \bar{d}$. Suppose, for example, that m is even and that $z(m) = 1$. Then at the end of stage $m + 1$, $\tilde{z} \restriction B_{m+1}$ has the form $\bar{c}\,1\bar{d}\,\bar{c} * \bar{d}$. Thus

$$W_{m+1}(\tilde{z}) = \{\, \bar{c}\,1\bar{d}\,\bar{c}\,0\bar{d}, \bar{c}\,1\bar{d}\,\bar{c}\,1\bar{d} \,\}$$

and the result follows. The other cases are similar. $\qquad\square$

Remark 6.15. It is not the case that there is a simultaneous ordering of all of the vertex sets V_m with this property.

We now easily obtain the following result.

Lemma 6.16. $K^0(X_z, T_z)/2K^0(X_z, T_z)$ *is cyclic of order 2.*

Proof. Recall that $K_0(V_{\mathcal{P}}, E_{\mathcal{P}})$ is the inductive limit of

$$\mathbb{Z}^{V_0} \overset{\varphi_0}{\to} \mathbb{Z}^{V_1} \overset{\varphi_1}{\to} \mathbb{Z}^{V_2} \overset{\varphi_2}{\to} \cdots \overset{\varphi_{m-1}}{\to} \mathbb{Z}^{V_m} \overset{\varphi_m}{\to} \mathbb{Z}^{V_{m+1}} \overset{\varphi_{m+1}}{\to} \cdots$$

where the homomorphism φ_n is given by matrix multiplication by the mth incidence matrix M_m. Applying Lemma 6.14, it follows that for each $m > 0$,

$$[\, \varphi_m(\mathbb{Z}^{V_m}) : \varphi_m(\mathbb{Z}^{V_m}) \cap 2\mathbb{Z}^{V_{m+1}} \,] = 2.$$

Together with Theorem 5.4, this implies that

$$K^0(X_z, T_z)/2K^0(X_z, T_z) \cong K_0(V_\mathcal{P}, E_\mathcal{P})/2K_0(V_\mathcal{P}, E_\mathcal{P}) \cong \mathbb{Z}_2.$$

$\square$

In particular, since $[[T_z]]_0/[[T_z]]' \cong K^0(X_z, T_z)/2K^0(X_z, T_z)$, it follows that if $\Pi_z \in [[T_z]]$ is *any* element such that $\Pi_z \in [[T_z]]_0 \smallsetminus [[T_z]]'$, then $[[T_z]]' \cup \{\Pi_z\}$ generates $[[T_z]]_0$. We will next describe how to find such an element Π_z.

For each $m \geq 1$ and $w \in W_m(\tilde{z})$, let $\Delta_w = \{T_z^k(B_w^m) \mid 0 \leq k < 2^m\}$. Let $G_\mathcal{P} = \bigcup_{m \geq 1} G_m$ be the group of permutations associated with $\mathcal{P} = (\mathcal{P}_n)_{n \geq 0}$, where

$$G_m = \prod_{w \in W_m(\tilde{z})} \operatorname{Sym}(\Delta_w).$$

Let $w = z(0)\,0 \in W_1(\tilde{z})$. Then $B_w^1 \cap T_z(B_z^1) = \emptyset$ and hence we can define a 1-permutation $\Pi_z \in \operatorname{Sym}(\Delta_w) \leqslant G_1$ by

$$\Pi_z(x) = \begin{cases} T_z(x) & \text{if } x \in B_w^1; \\ T_z^{-1}(x) & \text{if } x \in T_z(B_w^1); \\ x & \text{otherwise.} \end{cases}$$

Lemma 6.17. $\Pi_z \in [[T_z]]_0 \smallsetminus [[T_z]]'$.

Proof. Since $\Pi_z \in G_\mathcal{P} \leqslant [[T_z]]_0$, it is enough to show that $\Pi_z \notin [[T_z]]'$. To see this, note that Π_z is an odd permutation when regarded as an element of $\operatorname{Sym}(\Delta_w) \leqslant G_1$ and so $\Pi_z \notin G_1'$. Applying Lemma 6.14, we inductively see that $\Pi_z \notin G_m'$ for each $m > 1$ and so $\Pi_z \notin G_\mathcal{P}'$. Hence, by Proposition 5.9, we obtain that $\Pi_z \notin [[T_z]]'$. $\square$

Corollary 6.18. $[[T_z]]_0$ *is generated by* $[[T_z]]' \cup \{\Pi_z\}$.

Proof. Applying Theorem 4.6 and Lemma 6.16, we obtain that

$$[\,[[T_z]]_0 : [[T_z]]'\,] = [\,K^0(X_z, T_z) : 2K^0(X_z, T_z)\,] = 2.$$

Since $\Pi_z \in [[T_z]]_0 \smallsetminus [[T_z]]'$, the result follows. $\square$

We will next explicitly describe a finite generating set for $[[T_z]]'$. (The following generators were originally extracted from the proof of Matui [15, Theorem 5.4] by Grigorchuk-Medynets in an early version of their paper [11].)

Definition 6.19. Suppose that $A \subseteq X_z$ is a clopen subset such that the sets A, $\sigma(A)$ and $\sigma^2(A)$ are pairwise disjoint. Then the homeomorphism $\gamma_A \in [[\, T_z \,]]$ is defined by

$$
\gamma_A(x) = \begin{cases} \sigma(x) & \text{if } x \in A \cup \sigma(A); \\[2mm] \sigma^{-2}(x) & \text{if } x \in \sigma^2(A); \\[2mm] x & \text{otherwise.} \end{cases}
$$

By Matui [15, Section 5], each such homeomorphism γ_A is an element of $[[\, T_z \,]]'$; and, furthermore, $[[\, T_z \,]]'$ is generated by a suitably chosen finite subset of these homeomorphisms. In more detail, for each $m \geq 1$, let $\mathcal{B}_m(X_z)$ be the set of all *m-blocks* that occur in sequences $x \in X_z$; i.e. the words of the form

$$
x \restriction [k, k+m-1] = x_k \, x_{k+1} \cdots x_{k+m-1}
$$

for some $x \in X_z$ and $k \in \mathbb{Z}$. And for each $w \in \mathcal{B}_m(X_z)$ and $k \in \mathbb{Z}$, let

$$
S_k(w) = \{\, x \in X_z \mid x \restriction [k, k+m-1] = w \,\}.
$$

Then there exists an integer $m_z \geq 1$ such that for each $w \in \mathcal{B}_{m_z}(X_z)$, $k \in \mathbb{Z}$ and $1 \leq i \leq 4$,

$$
\sigma^i(\, S_k(w)\,) \cap S_k(w) = S_{k-i}(w) \cap S_k(w) = \emptyset.
$$

(If not, then an easy compactness argument yields an element $x \in X_z$ such that $\sigma^i(x) = x$ for some $1 \leq i \leq 4$, which contradicts the fact that minimal subshifts contain no periodic points.) Finally, the proof of Matui [15, Theorem 5.4] shows that the following result holds.

Lemma 6.20. $[[\, T_z \,]]'$ *is generated by* $D_{m_z} = \{\, \gamma_{S_0(w)} \mid w \in \mathcal{B}_{m_z+3}(X_z) \,\}$.

Combining Lemma 6.9, Corollary 6.18 and Lemma 6.20, we obtain the following explicit finite generating set for $[[\, T_z \,]]_0 \rtimes C^\varepsilon(T_z)$.

Proposition 6.21. $[[\, T_z \,]]_0 \rtimes C^\varepsilon(T_z)$ *is generated by* $D_{m_z} \cup \{\, T_z, \tau_z, \Pi_z \,\}$.

Combined with Proposition 6.5, the following result completes the proof of Lemma 6.1.

Lemma 6.22. *There exists a Borel map* $z \mapsto G_z$ *from* $\mathrm{Nec}(2^{\mathbb{N}})$ *to* $\mathcal{FG}$ *such that*

$$
G_z \cong \mathrm{Aut}([[\, T_z \,]]').
$$

Proof. Let $z \in \mathrm{Nec}(2^{\mathbb{N}})$. Then, in a Borel manner, we can choose an integer $m_z \geq 1$ such that $D_{m_z} = \{\, \gamma_{S_0(w)} \mid w \in \mathcal{B}_{m_z+3}(X_z) \,\}$ generates $[[T_z]]'$, together with an ordering $\varphi_1, \cdots, \varphi_{t_z}$ of the elements of D_{m_z}. Let $z \mapsto N_z \in \mathcal{N}_{t_z+3} \subseteq \mathcal{N}$ be the Borel map defined by letting N_z be the normal subgroup of the free group $\mathbb{F}_{t_z+3}$ consisting of the words $w(x_1, \cdots, x_{t_z}, x_{t_z+1}, x_{t_z+2}, x_{t_z+3})$ such that

$$w(\varphi_1, \cdots, \varphi_{t_z}, T_z, \tau_z, \Pi_z)(\sigma^n(\tilde{z})) = \sigma^n(\tilde{z}) \text{ for all } n \in \mathbb{Z}.$$

Since $\{\, \sigma^n(\tilde{z}) \mid n \in \mathbb{Z} \,\}$ is dense in X_z and each $\varphi_i, T_z, \tau_z, \Pi_z$ is a homeomorphism of X_z, it follows that

$$\mathbb{F}_{t_z+3}/N_z \cong [[T_z]]_0 \rtimes C^\varepsilon(T_z) \cong \mathrm{Aut}([[T_z]]').$$

Thus the Borel map $z \mapsto \mathbb{F}_{t_z+3}/N_z \in \mathcal{FG}$ satisfies our requirements. $\square$

This completes the proof of Theorem 1.1.

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