

INVARIANT RANDOM SUBGROUPS OF STRICTLY DIAGONAL LIMITS OF FINITE SYMMETRIC GROUPS

SIMON THOMAS AND ROBIN TUCKER-DROB

ABSTRACT. We classify the ergodic invariant random subgroups of strictly diagonal limits of finite symmetric groups.

1. INTRODUCTION

Let G be a countable discrete group and let Sub_G be the compact space of subgroups $H \leq G$. Then a Borel probability measure ν on Sub_G which is invariant under the conjugation action of G on Sub_G is called an *invariant random subgroup* or *IRS*. For example, if $N \trianglelefteq G$ is a normal subgroup, then the Dirac measure δ_N is an IRS of G . Further examples arise from the stabilizer distributions of measure preserving actions, which are defined as follows. Suppose that G acts via measure preserving maps on the Borel probability space (Z, μ) and let $f : Z \rightarrow \text{Sub}_G$ be the G -equivariant map defined by

$$z \mapsto G_z = \{g \in G \mid g \cdot z = z\}.$$

Then the corresponding *stabilizer distribution* $\nu = f_*\mu$ is an IRS of G . In fact, by a result of Abert-Glasner-Virag [1], every IRS of G can be realized as the stabilizer distribution of a suitably chosen measure preserving action. Moreover, by Creutz-Peterson [4], if ν is an ergodic IRS of G , then ν is the stabilizer distribution of an ergodic action $G \curvearrowright (Z, \mu)$.

A number of recent papers have focused on the problem of studying the IRS's of certain specific countably infinite groups. For example, Bowen [2] has shown that each free group \mathbb{F}_m of rank $m \geq 2$ has a huge “zoo” of IRS's; and Bowen-Grigorchuk-Kravchenko [3] have proved that the same is true of the lamplighter groups $(\mathbb{Z}/p\mathbb{Z})^n \wr \mathbb{Z}$, where p is a prime and $n \geq 1$. On the other hand, Vershik [13] has given a complete classification of the ergodic invariant random subgroups of the group of finitary permutations of the natural numbers.¹ In this paper, we will classify the ergodic invariant subgroups of the strictly diagonal limits of finite symmetric groups, which are defined as follows.

Suppose that $\text{Sym}(\Delta)$, $\text{Sym}(\Omega)$ are finite symmetric groups and that $|\Omega| = \ell|\Delta|$. Then an embedding $\varphi : \text{Sym}(\Delta) \rightarrow \text{Sym}(\Omega)$ is said to be an ℓ -fold *diagonal embedding* if $\varphi(\text{Sym}(\Delta))$ acts via its natural permutation representation on each of its orbits in Ω . The countable locally finite group G is a *strictly diagonal limit* of finite symmetric groups if we can express $G = \bigcup_{n \in \mathbb{N}} G_n$ as the union of an increasing chain of finite symmetric groups $G_n = \text{Sym}(X_n)$, where each embedding $G_n \hookrightarrow G_{n+1}$ is an $|X_{n+1}|/|X_n|$ -fold diagonal embedding. In this case, we say that G is an *SDS-group*. Here, letting $[k] = \{0, 1, \dots, k-1\}$, we can suppose that for

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¹There is a slight inaccuracy in the statement [13] of Vershik's classification theorem.

some sequence (k_n) of natural numbers $k_n \geq 2$, we have that $X_n = \prod_{0 \leq m \leq n} [k_m]$ and that the embedding $\varphi_n : G_n \rightarrow G_{n+1}$ is defined by

$$\varphi_n(g) \cdot (i_0, \dots, i_n, i_{n+1}) = (g(i_0, \dots, i_n), i_{n+1}).$$

Let $X = \prod_{n \geq 0} [k_n]$ and let μ be the product probability measure of the uniform probability measures on the $[k_n]$. Then G acts naturally on (X, μ) as a group of measure preserving transformations via

$$g \cdot (i_0, \dots, i_n, i_{n+1}, i_{n+2}, \dots) = (g(i_0, \dots, i_n), i_{n+1}, i_{n+2}, \dots), \quad g \in G_n.$$

It is easily checked that the action $G \curvearrowright (X, \mu)$ is weakly mixing and it follows that the diagonal action of G on the product space $(X^r, \mu^{\otimes r})$ is ergodic for each $r \in \mathbb{N}^+$. Hence the stabilizer distribution σ_r of $G \curvearrowright (X^r, \mu^{\otimes r})$ is an ergodic IRS of G . We will show that if G is simple, then $\{\delta_1, \delta_G\} \cup \{\sigma_r \mid r \in \mathbb{N}^+\}$ is a complete list of the ergodic IRS's of G . A moment's thought shows that G is simple if and only if k_n is even for infinitely many $n \in \mathbb{N}$. Suppose now that k_n is odd for all but finitely many $n \in \mathbb{N}$. Then clearly $A(G) = \bigcup_{n \in \mathbb{N}} \text{Alt}(X_n)$ is a simple subgroup of G such that $[G : A(G)] = 2$. For each $r \in \mathbb{N}^+$, let $\tilde{f}_r : X^r \rightarrow \text{Sub}_G$ be the G -equivariant map defined

$$(x_0, \dots, x_{r-1}) = \tilde{x} \mapsto G_{\tilde{x}} \cap A(G),$$

where $G_{\tilde{x}} = \{g \in G \mid g \cdot x_i = x_i \text{ for } 0 \leq i < r\}$. Then $\tilde{\sigma}_r = (\tilde{f}_r)_* \mu^{\otimes r}$ is also an ergodic IRS of G .

Theorem 1.1. *With the above notation, if G is a simple SDS-group, then the ergodic IRS's of G are*

$$\{\delta_1, \delta_G\} \cup \{\sigma_r \mid r \in \mathbb{N}^+\};$$

while if G is a non-simple SDS-group, then the ergodic IRS's of G are

$$\{\delta_1, \delta_{A(G)}, \delta_G\} \cup \{\sigma_r \mid r \in \mathbb{N}^+\} \cup \{\tilde{\sigma}_r \mid r \in \mathbb{N}^+\}.$$

By Creutz-Peterson [4], in order to prove Theorem 1.1, it is enough to show that the stabilizer distribution σ of each ergodic action $G \curvearrowright (Z, \mu)$ is included in the above list of invariant random subgroups. Our analysis of the action $G \curvearrowright (Z, \mu)$ will proceed via an application of the Pointwise Ergodic Theorem to the associated character $\chi(g) = \mu(\text{Fix}_Z(g))$, which will enable us to regard $G \curvearrowright (Z, \mu)$ as the “limit” of a suitable sequence of finite permutation groups $G_n \curvearrowright (\Omega_n, \mu_n)$, where μ_n is the uniform probability measure on Ω_n . (In other words, we will follow the *asymptotic approach to characters* of Kerov-Vershik [14, 15].)

Definition 1.2. If Γ is a countable discrete group, then the function $\chi : \Gamma \rightarrow \mathbb{C}$ is a *character* if the following conditions are satisfied:

- (i) $\chi(hgh^{-1}) = \chi(g)$ for all $g \in \Gamma$.
- (ii) $\sum_{i,j=1}^n \lambda_i \bar{\lambda}_j \chi(g_j^{-1} g_i) \geq 0$ for all $\lambda_1, \dots, \lambda_n \in \mathbb{C}$ and $g_1, \dots, g_n \in \Gamma$.
- (iii) $\chi(1_G) = 1$.

A character χ is said to be *indecomposable* or *extremal* if it is not possible to express $\chi = r\chi_1 + (1-r)\chi_2$, where $0 < r < 1$ and $\chi_1 \neq \chi_2$ are distinct characters.

In earlier work, Leinen-Puglisi [8] and Dudko-Medynets [6] classified the characters of the SDS-groups; and combining their classification and Theorem 1.1, we obtain that if G is a simple SDS-group, then the indecomposable characters of G are precisely the associated characters of the ergodic IRS's of G . However, it should

be stressed that our work neither makes use of nor implies the classification theorem of Leinen-Puglisi and Dudko-Medynets. (See Vershik [12] for some fascinating conjectures concerning the relationship between invariant random subgroups and characters.)

This paper is organized as follows. In Section 2, we will discuss the Pointwise Ergodic Theorem for ergodic actions of countably infinite locally finite groups. In Section 3, we will briefly outline the strategy of the proof of Theorem 1.1. In Sections 4 and 5, we will prove a series of key lemmas concerning the asymptotic values of the normalized permutation characters of various actions $S \curvearrowright S/H$, where S is a suitable finite symmetric group. Finally, in Section 6, we will present the proof of Theorem 1.1.

2. THE POINTWISE ERGODIC THEOREM

In this section, we will discuss the Pointwise Ergodic Theorem for ergodic actions of countably infinite locally finite groups. Throughout $G = \bigcup G_n$ is the union of a strictly increasing chain of finite subgroups G_n and $G \curvearrowright (Z, \mu)$ is an ergodic action on a Borel probability space. The following is a special case of more general results of Vershik [11, Theorem 1] and Lindenstrauss [7, Theorem 1.3].

The Pointwise Ergodic Theorem. *With the above hypotheses, if $f \in L^1(Z, \mu)$, then for μ -a.e. $z \in Z$,*

$$\int f d\mu = \lim_{n \rightarrow \infty} \frac{1}{|G_n|} \sum_{g \in G_n} f(g \cdot z).$$

In particular, the Pointwise Ergodic Theorem applies when f is the characteristic function of the Borel subset $\text{Fix}_Z(g) = \{z \in Z \mid g \cdot z = z\}$ for some $g \in G$. From now on, for each $z \in Z$ and $n \in \mathbb{N}$, let $\Omega_n(z) = \{g \cdot z \mid g \in G_n\}$ be the corresponding G_n -orbit.

Theorem 2.1. *With the above hypotheses, for μ -a.e. $z \in Z$, for all $g \in G$,*

$$\mu(\text{Fix}_Z(g)) = \lim_{n \rightarrow \infty} |\text{Fix}_{\Omega_n(z)}(g)| / |\Omega_n(z)|.$$

Proof. Fix some $g \in G$. Then by the Pointwise Ergodic Theorem, for μ -a.e. $z \in Z$,

$$\mu(\text{Fix}_Z(g)) = \lim_{n \rightarrow \infty} \frac{1}{|G_n|} |\{h \in G_n \mid h \cdot z \in \text{Fix}_Z(g)\}|.$$

Fix some such $z \in Z$; and for each $n \in \mathbb{N}$, let $H_n = \{h \in G_n \mid h \cdot z = z\}$ be the corresponding point stabilizer. Then clearly,

$$|\{h \in G_n \mid h \cdot z \in \text{Fix}_Z(g)\}| = |\text{Fix}_{\Omega_n(z)}(g)| |H_n|;$$

and so we have that

$$\begin{aligned} \frac{1}{|G_n|} |\{h \in G_n \mid h \cdot z \in \text{Fix}_Z(g)\}| &= |\text{Fix}_{\Omega_n(z)}(g)| / [G_n : H_n] \\ &= |\text{Fix}_{\Omega_n(z)}(g)| / |\Omega_n(z)|. \end{aligned}$$

The result now follows easily. \square

Clearly the normalized permutation character $|\text{Fix}_{\Omega_n(z)}(g)| / |\Omega_n(z)|$ is the probability that an element of $(\Omega_n(z), \mu_n)$ is fixed by $g \in G_n$, where μ_n is the uniform probability measure on $\Omega_n(z)$; and, in this sense, we can regard $G \curvearrowright (Z, \mu)$ as the “limit” of the sequence of finite permutation groups $G_n \curvearrowright (\Omega_n(z), \mu_n)$. Of course,

the permutation group $G_n \curvearrowright \Omega_n(z)$ is isomorphic to $G_n \curvearrowright G_n/H_n$, where G_n/H_n is the set of cosets of $H_n = \{h \in G_n \mid h \cdot z = z\}$ in G_n . The following simple observation will play a key role in our later applications of Theorem 2.1.

Proposition 2.2. *If $H \leq S$ are finite groups and θ is the normalized permutation character corresponding to the action $G \curvearrowright S/H$, then*

$$\theta(g) = \frac{|g^S \cap H|}{|g^S|} = \frac{|\{s \in S \mid sgs^{-1} \in H\}|}{|S|}.$$

Proof. Fix some $g \in S$. In order to see that the first equality holds, note that for each $a \in S$, we have that

$$aH \in \text{Fix}(g) \iff a^{-1}ga \in H.$$

Hence, counting the number of such $a \in S$, we see that

$$|\text{Fix}(g)| |H| = |g^S \cap H| |C_S(g)| = |g^S \cap H| |S|/|g^S|.$$

It follows that

$$\theta(g) = |\text{Fix}(g)|/|S:H| = |g^S \cap H|/|g^S|.$$

To see that the second inequality holds, note that

$$\frac{|\{s \in S \mid sgs^{-1} \in H\}|}{|S|} = \frac{|g^S \cap H| |C_S(g)|}{|g^S| |C_S(g)|}.$$

□

3. AN OUTLINE OF THE PROOF OF THEOREM 1.1

In this section, we will briefly outline the strategy of the proof of Theorem 1.1. Let (k_n) be a sequence of natural numbers $k_n \geq 2$, let $X_n = \prod_{0 \leq m \leq n} [k_m]$, and let $G = \bigcup_{n \in \mathbb{N}} G_n$ be the corresponding SDS-group with $G_n = \text{Sym}(X_n)$. Let ν be an ergodic IRS of G . Then we can suppose that ν is not a Dirac measure δ_N concentrating on a normal subgroup $N \trianglelefteq G$. Let ν be the stabilizer distribution of the ergodic action $G \curvearrowright (Z, \mu)$ and let $\chi(g) = \mu(\text{Fix}_Z(g))$ be the corresponding character. For each $z \in Z$ and $n \in \mathbb{N}$, let $\Omega_n(z) = \{g \cdot z \mid g \in G_n\}$. Then, by Theorem 2.1, for μ -a.e. $z \in Z$, for all $g \in G$,

$$\mu(\text{Fix}_Z(g)) = \lim_{n \rightarrow \infty} |\text{Fix}_{\Omega_n(z)}(g)|/|\Omega_n(z)|.$$

Fix such an element $z \in Z$ and let $H = \{h \in G \mid h \cdot z = z\}$ be the corresponding point stabilizer. Clearly we can suppose that z has been chosen so that if $g \in H$, then $\chi(g) > 0$.

For each $n \in \mathbb{N}$, let $H_n = H \cap G_n$. Then, examining the list of ergodic IRS's in the statement of Theorem 1.1, we see that it is necessary to show that there exists a *fixed* integer $r \geq 1$ such that for all but finitely many $n \in \mathbb{N}$, there is a subset $U_n \subseteq X_n$ of cardinality r such that H_n fixes U_n pointwise and induces at least the alternating group on $X_n \setminus U_n$. Most of our effort will be devoted to eliminating the possibility that H_n acts transitively on X_n for infinitely many $n \in \mathbb{N}$. In more detail, we will show that if H_n acts transitively on X_n for infinitely many $n \in \mathbb{N}$, then there exists an element $g \in H$ such that

$$\mu(\text{Fix}_Z(g)) = \lim_{n \rightarrow \infty} |g^{G_n} \cap H_n|/|g^{G_n}| = |\{s \in G_n \mid sgs^{-1} \in H_n\}|/|G_n| = 0,$$

which is a contradiction. Our analysis will split into three cases, depending on whether for infinitely many $n \in \mathbb{N}$,

- (i) H_n acts primitively on X_n ; or
- (ii) H_n acts imprimitively on X_n with a *fixed* maximal block-size d ; or
- (iii) H_n acts imprimitively on X_n with maximal blocksize $d_n \rightarrow \infty$.

Cases (i) and (ii) are easily dealt with via a straightforward counting argument based on Stirling's Approximation, which will be presented in Section 4. Case (iii) requires a more involved probabilistic argument which will be presented in Section 5. Essentially the same probabilistic argument will then show that there exists a fixed integer $r \geq 1$ such that for all but finitely many $n \in \mathbb{N}$, there is an H_n -invariant subset $U_n \subseteq X_n$ of cardinality r such that H_n acts transitively on $X_n \setminus U_n$. Repeating the above analysis for the action $H_n \curvearrowright X_n \setminus U_n$, we will obtain that H_n induces at least the alternating group on $X_n \setminus U_n$; and an easy application of Nadkarni's Theorem on compressible group actions, which we will present in Section 6, will show that H_n fixes U_n pointwise.

At this point, we will have shown that the ergodic IRS ν concentrates on the same space of subgroups $\mathcal{S}_r \subseteq \text{Sub}_G$ as one of the target IRS's σ_r or $\tilde{\sigma}_r$. Finally, via another application of the Pointwise Ergodic Theorem, we will show that the action of G on \mathcal{S}_r is uniquely ergodic and hence that $\nu = \sigma_r$ or $\nu = \tilde{\sigma}_r$, as required.

4. ALMOST PRIMITIVE ACTIONS

In Sections 4 and 5, we will fix an element $1 \neq g \in \text{Sym}(a)$ having a cycle decomposition consisting of k_i m_i -cycles for $1 \leq i \leq t$, where each $k_i \geq 1$. Let $\ell \gg a$ and let $\varphi : \text{Sym}(a) \rightarrow S = \text{Sym}(a\ell)$ be an ℓ -fold diagonal embedding. Then identifying g with $\varphi(g)$, the cyclic decomposition of $g \in S$ consists of $k_i\ell$ m_i -cycles for $1 \leq i \leq t$ and hence

$$(4.1) \quad |g^S| = \frac{(a\ell)!}{\prod_{1 \leq i \leq t} (k_i\ell)! m_i^{k_i\ell}}.$$

In this section, we will consider the value of the normalized permutation character $|g^S \cap H|/|g^S|$ for the action $S \curvearrowright S/H$ as $\ell \rightarrow \infty$ in the cases when:

- (i) H is a primitive subgroup of S ; or
- (ii) H is an imprimitive subgroup of S preserving a maximal system of imprimitivity of fixed block-size d .

In both cases, we will make use of the following theorem of Praeger-Saxl [10]. (It is perhaps worth mentioning that the proof of Theorem 4.1 does not rely upon the classification of the finite simple groups.)

Theorem 4.1. *If $H < \text{Sym}(n)$ is a primitive subgroup which does not contain $\text{Alt}(n)$, then $|H| < 4^n$.*

We will also make use of the following variant of Stirling's Approximation:

$$(4.2) \quad 1 \leq \frac{n!}{\sqrt{2\pi n} \left(\frac{n}{e}\right)^n} \leq \frac{e}{\sqrt{2\pi}} \quad \text{for all } n \geq 1.$$

Lemma 4.2. *There exist constants $r, s > 0$ (which only depend on the parameters $a, k_1, \dots, k_t, m_1, \dots, m_t$) such that*

$$|g^S| > r s^\ell \ell^{(a - \sum k_i)\ell} \geq r s^\ell \ell^\ell.$$

Proof. Combining (4.1) and (4.2), it follows that there exists a constant $c > 0$ such that

$$|g^S| > c \frac{(a\ell/e)^{a\ell} \sqrt{2\pi a\ell}}{\prod_{1 \leq i \leq t} m_i^{k_i\ell} \prod_{1 \leq i \leq t} (k_i\ell/e)^{k_i\ell} \sqrt{2\pi k_i\ell}}$$

and this implies that

$$|g^S| > c d^\ell \ell^{(a-\sum k_i)\ell} \frac{\sqrt{2\pi a\ell}}{\prod_{1 \leq i \leq t} \sqrt{2\pi k_i\ell}}$$

for a suitably chosen constant $d > 0$. The result now follows easily. \square

The following lemma will eliminate the possibility of primitive actions in the proof of Theorem 1.1.

Lemma 4.3. *For each $\varepsilon > 0$, there exists an integer ℓ_ε such that if $\ell \geq \ell_\varepsilon$ and $H < S = \text{Sym}(a\ell)$ is a primitive subgroup which does not contain $\text{Alt}(a\ell)$, then $|g^S \cap H|/|g^S| < \varepsilon$.*

Proof. Suppose that $\ell \gg a$ and that $H < S = \text{Sym}(a\ell)$ is a primitive subgroup which does not contain $\text{Alt}(a\ell)$. Applying Lemma 4.2 and Theorem 4.1, there exist constants $r, s > 0$ such that

$$|g^S \cap H|/|g^S| < |H|/|g^S| < \frac{4^{a\ell}}{r s^\ell \ell^\ell}$$

and the result follows easily. \square

Next we consider the case when H is an imprimitive subgroup of S preserving a maximal system of imprimitivity of fixed block-size d . Of course, we can suppose that $H \leq \text{Sym}(d) \text{ wr } \text{Sym}(a\ell/d) < S = \text{Sym}(a\ell)$. The following result is another easy consequence of Stirling's Approximation.

Lemma 4.4. *There exist constants $b, c > 0$ (which only depend on the parameters a, d) such that $|\text{Sym}(d) \text{ wr } \text{Sym}(a\ell/d)| < b c^\ell \ell^{a\ell/d}$.*

In this case, we can only show that $|g^S \cap H|/|g^S|$ is small for those elements $g \in \text{Sym}(a)$ such that $a/d < a - \sum k_i$. Fortunately, clause (ii) of the following lemma will guarantee the existence of a “suitable such” element during the proof of Theorem 1.1.

Lemma 4.5. *For each $d \geq 2$ and $\varepsilon > 0$, there exists an integer $\ell_{d,\varepsilon}$ such that if $\ell \geq \ell_{d,\varepsilon}$ and $H < S = \text{Sym}(a\ell)$ is an imprimitive subgroup with a maximal system of imprimitivity \mathcal{B} of blocksize d , then either:*

- (i) $|g^S \cap H|/|g^S| < \varepsilon$; or
- (ii) *the induced action of H on \mathcal{B} contains $\text{Alt}(\mathcal{B})$.*

Proof. Suppose that $\ell \gg a$ and that $H < S = \text{Sym}(a\ell)$ is an imprimitive subgroup with a maximal system of imprimitivity \mathcal{B} of blocksize d . Let $\Gamma \leq \text{Sym}(\mathcal{B})$ be the group induced by the action of H on \mathcal{B} and suppose that Γ does not contain $\text{Alt}(\mathcal{B})$. Since \mathcal{B} is a maximal system of imprimitivity, it follows that Γ is a primitive subgroup of $\text{Sym}(\mathcal{B})$; and hence by Theorem 4.1, we obtain that $|\Gamma| < 4^{a\ell/d}$. Since H is isomorphic to a subgroup of $\text{Sym}(d) \text{ wr } \Gamma$, it follows that

$$|H| < (d!)^{a\ell/d} 4^{a\ell/d} = c^\ell,$$

where $c = (d! 4)^{a/d}$. Arguing as in the proof of Lemma 4.3, the result follows easily. \square

5. IMPRIMITIVE AND INTRANSITIVE ACTIONS

In this section, we will continue to fix an element $1 \neq g \in \text{Sym}(a)$. Suppose that the cycle decomposition of $g \in \text{Sym}(a)$ has k nontrivial cycles. Let $\ell \gg a$ and let $\varphi : \text{Sym}(a) \rightarrow S = \text{Sym}(a\ell)$ be an ℓ -fold diagonal embedding. Then identifying g with $\varphi(g)$, the cyclic decomposition of $g \in S$ has $k\ell$ nontrivial cycles. In this section, we will consider the value of the normalized permutation character for the action $S \curvearrowright S/H$ as $\ell \rightarrow \infty$ in the following two cases.

- (i) There exists an H -invariant subset $U \subset [a\ell]$ of *fixed* cardinality $r \geq 0$ such that H acts imprimitively on $T = [a\ell] \setminus U$ with a proper system of imprimitivity \mathcal{B} of blocksize d with $d \rightarrow \infty$ as $\ell \rightarrow \infty$.
- (ii) H is an intransitive subgroup of S with an H -invariant subset $U \subset [a\ell]$ of cardinality $r = |U| \leq a\ell/2$ such that $r \rightarrow \infty$ as $\ell \rightarrow \infty$.

Our approach in this section will be probabilistic; i.e. we will regard the normalized permutation character $|\{s \in S \mid sgs^{-1} \in H\}|/|S|$ as the probability that a uniformly random permutation $s \in S$ satisfies $sgs^{-1} \in H$. Our probability theoretic notation is standard. In particular, if E is an event, then $\mathbb{P}[E]$ denotes the corresponding probability and 1_E denotes the indicator function; and if N is a random variable, then $\mathbb{E}[N]$ denotes the expectation, $\text{Var}[N]$ denotes the variance and $\sigma = (\text{Var}[N])^{1/2}$ denotes the standard deviation. We will make use of the following easy consequence of Chebyshev's inequality.

Lemma 5.1. *Let (N_ℓ) be a sequence of non-negative random variables such that $\mathbb{E}[N_\ell] = \mu_\ell > 0$ and $\text{Var}[N_\ell] = \sigma_\ell^2 > 0$. If $\lim_{\ell \rightarrow \infty} \mu_\ell/\sigma_\ell = \infty$, then $\mathbb{P}[N_\ell > 0] \rightarrow 1$ as $\ell \rightarrow \infty$.*

Proof. Let $K_\ell = (\mu_\ell/\sigma_\ell)^{1/2}$ and let $L_\ell = \mu_\ell - K_\ell\sigma_\ell$. By Chebyshev's inequality,

$$\mathbb{P}[N_\ell > L_\ell] \geq 1 - \frac{1}{K_\ell^2}$$

and so $\mathbb{P}[N_\ell > L_\ell] \rightarrow 1$ as $\ell \rightarrow \infty$. In addition, for all sufficiently large ℓ , we have that $K_\ell > 1$ and hence $L_\ell = (\mu_\ell\sigma_\ell)^{1/2}(K_\ell - 1) > 0$. \square

In our arguments, it will be convenient to make use of big O notation. Recall that if (a_m) and (x_m) are sequences of real numbers, then $a_m = O(x_m)$ means that there exists a constant $C > 0$ and an integer $m_0 \in \mathbb{N}$ such that $|a_m| \leq C|x_m|$ for all $m \geq m_0$. Also if (c_m) is another sequence of real numbers, then we write $a_m = c_m + O(x_m)$ to mean that $a_m - c_m = O(x_m)$.

Lemma 5.2. *For each $r \geq 0$ and $\varepsilon > 0$, there exists an integer $d_{r,\varepsilon}$ such that if $d \geq d_{r,\varepsilon}$ and $H < S = \text{Sym}(a\ell)$ is a subgroup such that:*

- (i) *there exists an H -invariant subset $U \subset [a\ell]$ of cardinality r , and*
- (ii) *H acts imprimitively on $T = [a\ell] \setminus U$ with a proper system of imprimitivity \mathcal{B} of blocksize d ,*

then $|\{s \in S \mid sgs^{-1} \in H\}|/|S| < \varepsilon$.

Proof. Let $m = a\ell$ and let $Z \subset [m]$ be a subset which contains one element from every non-trivial cycle of g . Then $|Z| = cm$ where $0 < c = k/a \leq 1/2$ is a fraction which is independent of ℓ . Let $Y = g(Z)$ so that $Z \cap Y = \emptyset$. Fix an element $z_0 \in Z$ and let $y_0 = g(z_0)$. Let $s \in S$ be a uniformly random permutation. If $s(z_0), s(y_0) \in T$, let $B_0, C_0 \in \mathcal{B}$ be the blocks in \mathcal{B} containing $s(z_0)$ and $s(y_0)$

respectively; otherwise, let $B_0 = C_0 = \emptyset$. Let E be the event that either $s(z_0) \notin T$ or $s(y_0) \notin T$. Then clearly $\mathbb{P}[E] \leq 2r/m$. Let

$$J(s) = \{z \in Z \setminus \{z_0\} \mid s(z) \in B_0 \text{ and } s(g(z)) \notin C_0\}.$$

Note that if $J(s) \neq \emptyset$, then $sgs^{-1}(B_0)$ intersects at least two of the blocks of \mathcal{B} and thus $sgs^{-1} \notin H$. Hence it suffices to show that

$$(5.1) \quad \mathbb{P}[|J(s)| > 0] \rightarrow 1 \quad \text{as } d \rightarrow \infty.$$

Since we will be using Lemma 5.1, we need to compute the asymptotics of the expectation and variance of the random variable

$$|J(s)| = \sum_{z \in Z \setminus \{z_0\}} 1_{[s(z) \in B_0, s(g(z)) \notin C_0]}.$$

We will often implicitly use that $0 < \frac{d}{n} \leq \frac{1}{2}$, and that $r = O(1)$, $\frac{d}{n} = O(1)$ and $\frac{n-d}{n} = O(1)$. To compute both the expectation $\mathbb{E}[|J(s)|]$ and the second moment $\mathbb{E}[|J(s)|^2]$, we will separately condition on the events E , $[C_0 = B_0] \setminus E$ and $[C_0 \neq B_0] \setminus E$. On E the expectation is 0, and otherwise we have that

$$\begin{aligned} \mathbb{E}[|J(s)| \mid [C_0 = B_0] \setminus E] &= \sum_{z \in Z \setminus \{z_0\}} \mathbb{P}[s(z) \in B_0, s(g(z)) \notin B_0 \mid s(y_0) \in B_0] \\ &= (cm - 1) \frac{(d-2)(m-d)}{(m-2)(m-3)} \\ &= cd(1 - \frac{d}{m}) + O(1) \end{aligned}$$

and that

$$\begin{aligned} \mathbb{E}[|J(s)| \mid [C_0 \neq B_0] \setminus E] &= \sum_{z \in Z \setminus \{z_0\}} \mathbb{P}[s(z) \in B_0, s(g(z)) \notin C_0 \mid s(y_0) \notin B_0] \\ &= (cm - 1) \frac{(d-1)(m-(d-2))}{(m-2)(m-3)} \\ &= cd(1 - \frac{d}{m}) + O(1). \end{aligned}$$

Since $\mathbb{P}[E] \leq 2r/m$, we have that $cd(1 - \frac{d}{m})(1 - \mathbb{P}[E]) = cd(1 - \frac{d}{m}) + O(1)$ and hence we obtain that

$$(5.2) \quad \mathbb{E}[|J(s)|] = cd(1 - \frac{d}{m}) + O(1)$$

and that

$$(5.3) \quad \mathbb{E}[|J(s)|^2] = [cd(1 - \frac{d}{m})]^2 + O(d),$$

where in the second equality we use the fact that $\mathbb{E}[|J(s)|] = O(d)$. For the second moment, we have that

$$\begin{aligned} &\mathbb{E}[|J(s)|^2 \mid [C_0 = B_0] \setminus E] - \mathbb{E}[|J(s)| \mid [C_0 = B_0] \setminus E]^2 \\ &= \sum_{w \neq z \in Z \setminus \{z_0\}} \mathbb{P}[s(w), s(z) \in B_0, s(g(w)), s(g(z)) \notin B_0 \mid s(y_0) \in B_0] \\ &= (cm - 1)(cm - 2) \frac{(d-2)(d-3)(m-d)(m-(d-1))}{(m-2)(m-3)(m-4)(m-5)} \\ &= [cd(1 - \frac{d}{m})]^2 + O(d) \end{aligned}$$

and

$$\begin{aligned}
& \mathbb{E} [|J(s)|^2 \mid [C_0 \neq B_0] \setminus E] - \mathbb{E} [|J(s)| \mid [C_0 \neq B_0] \setminus E] \\
&= \sum_{w \neq z \in Z \setminus \{z_0\}} \mathbb{P} [s(w), s(z) \in B_0, s(g(w)), s(g(z)) \notin C_0 \mid s(y_0) \notin B_0] \\
&= (cm - 1)(cm - 2) \frac{(d-1)(d-2)(m-(d-1))(m-(d-2))}{(m-2)(m-3)(m-4)(m-5)} \\
&= [cd(1 - \frac{d}{m})]^2 + O(d).
\end{aligned}$$

Since $[cd(1 - \frac{d}{n})]^2(1 - \mathbb{P}[E]) = [cd(1 - \frac{d}{n})]^2 + O(d)$, it follows that

$$(5.4) \quad \mathbb{E} [|J(s)|^2] = [cd(1 - \frac{d}{m})]^2 + O(d).$$

Combining (5.3) and (5.4) we obtain that

$$\text{Var}(|J(s)|) = \mathbb{E} [|J(s)|^2] - \mathbb{E} [|J(s)|]^2 = O(d),$$

and hence $\text{Var}(|J(s)|)^{1/2} = O(\sqrt{d})$. Of course, (5.2) implies that $d = O(\mathbb{E} [|J(s)|])$. Thus there exists a constant $C > 0$ such that $\sigma = \text{Var}(|J(s)|)^{1/2} \leq C\sqrt{d}$ and $d \leq C\mathbb{E} [|J(s)|] = C\mu$ for all sufficiently large d . It follows that

$$\mu/\sigma \geq C^{-1}d/C\sqrt{d} = C^{-2}\sqrt{d} \rightarrow \infty \quad \text{as } d \rightarrow \infty.$$

Applying Lemma 5.1, we conclude that $\mathbb{P} [|J(s)| > 0] \rightarrow 1$ as $d \rightarrow \infty$, which proves (5.1). This completes the proof of Lemma 5.2. \square

Lemma 5.3. *For each $\varepsilon > 0$, there exists an integer r_ε such that if $r \geq r_\varepsilon$ and $H < S = \text{Sym}(a\ell)$ is an intransitive subgroup with an H -invariant subset $U \subset [a\ell]$ such that $r = |U| \leq a\ell/2$, then $|\{s \in S \mid sgs^{-1} \in H\}|/|S| < \varepsilon$.*

Proof. The proof is similar to that of Lemma 5.2, although the computations are simpler. Let $m = a\ell$ and let Z, Y , and c be as in Lemma 5.2. Fix some H -invariant $U \subseteq [m]$ with $|U| = r$. Let $s \in S$ be a uniformly random permutation and let

$$I(s) = \{z \in Z \mid s(z) \in U \text{ and } s(g(z)) \notin U\}.$$

If $I(s) \neq \emptyset$, then U is not sgs^{-1} -invariant and thus $sgs^{-1} \notin H$. Hence it suffices to show that

$$(5.5) \quad \mathbb{P} [|I(s)| > 0] \rightarrow 1 \text{ as } r \rightarrow \infty.$$

Computations similar to those in the proof of Lemma 5.2 show that

$$(5.6) \quad \mathbb{E} [|I(s)|] = cr(1 - \frac{r}{m}) + O(1) \text{ and } \mathbb{E} [|I(s)|^2] = [cr(1 - \frac{r}{m})]^2 + O(r).$$

It follows that $\text{Var}(|I(s)|)^{1/2} = O(\sqrt{r})$ and $r = O(\mathbb{E} [|I(s)|])$; and another application of Lemma 5.1 shows that $\mathbb{P} [|I(s)| > 0] \rightarrow 1$ as $r \rightarrow \infty$. \square

6. THE PROOF OF THEOREM 1.1

In this section, we will present the proof of Theorem 1.1. Let (k_n) be a sequence of natural numbers $k_n \geq 2$, let $X_n = \prod_{0 \leq m \leq n} [k_m]$, and let $G = \bigcup_{n \in \mathbb{N}} G_n$ be the corresponding SDS-group with $G_n = \text{Sym}(X_n)$. Let ν be an ergodic IRS of G . Then we can suppose that ν is not a Dirac measure δ_N concentrating on a normal subgroup $N \trianglelefteq G$. Applying Creutz-Peterson [4, Proposition 3.3.1], let ν be the stabilizer distribution of the ergodic action $G \curvearrowright (Z, \mu)$ and let $\chi(g) = \mu(\text{Fix}_Z(g))$

be the corresponding character. Then, since $\nu \neq \delta_1$, it follows that $\chi \neq \chi_{\text{reg}}$, where χ_{reg} is the *regular character* defined by

$$\chi_{\text{reg}}(g) = \begin{cases} 1 & \text{if } g = 1; \\ 0 & \text{if } g \neq 1. \end{cases}$$

For each $z \in Z$ and $n \in \mathbb{N}$, let $\Omega_n(z) = \{g \cdot z \mid g \in G_n\}$. Then, by Theorem 2.1, for μ -a.e. $z \in Z$, for all $g \in G$, we have that

$$(6.1) \quad \mu(\text{Fix}_Z(g)) = \lim_{n \rightarrow \infty} |\text{Fix}_{\Omega_n(z)}(g)| / |\Omega_n(z)|.$$

Fix such an element $z \in Z$ and let $H = \{h \in G \mid h \cdot z = z\}$ be the corresponding point stabilizer. Clearly we can suppose that the element $z \in Z$ has been chosen so that H is not a normal subgroup of G and so that if $g \in H$, then $\chi(g) > 0$. For each $n \in \mathbb{N}$, let $H_n = H \cap G_n$. Then, by Proposition 2.2, for each $g \in G$, we have that

$$\mu(\text{Fix}_Z(g)) = \lim_{n \rightarrow \infty} |g^{G_n} \cap H_n| / |g^{G_n}| = |\{s \in G_n \mid sgs^{-1} \in H_n\}| / |G_n|.$$

We will consider the various possibilities for the action of $H_n \leq G_n = \text{Sym}(X_n)$ on the set X_n .

Lemma 6.1. *There exist only finitely many $n \in \mathbb{N}$ such that H_n acts transitively on X_n .*

Proof. Suppose, on the contrary, that $T = \{n \in \mathbb{N} \mid H_n \text{ acts transitively on } X_n\}$ is infinite. First consider the case when there are infinitely many $n \in T$ such that H_n acts primitively on X_n . If $\text{Alt}(X_n) \leq H_n$ for infinitely many $n \in T$, then either $H = G$ or $H = A(G)$, which contradicts the fact that H is not a normal subgroup of G ; and hence there are only finitely many such $n \in T$. But then Lemma 4.3 implies that

$$\chi(g) = \lim_{n \rightarrow \infty} |g^{G_n} \cap H_n| / |g^{G_n}| = 0$$

for all $1 \neq g \in G$, which contradicts the fact that $\chi \neq \chi_{\text{reg}}$. Thus H_n acts imprimitively on X_n for all but finitely many $n \in T$. For each $d \geq 2$, let B_d be the set of $n \in T$ such that:

- (a) each nontrivial proper system of imprimitivity for the action of H_n on X_n has blocksize at most d ; and
- (b) there exists a (necessarily maximal) system of imprimitivity \mathcal{B}_n of blocksize d for the action of H_n on X_n .

If each B_d is finite, then Lemma 5.2 (in the case when $r = 0$) implies that

$$\chi(g) = \lim_{n \rightarrow \infty} |\{s \in G_n \mid sgs^{-1} \in H_n\}| / |G_n| = 0$$

for all $1 \neq g \in G$, which again contradicts the fact that $\chi \neq \chi_{\text{reg}}$. Thus there exists $d \geq 2$ such that B_d is infinite. Since $\chi \neq \chi_{\text{reg}}$, Lemma 4.5 implies that for all but finitely many $n \in B_d$, the induced action of H_n on \mathcal{B}_n contains $\text{Alt}(\mathcal{B}_n)$. Choose some $m \in B_d$ such that $|X_m| = a \gg d$ and let $g \in H_m$ induce either an a/d -cycle or an $(a/d-1)$ -cycle on \mathcal{B}_m (depending upon whether a/d is odd or even). Suppose that g has a cycle decomposition as an element of $\text{Sym}(X_n)$ consisting of k_i m_i -cycles for $1 \leq i \leq t$, where each $k_i \geq 1$. Since g has at most $2d$ orbits on X_m , it follows that $\sum k_i \leq 2d$ and so $a - \sum k_i > a/d$. Next suppose that $n \in B_d$ and

that $n \gg m$. Let $|X_n| = a\ell$. Then, applying Lemma 4.4, there exist constants $b, c > 0$ such that

$$|H_n| \leq |\text{Sym}(d) \text{ wr } \text{Sym}(\ell a/d)| < b c^{a\ell} \ell^{a\ell/d}.$$

Also, by Lemma 4.2, there exist constants $r, s > 0$ such that $|g^{G_n}| > r s^{a\ell} \ell^{(a-\sum k_i)\ell}$. Since $a - \sum k_i > a/d$, it follows that

$$\chi(g) = \lim_{n \rightarrow \infty} |g^{G_n} \cap H_n| / |g^{G_n}| \leq \lim_{n \rightarrow \infty} |H_n| / |g^{G_n}| = 0,$$

which contradicts the fact that $\chi(g) > 0$ for all $g \in H$. \square

Let $I = \{n \in \mathbb{N} \mid H_n \text{ acts intransitively on } X_n\}$; and for each $n \in I$, let

$$r_n = \max\{|U| : U \subseteq X_n \text{ is } H_n\text{-invariant and } |U| \leq \frac{1}{2}|X_n|\}.$$

Then each $r_n \geq 1$. Furthermore, since $\chi \neq \chi_{\text{reg}}$, Lemma 5.3 implies that the sequence $(r_n)_{n \in I}$ is bounded above. Let $r = \liminf r_n$ and let I_r be the set of integers $n \in I$ such that:

- (i) $r_n = r$;
- (ii) $n > \max\{m \in I \mid r_m < r\}$; and
- (iii) $n > r + 1$.

Here we have chosen $n > r + 1$ in order to ensure that $|X_n| > 4r$.

Lemma 6.2. *If $n \in I_r$, then there exists a unique H_n -invariant subset $U_n \subset X_n$ of cardinality r and H_n acts transitively on $X_n \setminus U_n$.*

Proof. By definition, there exists at least one H_n -invariant subset $U_n \subset X_n$ of cardinality r . Suppose that $V \subseteq X_n \setminus U_n$ is an H_n -orbit. If $|V| \leq \frac{1}{2}|X_n|$, then $|V| \leq r$ and so the H_n -invariant subset $U_n \cup V$ satisfies $r < |U_n \cup V| \leq 2r < \frac{1}{2}|X_n|$, which is a contradiction. Thus each H_n -orbit $V \subseteq X_n \setminus U_n$ satisfies $|V| > \frac{1}{2}|X_n|$, and this clearly implies that H_n acts transitively on $X_n \setminus U_n$. \square

For each $n \in I_r$, let $K_n = \{g \in H_n \mid g \cdot u = u \text{ for all } u \in U_n\}$ be the pointwise stabilizer of U_n and let $Y_n = X_n \setminus U_n$. As usual, we will identify K_n with the corresponding subgroup of $\text{Sym}(Y_n)$.

Lemma 6.3. *If $n \in I_r$, then $\text{Alt}(Y_n) \leq K_n \leq \text{Sym}(Y_n)$.*

Proof. Let \bar{H}_n be the subgroup of $\text{Sym}(Y_n)$ induced by the action of H_n on Y_n . Then, arguing as in the proof of Lemma 6.1, we see first that \bar{H}_n must act primitively on Y_n and then that $\text{Alt}(Y_n) \leq \bar{H}_n$. Let $\pi_n : H_n \rightarrow \text{Sym}(U_n)$ be the homomorphism defined by $g \mapsto g \upharpoonright U_n$. Then $K_n = \ker \pi_n \leq H_n$; and identifying K_n with the corresponding subgroup of $\text{Sym}(Y_n)$, we have that $K_n \leq \bar{H}_n$. Since $|Y_n| = |X_n| - r > 3r$, it follows that $[\bar{H}_n : K_n] \leq r! < |\text{Alt}(Y_n)|$ and hence $\text{Alt}(Y_n) \leq K_n$. \square

From now on, let $n_0 = \min I_r$.

Lemma 6.4. (i) $I_r = \{n \in \mathbb{N} \mid n \geq n_0\}$.

(ii) *For each $n \in I_r$ and $w \in U_n$, there exists a unique $i \in [k_{n+1}]$ such that $w \hat{=} i \in U_{n+1}$.*

Proof. Suppose that $n_0 < n \in I_r$. Let $U = \{u \upharpoonright n-1 \mid u \in U_n\} \subseteq X_{n-1}$ and let

$$U_n^+ = \{w \in X_n \mid (\exists u \in U_n) w \upharpoonright n-1 = u \upharpoonright n-1\}.$$

Then $\text{Alt}(X_n \setminus U_n^+) \leq K_n \leq H_n$ and this implies that $\text{Alt}(X_{n-1} \setminus U) \leq H_{n-1}$. Thus $r_{n-1} \leq |U| \leq |U_n| = r$. Since $n-1 \geq n_0 > \max\{m \in I \mid r_m < r\}$, it follows that $r_{n-1} = r$ and this implies that the map $u \mapsto u \upharpoonright n-1$ from U_n to U is injective. The result follows. \square

Let $X = \prod_{n \geq 0} [k_n]$ and let $\mathcal{B}_H = \{x \in X \mid x \upharpoonright n \in U_n \text{ for all } n \in I_r\}$. Then clearly \mathcal{B}_H is an element of the standard Borel space $[X]^r$ of r -element subsets of X ; and it is easily checked that \mathcal{B}_H is precisely the set of $x \in X$ such that the corresponding orbit $H \cdot x$ is finite. It follows that the Borel map $H \xrightarrow{\varphi} \mathcal{B}_H$, defined on the ν -measure 1 subset of those $H \in \text{Sub}_G$ such that

- (a) H satisfies (6.1),
- (b) H is not a normal subgroup of G , and
- (c) if $g \in H$, then $\chi(g) > 0$,

is G -equivariant; and so $\varphi_*\nu$ is an ergodic G -invariant probability measure on $[X]^r$.

Suppose now that there exists $n \in I_r$ such that H_n acts nontrivially on U_n ; say, $u \neq g \cdot u = v$, where $u, v \in U_n$ and $g \in H_n$. Then, regarding g as an element of H_{n+1} , we have that $u \hat{\cdot} i \neq g \cdot u \hat{\cdot} i = v \hat{\cdot} i$ for all $i \in [k_{n+1}]$ and it follows that there exists $i \in [k_{n+1}]$ such that $u \hat{\cdot} i, v \hat{\cdot} i \in U_{n+1}$. Continuing in this fashion, we see that there exist $x \neq y \in \mathcal{B}_H$ such that $x E_0 y$, where E_0 is the Borel equivalence relation defined on X by

$$x E_0 y \iff x(n) = y(n) \text{ for all but finitely many } n \in \mathbb{N}.$$

Consequently, the following result implies that ν concentrates on those $H \in \text{Sub}_G$ such that H_n acts trivially on U_n for all $n \in I_r$.

Lemma 6.5. *There does not exist a G -invariant Borel probability measure on the standard Borel space $S = \{F \in [X]^r : E_0 \upharpoonright F \text{ is not the identity relation}\}$.*

Before proving Lemma 6.5, we will complete the proof of Theorem 1.1. Continuing our analysis of the ν -generic subgroup $H < G$, we can suppose that H_n acts trivially on U_n for all $n \in I_r$. Hence, applying Lemmas 6.3 and 6.4, we see that $\text{Alt}(Y_n) \leq H_n \leq \text{Sym}(Y_n)$ for all $n \geq n_0$. If $n_0 \leq n < m$ and k_m is even, then identifying $G_n = \text{Sym}(X_n)$ with the corresponding subgroup of $G_m = \text{Sym}(X_m)$, we have that $\text{Sym}(Y_n) \leq \text{Alt}(Y_m)$ and so $H_n = \text{Sym}(Y_n)$. In particular, if G is a simple SDS -group, then $H_n = \text{Sym}(Y_n)$ for all $n \geq n_0$. Similarly, if G is not simple, then either $H_n = \text{Sym}(Y_n)$ for all $n \geq n_0$, or else $H_n = \text{Alt}(Y_n)$ for all but finitely many $n \geq n_0$.

Notation 6.6. For each finite set Y , let $S^+(Y) = \text{Sym}(Y)$ and $S^-(Y) = \text{Alt}(Y)$.

Definition 6.7. For each $r \geq 1$ and $\varepsilon = \pm$, let $\mathcal{S}_r^\varepsilon$ be the standard Borel space of subgroups $H < G$ such that there is an integer n_0 such that for all $n \geq n_0$, there exists a subset $U_n \subset X_n$ of cardinality r such that $H_n = S^\varepsilon(X_n \setminus U_n)$.

Summing up, we have shown that there exists $r \geq 1$ and $\varepsilon = \pm$ such that the ergodic IRS ν concentrates on $\mathcal{S}_r^\varepsilon$. Thus the following lemma completes the proof of Theorem 1.1.

Proposition 6.8. *$\sigma_r, \tilde{\sigma}_r$ are the unique ergodic probability measures on $\mathcal{S}_r^+, \mathcal{S}_r^-$ under the conjugation action of G .*

Proof. Suppose, for example, that m is an ergodic probability measure on \mathcal{S}_r^+ . Then it is enough to show that if $B \subseteq \text{Sub}_G$ is a basic clopen subset, then $m(B) = \sigma_r(B)$. Let $B = \{K \in \text{Sub}_G \mid K \cap G_m = L\}$, where $m \in \mathbb{N}$ and $L \leq G_m$ is a subgroup. By the Pointwise Ergodic Theorem, there exists $H \in \mathcal{S}_r^+$ such that

$$\begin{aligned} m(B) &= \lim_{n \rightarrow \infty} |\{g \in G_n \mid gHg^{-1} \in B\}|/|G_n| \\ &= \lim_{n \rightarrow \infty} |\{g \in G_n \mid gH_ng^{-1} \cap G_m = L\}|/|G_n|. \end{aligned}$$

Similarly, there exists $H' \in \mathcal{S}_r^+$ such that

$$\sigma_r(B) = \lim_{n \rightarrow \infty} |\{g \in G_n \mid gH'_ng^{-1} \cap G_m = L\}|/|G_n|.$$

Since $H, H' \in \mathcal{S}_r^+$, there exists $a \in \mathbb{N}$ such that H_n and H'_n are conjugate in G_n for all $n \geq a$; and this implies that

$$\begin{aligned} \lim_{n \rightarrow \infty} |\{g \in G_n \mid gH_ng^{-1} \cap G_m = L\}|/|G_n| \\ = \lim_{n \rightarrow \infty} |\{g \in G_n \mid gH'_ng^{-1} \cap G_m = L\}|/|G_n|. \end{aligned}$$

□

The remainder of this section will be devoted to the proof of Lemma 6.5. First we need to recall Nadkarni's Theorem on compressible group actions. (Here we follow Dougherty-Jackson-Kechris [5].) Suppose that Γ is a countable discrete group and that $E = E_\Gamma^Y$ is the orbit equivalence relation of a Borel action of Γ on a standard Borel space Y . Then $[[E]]$ denotes the set of Borel bijections $f : A \rightarrow B$, where $A, B \subseteq Y$ are Borel subsets, such that $f(y) E y$ for all $y \in A$. If $A, B \subseteq Y$ are Borel subsets, then we write $A \sim B$ if there exists an $f \in [[E]]$ with $f : A \rightarrow B$; and we write $A \preceq B$ if there exists a Borel subset $B' \subseteq B$ with $A \sim B'$. The usual Schröder-Bernstein argument shows that

$$A \sim B \iff A \preceq B \text{ and } B \preceq A.$$

The orbit equivalence relation E is *compressible* if there exists a Borel subset $A \subseteq Y$ such that $Y \sim A$ and $Y \setminus A$ intersects every E -class. We will make use of the easy direction of the following theorem; i.e. the observation that (ii) implies (i).

Theorem 6.9 (Nadkarni [9]). *If $E = E_\Gamma^Y$ is the orbit equivalence relation of a Borel action of a countable group Γ on a standard Borel space Y , then the following are equivalent:*

- (i) $E = E_\Gamma^Y$ is not compressible.
- (ii) There exists a Γ -invariant Borel probability measure on Y .

Thus to prove Lemma 6.5, it is enough to show that the orbit equivalence relation E for the action of G on S is compressible. To see this, for each $\ell \in \mathbb{N}$, let S_ℓ be the Borel subset of those $F \in S$ for which ℓ is the least integer such that if $x, y \in F$ and $x E_0 y$, then $x(n) = y(n)$ for all $n > \ell$. Clearly if $\ell < m$, then $S_\ell \preceq S_m$; and it follows that if $I \subseteq \mathbb{N}$ is an infinite co-infinite subset, then $A = \bigcup_{\ell \in I} S_\ell$ is a Borel subset such that $S \sim A$ and $S \setminus A$ is full.

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MATHEMATICS DEPARTMENT, RUTGERS UNIVERSITY, 110 FRELINGHUYSEN ROAD, PISCATAWAY, NEW JERSEY 08854-8019, USA

E-mail address: `stthomas@math.rutgers.edu`

E-mail address: `rtuckerd@gmail.com`