A REMARK ON THE HIGMAN-NEUMANN-NEUMANN EMBEDDING THEOREM

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Abstract. There does not exist an isomorphism-invariant Borel version of the Higman-Neumann-Neumann Embedding Theorem.

1. Introduction

The Higman-Neumann-Neumann Embedding Theorem [3] states that any countable group G can be embedded into a 2-generator group K. In the standard proof of this classical theorem, the construction of the group K involves an enumeration of a set $\{g_n \mid n \in \mathbb{N}\}$ of generators of the group G; and it is clear that the isomorphism type of K usually depends upon both the generating set and the particular enumeration that is used. Consequently, it is natural to ask whether there is a more uniform construction with the property that the isomorphism type of K only depends upon the isomorphism type of G. The main result of this paper implies that no such construction exists.

Before we can give an exact statement of our main result, we first need to recall how to represent the class of countably infinite groups by the elements of a standard Borel space; i.e., a Polish space equipped with its associated σ-algebra of Borel subsets. Let $\mathcal{G}$ be the set of countably infinite groups G with underlying set $\mathbb{N}$; and let $2^{\mathbb{N}^3}$ be the Polish space of all 3-ary functions $f : \mathbb{N}^3 \to \{0,1\}$ with the natural product topology. Then, identifying each group $G \in \mathcal{G}$ with the graph of its multiplication operation $m_G \in 2^{\mathbb{N}^3}$, it is easily checked that $\mathcal{G}$ is a Borel subset of $2^{\mathbb{N}^3}$. It follows that $\mathcal{G}$ is a standard Borel space; and since

$$\mathcal{G}_{fg} = \{G \in \mathcal{G} \mid G \text{ is finitely generated } \}$$

is a Borel subset of $\mathcal{G}$, it follows that $\mathcal{G}_{fg}$ is also a standard Borel space. (For more details, see Hjorth-Kechris [5] or Thomas-Velickovic [9].)

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Theorem 1.1. *There does* not *exist a Borel function* $\varphi : \mathcal{G} \to \mathcal{G}_{fg}$ *such that for all* $G, H \in \mathcal{G}$,

(a) G *embeds into* $\varphi(G)$*; and*
(b) *if* $G \cong H$*, then* $\varphi(G) \cong \varphi(H)$.

Here a function $\psi : X \to Y$ between standard Borel spaces X, Y is said to be *Borel* iff graph(ψ) is a Borel subset of $X \times Y$. Equivalently, $\psi : X \to Y$ is a Borel function iff $\psi^{-1}(B)$ is a Borel subset of X for every Borel subset B of Y.

Remark 1.2. The proof of Theorem 1.1 relies upon the fact that the isomorphism relation on the space $\mathcal{G}$ of arbitrary countable groups is much more complicated than that on the space $\mathcal{G}_{fg}$ of finitely generated groups. Hence, letting $\mathcal{G}_2$ be the standard Borel space of 2-generator groups, it is natural to ask whether there exists a Borel map $\psi : \mathcal{G}_{fg} \to \mathcal{G}_2$ such that:

- G embeds into $\psi(G)$; and
- if $G \cong H$, then $\psi(G) \cong \psi(H)$.

Perhaps surprisingly, such a map does indeed exist. More specifically, Friedman [1] has constructed such a map into the space of 4-generator groups; and, making use of the techniques of Galvin [2], it is easy to modify Friedman's map so that it takes values in $\mathcal{G}_2$.

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2. Borel equivalence relations

In this section, we shall present the proof of Theorem 1.1, modulo a key lemma which will be proved in Section 3. But first we need to recall some of the basic notions from the theory of Borel equivalence relations.

Let X be a standard Borel space. Then an equivalence relation E on X is said to be *Borel* iff E is a Borel subset of X^2. More generally, E is said to be *analytic* iff E is an analytic subset of X^2. For example, the isomorphism relation on the space $\mathcal{G}_{fg}$ of finitely generated groups is a Borel equivalence relation, while the isomorphism relation on the space $\mathcal{G}$ of countable groups is analytic but not Borel. Suppose that E, F are analytic equivalence relations on the standard Borel spaces

X, Y respectively. Then a Borel map $\psi : X \to Y$ is said to be a *homomorphism* from E to F iff $x \, E \, y$ implies $\psi(x) \, F \, \psi(y)$ for all $x, y \in X$. If ψ satisfies the stronger condition that $x \; E \; y$ iff $\psi(x) \; F \; \psi(y)$ for all $x, y \in X$, then ψ is said to be a *Borel reduction* and we write $E \leq_B F$.

The following Borel equivalence relation will play a central role in the proof of Theorem 1.1.

Definition 2.1. Let $\mathrm{I}(\mathbb{N}, 2^{\mathbb{N}})$ be the standard Borel space of all injective maps $z : \mathbb{N} \to 2^{\mathbb{N}}$. Then E_{cntble} is the Borel equivalence relation on $\mathrm{I}(\mathbb{N}, 2^{\mathbb{N}})$ defined by

$$z \, E_{cntble} \, z' \quad \text{iff} \quad \{z(n) \mid n \in \mathbb{N}\} = \{z'(n) \mid n \in \mathbb{N}\}.$$

The following key lemma will be proved in Section 3.

Lemma 2.2. *Suppose that $\theta : \mathrm{I}(\mathbb{N}, 2^{\mathbb{N}}) \to \mathcal{G}_{fg}$ is a Borel homomorphism from E_{cntble} to the isomorphism relation $\cong$. Then there exists a group $G \in \mathcal{G}_{fg}$ such that for all $x \in 2^{\mathbb{N}}$, there exists $z \in \mathrm{I}(\mathbb{N}, 2^{\mathbb{N}})$ such that $x \in \{z(n) \mid n \in \mathbb{N}\}$ and $\theta(z) \cong G$.*

The following lemma is implicitly contained in the classical paper [8] of B.H. Neumann.

Lemma 2.3. *There exists a Borel family $\{H_x \mid x \in 2^{\mathbb{N}}\} \subseteq \mathcal{G}$ of pairwise nonisomorphic infinite 2-generator groups.*

Proof. For each strictly increasing sequence $\mathbf{d} = \langle d_n \mid n \in \omega \rangle$ of odd integers with $d_0 \geq 5$, let $X^n_{\mathbf{d}} = \{x^n_1, x^n_2, \cdots, x^n_{d_n}\}$ and let $\Gamma_{\mathbf{d}}$ be the subgroup of $\prod_{n \in \omega} \mathrm{Alt}(X^n_{\mathbf{d}})$ generated by the two permutations

$$\alpha_{\mathbf{d}} = \prod_{n \in \omega} (\, x^n_1 \;\; x^n_2 \;\; x^n_3 \;\; \cdots \;\; x^n_{d_n} \,)$$

$$\beta_{\mathbf{d}} = \prod_{n \in \omega} (\, x^n_1 \;\; x^n_2 \;\; x^n_3 \,).$$

Then by B.H. Neumann [8], the groups $\Gamma_{\mathbf{d}}$ are infinite and pairwise nonisomorphic. The result follows easily. □

We are now ready to present the proof of Theorem 1.1. Suppose that $\varphi : \mathcal{G} \to \mathcal{G}_{fg}$ is a Borel map such that for all $G, H \in \mathcal{G}$,

(a) G embeds into $\varphi(G)$; and

(b) if $G \cong H$, then $\varphi(G) \cong \varphi(H)$.

Let $\{H_x \mid x \in 2^{\mathbb{N}}\} \subseteq \mathcal{G}$ be the Borel family of pairwise nonisomorphic 2-generator groups given by Lemma 2.3 and let $\psi : \mathrm{I}(\mathbb{N}, 2^{\mathbb{N}}) \to \mathcal{G}$ be the Borel map defined by

$$\psi(z) = H_{z(0)} \times H_{z(1)} \times \cdots \times H_{z(n)} \times \cdots$$

i.e. $\psi(z)$ is the *restricted* direct product of the sequence $\langle H_{z(n)} \mid n \in \mathbb{N}\rangle$. Clearly if $z \, E_{cntble} \, z'$, then $\psi(z) \cong \psi(z')$. It follows that the map $\theta = \varphi \circ \psi$ is a Borel homomorphism from E_{cntble} to the isomorphism relation $\cong$ on $\mathcal{G}_{fg}$. By Lemma 2.2, there exists a group $G \in \mathcal{G}_{fg}$ such that for all $x \in 2^{\mathbb{N}}$, there exists $z \in \mathrm{I}(\mathbb{N}, 2^{\mathbb{N}})$ such that $x \in \{z(n) \mid n \in \mathbb{N}\}$ and $\theta(z) \cong G$. But this means that H_x embeds into G for every $x \in 2^{\mathbb{N}}$, which is impossible since G has only countably many finitely generated subgroups. This completes the proof of Theorem 1.1, modulo the proof of Lemma 2.2.

3. Pinned equivalence relations

As we shall see, Lemma 2.2 is an easy consequence of the basic theory of pinned equivalence relations. Throughout this section, we shall work within a fixed base universe V of set theory and consider extensions of analytic equivalence relations in various generic extensions $V^{\mathbb{P}}$. Suppose that E is an analytic equivalence relation on the Polish space X. If $V^{\mathbb{P}}$ is a generic extension, then $X^{V^{\mathbb{P}}}$ and $E^{V^{\mathbb{P}}}$ will denote the sets obtained by applying the definitions of X, E within $V^{\mathbb{P}}$. By the Shoenfield Absoluteness Theorem [6, Theorem 25.20], it follows that $X^{V^{\mathbb{P}}} \cap V = X$, $E^{V^{\mathbb{P}}} \cap V = E$ and that $E^{V^{\mathbb{P}}}$ is an analytic equivalence relation on $X^{V^{\mathbb{P}}}$.

The following notion of a pinned equivalence relation was abstracted by Kanovei-Reeken [7] from an argument in Hjorth [4, Section 5].

Definition 3.1 (Kanovei-Reeken [7])**.** Working in the base universe V, suppose that E is an analytic equivalence relation on the Polish space X. If $\mathbb{P}$ is a notion of forcing, then a *virtual E-class* is a $\mathbb{P}$-name $\boldsymbol{\tau}$ such that:

(i) $\Vdash_{\mathbb{P}} \boldsymbol{\tau} \in X^{V^{\mathbb{P}}}$; and

(ii) $\Vdash_{\mathbb{P}\times\mathbb{P}} \boldsymbol{\tau}_{\text{left}} \, E^{V^{\mathbb{P}\times\mathbb{P}}} \, \boldsymbol{\tau}_{\text{right}}$.

Here $\boldsymbol{\tau}_{\text{left}}$ and $\boldsymbol{\tau}_{\text{right}}$ are the $(\mathbb{P}\times\mathbb{P})$-names such that if $G \times H$ is $(\mathbb{P}\times\mathbb{P})$-generic, then $\boldsymbol{\tau}_{\text{left}}[G\times H] = \boldsymbol{\tau}[G]$ and $\boldsymbol{\tau}_{\text{right}}[G \times H] = \boldsymbol{\tau}[H]$.

The analytic equivalence relation E on the Polish space X is said to be *pinned* iff for every forcing notion $\mathbb{P}$ and every virtual E-class $\boldsymbol{\tau}$, there exists an element $x \in X$ such that $\Vdash_{\mathbb{P}} x\, E^{V^{\mathbb{P}}}\, \boldsymbol{\tau}$.

Example 3.2. By Hjorth [4, Section 5], if E is a countable Borel equivalence relation, then E is pinned. (Recall that a Borel equivalence relation E is said to be *countable* iff every E-equivalence class is countable.)

It is well-known that the isomorphism relation $\cong$ on the space $\mathcal{G}_{fg}$ of finitely generated groups is *essentially countable*; i.e. there exists a countable Borel equivalence relation E such that $\cong \leq_B E$. (For example, see Hjorth-Kechris [5].) By Kanovei-Reeken [7, Lemma 20], it follows that $\cong$ is also pinned. Hence Lemma 2.2 is an immediate consequence of the following result. (It should be stressed that Theorem 3.3 is implicitly contained within the proofs of both Hjorth [4, Section 5] and Kanovei-Reeken [7, Section 4].)

Theorem 3.3. *Suppose that F is a pinned analytic equivalence relation on the standard Borel space Y and that $\theta : \mathrm{I}(\mathbb{N}, 2^{\mathbb{N}}) \to Y$ is a Borel homomorphism from E_{cntble} to F. Then there exists an element $y \in Y$ such that for all $x \in 2^{\mathbb{N}}$, there exists $z \in \mathrm{I}(\mathbb{N}, 2^{\mathbb{N}})$ such that $x \in \{z(n) \mid n \in \mathbb{N}\}$ and $\theta(z)\, F\, y$.*

Proof. Working in the base universe V, suppose that $\theta : \mathrm{I}(\mathbb{N}, 2^{\mathbb{N}}) \to Y$ is a Borel homomorphism from E_{cntble} to F. Let $\mathbb{P}$ be the notion of forcing consisting of all finite injective partial functions $p : \mathbb{N} \to 2^{\mathbb{N}}$. Then if G is $\mathbb{P}$-generic, it follows that $g = \bigcup G$ is a bijection between $\mathbb{N}$ and $2^{\mathbb{N}} \cap V$. Thus if $\boldsymbol{\tau}$ is the canonical $\mathbb{P}$-name such that $\boldsymbol{\tau}[G] = g$, then $\boldsymbol{\tau}$ is a virtual E_{cntble}-class. By the Shoenfield Absoluteness Theorem, $\theta^{V^{\mathbb{P}\times\mathbb{P}}}$ is a Borel homomorphism from $E_{cntble}^{V^{\mathbb{P}\times\mathbb{P}}}$ to $F^{V^{\mathbb{P}\times\mathbb{P}}}$. Hence, letting $\boldsymbol{\sigma}$ be a $\mathbb{P}$-name for $\theta^{V^{\mathbb{P}}}(\boldsymbol{\tau})$, it follows that $\boldsymbol{\sigma}$ is a virtual F-class. Since F is pinned, there exists an element $y \in Y$ such that $\Vdash_{\mathbb{P}}\ y\, F^{V^{\mathbb{P}}}\, \boldsymbol{\sigma}$. In particular, for each $x \in 2^{\mathbb{N}} \cap V$,

$$V^{\mathbb{P}} \models (\exists z \in \mathrm{I}(\mathbb{N}, 2^{\mathbb{N}})^{V^{\mathbb{P}}})\, (\exists n \in \mathbb{N})\, (z(n) = x \,\wedge\, \theta^{V^{\mathbb{P}}}(z)\, F^{V^{\mathbb{P}}}\, y).$$

Applying the Shoenfield Absoluteness Theorem, it follows that y satisfies our requirements. $\square$

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