ON THE CONCEPT OF "LARGENESS" IN GROUP THEORY

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Abstract. In this paper, we will consider the Borel complexity of Pride's quasi-order $\succeq_p$ and Pride's equivalence relation $\approx_p$ on the space $\mathcal{G}$ of finitely generated groups. Our main results show that these relations are as complex as they conceivably could be.

1. Introduction

Following Pride [16, 2], the group G_1 is said to be *larger* than the group G_2, written $G_1 \succeq_p G_2$, if there exist subgroups $H_1 \leqslant G_1$ and $N_2 \leqslant H_2 \leqslant G_2$ such that the following conditions are satisfied:

(i) $[G_1 : H_1]$, $[G_2 : H_2] < \infty$.

(ii) N_2 is a finite normal subgroup of H_2.

(iii) There exists a surjective homomorphism $f : H_1 \to H_2/N_2$.

Let $\approx_p$ be the associated equivalence relation defined by

$$G_1 \approx_p G_2 \iff G_1 \succeq_p G_2 \text{ and } G_2 \succeq_p G_1;$$

and for each group G, let $[\,G\,]_{\approx_p}$ denote the corresponding $\approx_p$-equivalence class. Then $\preceq_p$ induces a partial ordering of the collection of $\approx_p$-equivalence classes, which we will also denote by $\preceq_p$. Throughout this paper, we will only be concerned with the restrictions of the relations $\preceq_p$ and $\approx_p$ to the space $\mathcal{G}$ of *finitely generated groups*. Here it is clear that $[\,1\,]_{\approx_p}$ is the $\preceq_p$-least class and that $[\,\mathbb{F}\,]_{\approx_p}$ is the $\preceq_p$-greatest class, where $\mathbb{F}$ is any finitely generated nonabelian free group. Similarly, if G is a finitely generated infinite simple group, then G is *atomic*; i.e. $[\,G\,]_{\approx_p}$ is an immediate successor of $[\,1\,]_{\approx_p}$ in the $\preceq_p$-partial order. (For more interesting examples of atomic groups, see Neumann [14] and Grigorchuk-Wilson [7].) In contrast, it is unknown whether $[\,\mathbb{F}\,]_{\approx_p}$ has an immediate predecessor under the $\preceq_p$-partial order. (For a discussion of this problem, see Edjvet-Pride [2].)

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In this paper, we will consider the Borel complexity of Pride's quasi-order $\succeq_p$ and Pride's equivalence relation $\approx_p$ on the space $\mathcal{G}$ of finitely generated groups. Our main results show that these relations are as complex as they conceivably could be. Before we can give exact statements of our main results, we first need to describe how to represent the class of finitely generated groups by the elements of a suitable Polish space $\mathcal{G}$ and then we need to recall some of the basic notions of the theory of Borel equivalence relations.

We will begin by describing the the space $\mathcal{G}$ of (marked) finitely generated groups, which was first introduced by Grigorchuk [5]. (For a fuller treatment, see Champetier [1] or Grigorchuk [6].) A *marked group* $(G, \bar{s})$ consists of a finitely generated group with a distinguished sequence $\bar{s} = (s_1, \cdots, s_m)$ of generators. (Here the sequence $\bar{s}$ is allowed to contain repetitions and we also allow the possibility that the sequence contains the identity element.) Two marked groups $(G, (s_1, \cdots, s_m))$ and $(H, (t_1, \cdots, t_n))$ are said to be *isomorphic* if $m = n$ and the map $s_i \mapsto t_i$ extends to a group isomorphism between G and H.

Definition 1.1. For each $m \geq 2$, let $\mathcal{G}_m$ be the set of *isomorphism types* of marked groups $(G, (s_1, \cdots, s_m))$ with m distinguished generators.

Let $\mathbb{F}_m$ be the free group on the generators $\{x_1, \cdots, x_m\}$. Then for each marked group $(G, (s_1, \cdots, s_m))$, we can define an associated surjective homomorphism $\theta_{G,\bar{s}} : \mathbb{F}_m \to G$ by $\theta_{G,\bar{s}}(x_i) = s_i$. It is easily checked that two marked groups $(G, (s_1, \cdots, s_m))$ and $(H, (t_1, \cdots, t_m))$ are isomorphic if and only if $\ker \theta_{G,\bar{s}} = \ker \theta_{H,\bar{t}}$. Thus we can naturally identify $\mathcal{G}_m$ with the set $\mathcal{N}_m$ of normal subgroups of $\mathbb{F}_m$. Note that $\mathcal{N}_m$ is a closed subset of the compact space $\mathcal{P}(\mathbb{F}_m)$ of all subsets of $\mathbb{F}_m$ and so $\mathcal{N}_m$ is also a compact space.[1] Hence, via the above identification, we can regard $\mathcal{G}_m$ as a compact space.

For each $m \geq 2$, there is a natural embedding of $\mathcal{N}_m$ into $\mathcal{N}_{m+1}$ defined by

$$N \mapsto \text{ the normal closure of } N \cup \{x_{m+1}\} \text{ in } \mathbb{F}_{m+1};$$

[1] If C is any countably infinite set, then the Cantor space $2^C = \{ f \mid f : C \to \{0,1\} \}$ with the natural product topology is a compact space. Hence, identifying each subset $B \subseteq C$ with its characteristic function $\chi_B \in 2^C$, the powerset $\mathcal{P}(C)$ is also a compact space.

and this enables us to regard $\mathcal{N}_m$ as a clopen subset of $\mathcal{N}_{m+1}$ and to form the locally compact Polish space $\mathcal{N} = \bigcup \mathcal{N}_m$. Note that $\mathcal{N}$ can be identified with the space of normal subgroups N of the free group $\mathbb{F}_\infty$ on countably many generators such that N contains all but finitely many elements of the basis $X = \{x_i \mid i \in \mathbb{N}^+\}$. Similarly, we can form the locally compact Polish space $\mathcal{G} = \bigcup \mathcal{G}_m$ of finitely generated groups via the corresponding natural embedding

$$(G, (s_1, \cdots, s_m)) \mapsto (G, (s_1, \cdots, s_m, 1))$$

In the literature, the Polish spaces $\mathcal{N}$ and $\mathcal{G}$ are usually completely identified. However, in this paper, it will be convenient to distinguish between these two spaces.

Next we need to recall some of the basic notions of the theory of Borel equivalence relations, including the notion of a Borel reduction which will provide us with a measure of the relative complexity of the commonly studied equivalence relations on the space $\mathcal{G}$ of finitely generated groups. If X is a Polish space, then a *Borel equivalence relation* on X is an equivalence relation $E \subseteq X \times X$ which is a Borel subset of $X \times X$. For example, the isomorphism relation, the virtual isomorphism relation and the quasi-isometry relation are all Borel equivalence relations on $\mathcal{G}$. (See Thomas [19].) If E, F are Borel equivalence relations on the Polish spaces X, Y respectively, then we say that E is *Borel reducible* to F and write $E \leq_B F$ if there exists a Borel map $f : X \to Y$ such that

$$x \, E \, y \iff f(x) \, F \, f(y).$$

We say that E and F are *Borel bireducible* and write $E \sim_B F$ if both $E \leq_B F$ and $F \leq_B E$. Finally we write $E <_B F$ if both $E \leq_B F$ and $F \nleq_B E$. The notion of a Borel reduction from E to F is intended to capture the idea of an *explicit reduction* from the E-classification problem to the F-classification problem.

A Borel equivalence relation E on a Polish space X is said to be *countable* if every E-class is countable; and a countable Borel equivalence relation E is said to be *universal* if $F \leq_B E$ for every countable Borel equivalence relation F. For example, by Thomas-Velickovic [20], the isomorphism relation $\cong$ on $\mathcal{G}$ is a universal countable Borel equivalence relation.

Remark 1.2. Two finitely generated groups $G_1, G_2 \in \mathcal{G}$ are *bi-embeddable*, written $G_1 \approx_{em} G_2$, if G_1 embeds into G_2 and G_2 embeds into G_1. It is easily checked that the bi-embeddability relation $\approx_{em}$ is a Borel equivalence relation on $\mathcal{G}$. In fact, since each finitely generated group has only countably many finitely generated subgroups, it follows that $\approx_{em}$ is a countable Borel equivalence relation and hence there exists a Borel reduction $\varphi : \mathcal{G} \to \mathcal{G}$ from the bi-embeddability relation $\approx_{em}$ to the isomorphism relation $\cong$. However, I do not know how to explicitly define an example of such a Borel reduction φ. (The proof that $\cong$ is a universal countable Borel equivalence relation ultimately relies on the Lusin-Novikov Uniformization Theorem [9, Theorem 18.10] and this does not provide an explicit example of such a Borel reduction.)

Of course, it is clear that Pride's equivalence relation $\approx_p$ is not a countable Borel equivalence relation; and, in fact, the main result of this paper implies that $\cong \, <_B \, \approx_p$. Hence if we wish to understand the precise Borel complexity of Pride's equivalence relation $\approx_p$, then we must work within a strictly larger class of Borel equivalence relations than the relatively well-understood class of countable Borel equivalence relations.

Definition 1.3. A binary relation R on a Polish space X is said to be K_σ if R is the union of countably many compact subsets of $X \times X$.

For example, the isomorphism relation, the virtual isomorphism relation and the quasi-isometry relation are all K_σ equivalence relations on $\mathcal{G}$. (See Thomas [19].) By Kechris [8] and Louveau-Rosendal [10], there also exists a universal K_σ equivalence relation. In fact, Rosendal [17] has recently shown that the relation of Lipschitz equivalence between compact metric spaces is a universal K_σ equivalence relation; and Thomas [19] has conjectured that the quasi-isometry relation on $\mathcal{G}$ is also universal K_σ. In Section 5, we will prove the following result, which provides the first purely group-theoretic example of a complete $\mathbf{K}_\sigma$ equivalence relation.

Theorem 1.4. *Pride's equivalence relation $\approx_p$ is a universal $\mathbf{K}_\sigma$ equivalence relation.*

Pride's equivalence relation $\approx_p$ can be regarded as a combination of two more basic equivalence relations; namely, the virtual isomorphism relation $\approx_{VI}$ and the bi-surjectability equivalence relation $\approx_s$, which are defined as follows.

Definition 1.5. Two finitely generated groups $G_1, G_2 \in \mathcal{G}$ are said to be *virtually isomorphic* or *commensurable up to finite kernels*, written $G_1 \approx_{VI} G_2$, if there exist subgroups $N_i \leqslant H_i \leqslant G_i$ for $i = 1, 2$ satisfying the following conditions:

(a) $[G_1 : H_1]$, $[G_2 : H_2] < \infty$.
(b) N_1, N_2 are finite normal subgroups of H_1, H_2 respectively.
(c) $H_1/N_1 \cong H_2/N_2$.

Definition 1.6. The *surjectability relation* $\succeq_s$ is the quasi-order on the space $\mathcal{G}$ of finitely generated groups defined by

- $G_1 \succeq_s G_2$ if there exists a surjective homomorphism $f : G_1 \to G_2$;

and the associated *bi-surjectability equivalence relation* $\approx_s$ is defined by

- $G_1 \approx_s G_2$ if both $G_1 \succeq_s G_2$ and $G_2 \succeq_s G_1$.

Combining Theorem 1.4 with the earlier results of Thomas [18, 19], it follows that $\cong \,<_B\, \approx_{VI} \,<_B\, \approx_p$. In particular, the Borel complexity of $\approx_{VI}$ is strictly less than that of Pride's equivalence relation $\approx_p$. However, the following result shows that the bi-surjectability equivalence relation $\approx_s$ has precisely the same Borel complexity as Pride's equivalence relation $\approx_p$.

Theorem 1.7. *The bi-surjectability equivalence relation $\approx_s$ is a universal $\mathbf{K}_\sigma$ equivalence relation.*

This paper is organised as follows. In Section 2, we will discuss some basic results concerning $\mathbf{K}_\sigma$ quasi-orders and equivalence relations; and in Section 3, we will recall a fundamental result from small cancellation theory which will play a key role in the proofs of the main results. In Section 4, we will prove Theorem 1.7; and in Section 5, we will prove Theorem 1.4.

Our group-theoretic notation is standard. For example, if G is a group and $A \subseteq G$, then $\langle A \rangle$ denotes the subgroup of G which is generated by A and $C_G(A)$ denotes the centralizer of A in G.

2. $\mathbf{K}_\sigma$ QUASI-ORDERS AND EQUIVALENCE RELATIONS

As we will soon see, Theorems 1.4 and 1.7 are immediate consequences of the analogous results for the quasi-orders $\succeq_p$ and $\succeq_s$. Consequently, it is next necessary to say a few words about the basic theory of $\mathbf{K}_\sigma$ quasi-orders.

Recall that a binary relation R is said to be a *quasi-order* if R is symmetric and transitive. The basic notions of the theory of countable Borel equivalence relations have natural generalizations to the more general setting of Borel quasi-orders. For example, if R, S are Borel quasi-orders on the Polish spaces X, Y respectively, then R is said to be *Borel reducible* to S, again written $R \leq_B S$, if there exists a Borel map $f : X \to Y$ such that

$$x \, R \, y \iff f(x) \, S \, f(y).$$

By Louveau-Rosendal [10], the class of $\mathbf{K}_\sigma$ quasi-orders also admits universal elements. Furthermore, if R is a universal $\mathbf{K}_\sigma$ quasi-order on the Polish space X, then the associated equivalence relation E_R, defined by

$$x \, E_R \, y \iff x \, R \, y \text{ and } y \, R \, x,$$

is a universal $\mathbf{K}_\sigma$ equivalence relation. To see this, suppose that E is a $\mathbf{K}_\sigma$ equivalence relation on the Polish space Z. Then E is also a $\mathbf{K}_\sigma$ quasi-order and hence there exists a Borel reduction $f : Z \to X$ from E to R. Clearly f is also a Borel reduction from E to E_R and hence $E \leq_B E_R$. Thus Theorems 1.4 and 1.7 are immediate consequences of the following two results.

Theorem 2.1. *Pride's quasi-order $\succeq_p$ is a universal $\mathbf{K}_\sigma$ quasi-order.*

Theorem 2.2. *The surjectability relation $\succeq_s$ is a universal $\mathbf{K}_\sigma$ quasi-order.*

The proofs of Theorems 2.1 and 2.2 make essential use of the following concrete example of a universal $\mathbf{K}_\sigma$ quasi-order.

Definition 2.3. Let $\subseteq^{\mathbb{Z}^2,t}_{\mathcal{P}(\mathbb{Z}^2)}$ be the quasi-order on the Polish space $\mathcal{P}(\mathbb{Z}^2)$ defined by

$$S \subseteq^{\mathbb{Z}^2,t}_{\mathcal{P}(\mathbb{Z}^2)} T \iff (\exists \, (m,n) \in \mathbb{Z}^2 \,) \;\; (m,n) + S \subseteq T.$$

Theorem 2.4 (Louveau-Rosendal [10]). $\subseteq^{\mathbb{Z}^2,t}_{\mathcal{P}(\mathbb{Z}^2)}$ *is a universal $\mathbf{K}_\sigma$ quasi-order.*

Remark 2.5. By considering the Borel map $S \mapsto (\mathbb{Z}^2 \smallsetminus S)$, it follows that the reverse quasi-order $\supseteq^{\mathbb{Z}^2,t}_{\mathcal{P}(\mathbb{Z}^2)}$ is also universal $\mathbf{K}_\sigma$. This easily implies that if $\preceq$ is any universal $\mathbf{K}_\sigma$ quasi-order, then $\succeq$ is also universal $\mathbf{K}_\sigma$.

For technical reasons, we will find it more convenient to work with the restriction of the quasi-order $\subseteq^{\mathbb{Z}^2,t}_{\mathcal{P}(\mathbb{Z}^2)}$ to the following Borel subset of $\mathcal{P}(\mathbb{Z}^2)$.

Definition 2.6. Let $\mathcal{P}^\infty(\mathbb{Z}^2)$ be the Borel subset of $\mathcal{P}(\mathbb{Z}^2)$ consists of those $S \subseteq \mathbb{Z}^2$ such that for all finite subsets $F \subseteq \mathbb{Z}$,

$$S \cap \{ (k, \ell) \in \mathbb{Z}^2 \mid k \notin F \text{ and } \ell \notin F \} \neq \emptyset.$$

In order to simplify notation, we will denote the restriction of the quasi-order $\subseteq^{\mathbb{Z}^2,t}_{\mathcal{P}(\mathbb{Z}^2)}$ to $\mathcal{P}^\infty(\mathbb{Z}^2)$ by $\subseteq^{\mathbb{Z}^2}$.

Proposition 2.7. *The quasi-orders $\subseteq^{\mathbb{Z}^2,t}_{\mathcal{P}(\mathbb{Z}^2)}$ and $\subseteq^{\mathbb{Z}^2}$ are Borel bireducible.*

Proof. Clearly the inclusion map $\mathcal{P}^\infty(\mathbb{Z}^2) \hookrightarrow \mathcal{P}(\mathbb{Z}^2)$ is a Borel reduction from $\subseteq^{\mathbb{Z}^2}$ to $\subseteq^{\mathbb{Z}^2,t}_{\mathcal{P}(\mathbb{Z}^2)}$. Conversely, it is easily checked that the map

$$S \mapsto \{ (3k, 3\ell) \in \mathbb{Z}^2 \mid (k, \ell) \in S \} \cup (\ \mathbb{Z}^2 \smallsetminus (3\mathbb{Z})^2\ )$$

is a Borel reduction from $\subseteq^{\mathbb{Z}^2,t}_{\mathcal{P}(\mathbb{Z}^2)}$ to $\subseteq^{\mathbb{Z}^2}$. □

Hence, for example, in order to prove that $\succeq_p$ is a universal $\mathbf{K}_\sigma$ quasi-order, it will be enough to show that $\succeq_p$ is a $\mathbf{K}_\sigma$ relation and that there exists a Borel reduction from $\subseteq^{\mathbb{Z}^2}$ to $\succeq_p$.

3. The $C'(1/6)$ Cancellation Condition

In this section, we will recall some basic notions of small cancellation theory, which will play a key role in the proof of Theorem 2.2. (For a fuller treatment, see Lyndon-Schupp [11, Chapter V].)

Let $\mathbb{F}_n = \langle\, x_1, \cdots, x_n \,\rangle$ be the free group on n generators. Then a nontrivial reduced word $w \in \mathbb{F}_n$ is said to be *cyclically reduced* if the first and last letters of w are not inverses of each other. In this paper, we will only consider presentations

$$G = \langle\, x_1, \cdots, x_n \ \mid \mathcal{R} \,\rangle \tag{3.4}$$

such that every relator $r \in \mathcal{R}$ is cyclically reduced. If $\mathcal{R} \subseteq \mathbb{F}_n$ is a set of cyclically reduced words, then the *symmetrization* $\mathcal{R}^*$ of $\mathcal{R}$ is defined to be the smallest subset $\mathcal{R} \subseteq \mathcal{R}^* \subseteq \mathbb{F}_n$ such that the following conditions are satisfied:

(a) if $r \in \mathcal{R}^*$, then $r^{-1} \in \mathcal{R}^*$; and

(b) whenever $r = uv \in \mathcal{R}^*$ is the freely reduced product of the subwords u and v, then the *cyclic conjugate* $r^* = vu \in \mathcal{R}^*$.

Of course, since $vu = u^{-1}uvu$, it follows that the presentation

$$\langle\, x_1, \cdots, x_n \mid \mathcal{R}^* \,\rangle$$

defines the same group G as the presentation (3.4). The presentation (3.4) is said to be *symmetrized* if $\mathcal{R} = \mathcal{R}^*$.

Definition 3.5. The presentation (3.4) is said to satisfy the $C'(1/6)$ *cancellation condition* if whenever $r_1 \neq r_2 \in \mathcal{R}^*$ are distinct elements with $r_1 = bc_1$ and $r_2 = bc_2$ as freely reduced words, then

$$|b| < 1/6 \min\{\, |r_1|, |r_2| \,\}.$$

Here $|w|$ denotes the length of the word $w \in \mathbb{F}_n$.

In the next section, we will make repeated use of the following fundamental result, which is due to Greendlinger [3]. (A proof can also be found in Lyndon-Schupp [11, Section V.4].)

Theorem 3.6. *Suppose that $G = \langle\, x_1, \cdots, x_n \mid \mathcal{R} \,\rangle$ is a symmetrized presentation which satisfies the $C'(1/6)$ cancellation condition. Let w be a nontrivial cyclically reduced word in $x_1, \cdots, x_n$ such that $w = 1$ in G. Then there exists a cyclically reduced conjugate w^* of w and a relator $r \in \mathcal{R}$ such that w^* contains a subword s of r with $|s| > 1/2\,|r|$.*

4. The surjectability relation

In this section, we will prove that the surjectability relation $\succeq_s$ is a universal $\mathbf{K}_\sigma$ quasi-order on the space $\mathcal{G}$ of finitely generated groups. As explained in Section 2, this implies that the associated surjectability equivalence relation $\approx_s$ is a universal $\mathbf{K}_\sigma$ equivalence relation. To prove that $\succeq_s$ is a universal $\mathbf{K}_\sigma$ quasi-order, it is enough to show that $\succeq_s$ is a $\mathbf{K}_\sigma$ relation and that there exists a Borel reduction from $\subseteq^{\mathbb{Z}^2}$ to $\succeq_s$.

Lemma 4.1. *$\succeq_s$ is a $\mathbf{K}_\sigma$ quasi-order on the space $\mathcal{G}$ of finitely generated groups.*

Instead of working directly with $\mathcal{G}$, it will be more convenient to work with the space $\mathcal{N}$ of normal subgroups N of the free group $\mathbb{F}_\infty$ on countably many generators such that N contains all but finitely many elements of the basis $X = \{x_i \mid i \in \mathbb{N}^+\}$. Let $\mathrm{Aut}_f(\mathbb{F}_\infty)$ be the subgroup of $\mathrm{Aut}(\mathbb{F}_\infty)$ generated by the elementary Nielsen transformations

$$\{\alpha_i \mid i \in \mathbb{N}^+\} \cup \{\beta_{ij} \mid i \neq j \in \mathbb{N}^+\},$$

where α_i is the automorphism sending x_i to x_i^{-1} and leaving $X \smallsetminus \{x_i\}$ fixed; and β_{ij} is the automorphism sending x_i to $x_i x_j$ and leaving $X \smallsetminus \{x_i\}$ fixed. Then the natural action of the countable group $\mathrm{Aut}_f(\mathbb{F}_\infty)$ on $\mathbb{F}_\infty$ induces a corresponding action as a group of homeomorphisms on the space $\mathcal{N}$. Furthermore, if N, $M \in \mathcal{N}$, then $\mathbb{F}_\infty/N \cong \mathbb{F}_\infty/M$ if and only if there exists $\varphi \in \mathrm{Aut}_f(\mathbb{F}_\infty)$ such that $\varphi[N] = M$. (For example, see Champetier [1].)

It is easily checked that the inclusion relation $\subseteq$ is a $\mathbf{K}_\sigma$ relation on $\mathcal{N}$. Hence Lemma 4.1 is an immediate consequence of the following result.

Lemma 4.2. *If N, $M \in \mathcal{N}$, then the following conditions are equivalent.*

(i) $\mathbb{F}_\infty/N \succeq_s \mathbb{F}_\infty/M$.

(ii) *There exists $\varphi \in \mathrm{Aut}_f(\mathbb{F}_\infty)$ such that $\varphi[N] \subseteq M$.*

Proof. It is clear that (ii) implies (i). Suppose that $\mathbb{F}_\infty/N \succeq_s \mathbb{F}_\infty/M$. Then there exists a surjective homomorphism $f : \mathbb{F}_\infty \to \mathbb{F}_\infty/M$ such that $N \leqslant L = \ker f$. Since $L \in \mathcal{N}$ and $\mathbb{F}_\infty/L \cong \mathbb{F}_\infty/M$, there exists $\varphi \in \mathrm{Aut}_f(\mathbb{F}_\infty)$ such that $\varphi[L] = M$. Of course, this means that $\varphi[N] \subseteq M$. □

In the remainder of this section, we will define a Borel reduction $S \mapsto K_S$ from $\subseteq^{\mathbb{Z}^2}$ to $\succeq_s$. The construction of the following auxiliary group G_S makes essential use of the ideas of Champetier [1, Section 4].

Definition 4.3. Let $\mathbb{F}_4$ be the free group on $\{a, b, c, d\}$ and let φ, $\psi \in \mathrm{Aut}(\mathbb{F}_4)$ be the automorphisms defined by

- $\varphi(a) = ab$ and $\varphi(x) = x$ for all $x \in \{b, c, d\}$;

- $\psi(c) = cd$ and $\psi(x) = x$ for all $x \in \{ a, b, d \}$.

Then for each $S \in \mathcal{P}^\infty(\mathbb{Z}^2)$, we define

$$G_S = \langle a, b, c, d \mid \mathcal{R}_S \rangle,$$

where $\mathcal{R}_S = \{ (\varphi^k(a) \, \psi^\ell(c) \, b \, d)^{17} \mid (k, \ell) \in S \}$.

For each $S \in \mathcal{P}^\infty(\mathbb{Z}^2)$, let $N_S \in \mathcal{N}_4$ be the normal closure of $\mathcal{R}_S$ in $\mathbb{F}_4$. Thus $G_S = \mathbb{F}_4 / N_S$. Consider the induced action of the subgroup

$$\langle \varphi, \psi \rangle = \{ \varphi^m \psi^n \mid (m, n) \in \mathbb{Z}^2 \} \leqslant \operatorname{Aut}(\mathbb{F}_4)$$

on $\mathcal{N}_4$. Suppose that S, $T \in \mathcal{P}^\infty(\mathbb{Z}^2)$ and that $S \subseteq^{\mathbb{Z}^2} T$. Choose $(m, n) \in \mathbb{Z}^2$ such that $(m, n) + S \subseteq T$. Let $S' = (m, n) + S$ and let $\tau = \varphi^m \psi^n \in \operatorname{Aut}(\mathbb{F}_4)$. Then for each $(k, \ell) \in S$,

$$(\varphi^k(a) \, \psi^\ell(c) \, b \, d)^{17} \overset{\tau}{\mapsto} (\varphi^{m+k}(a) \, \psi^{n+\ell}(c) \, b \, d)^{17}$$

and so $\tau[\mathcal{R}_S] = \mathcal{R}_{S'}$. Thus $\tau[N_S] \subseteq N_T$ and so $G_S \succeq_s G_T$. In summary, for all S, $T \in \mathcal{P}^\infty(\mathbb{Z}^2)$, we have that

$$S \subseteq^{\mathbb{Z}^2} T \quad \Longrightarrow \quad G_S \succeq_s G_T.$$

It is conceivable that the map $S \mapsto G_S$ is already a Borel reduction from $\subseteq^{\mathbb{Z}^2}$ to $\succeq_s$. However, when we we attempted to prove this, we found that our arguments were becoming unpleasantly complicated. So in order to keep the proofs as straightforward as possible, we decided to extend each group G_S to the slightly larger group K_S as follows.

Definition 4.4. For each $S \in \mathcal{P}^\infty(\mathbb{Z}^2)$, let H_S be the free product with amalgamation

$$H_S = G_S *_A (A \times V),$$

where $A = \langle a, b \rangle$ and V is an elementary abelian group of order 2^5; and let K_S be the free product with amalgamation

$$K_S = H_S *_D (D \times W),$$

where $D = \langle c, d \rangle * V$ and W is an elementary abelian group of order 3^{10}.

Remark 4.5. For later use, note that since G_S is a 4-generator group, it follows that there does not exist a surjective homomorphism from G_S onto V; and hence if S, $T \in \mathcal{P}^\infty(\mathbb{Z}^2)$, then there does not exist a surjective homomorphism from G_S onto H_T. Similarly, if S, $T \in \mathcal{P}^\infty(\mathbb{Z}^2)$, then there does not exist a surjective homomorphism from H_S onto K_T.

The remainder of this section is devoted to the proof of the following result.

Theorem 4.6. *The map $S \mapsto K_S$ is a Borel reduction from $\subseteq^{\mathbb{Z}^2}$ to $\succeq_s$.*

We will initially focus our attention on the group G_S. We have already noted that if $S \subseteq^{\mathbb{Z}^2} T$, then $G_S \succeq_s G_T$. In fact, our argument proves the following slightly stronger result.

Lemma 4.7. *If $S \subseteq^{\mathbb{Z}^2} T$, then there exists a surjective homomorphism*

$$\theta : G_S \to G_T$$

with the property that $\theta[\langle a, b \rangle] \leqslant \langle a, b \rangle$ and $\theta[\langle c, d \rangle] \leqslant \langle c, d \rangle$.

□

Most of our effort will be devoted to proving that the converse also holds.

Theorem 4.8. *If S, $T \in \mathcal{P}^\infty(\mathbb{Z}^2)$, then the following conditions are equivalent.*

(i) $S \subseteq^{\mathbb{Z}^2} T$.

(ii) *There exists a surjective homomorphism $\theta : G_S \to G_T$ with the property that $\theta[\langle a, b \rangle] \leqslant \langle a, b \rangle$ and $\theta[\langle c, d \rangle] \leqslant \langle c, d \rangle$.*

The following easy observation will play a key role in the proof of Theorem 4.8.

Lemma 4.9. *For each $S \in \mathcal{P}^\infty(\mathbb{Z}^2)$, the presentation*

$$G_S = \langle a, b, c, d \mid \mathcal{R}_S \rangle$$

satisfies the $C'(1/6)$ cancellation condition.

□

In particular, the symmetrized presentation $G_S = \langle a, b, c, d \mid \mathcal{R}_S^* \rangle$ satisfies the conclusion of Theorem 3.6.

Lemma 4.10. *For each $S \in \mathcal{P}^{\infty}(\mathbb{Z}^2)$, the subgroups $\langle\, a, b\,\rangle$ and $\langle\, c, d\,\rangle$ of G_S are* freely *generated by $\{\, a, b\,\}$, $\{\, c, d\,\}$ respectively.*

Proof. For example, to see that $\langle\, a, b\,\rangle$ is freely generated by $\{\, a, b\,\}$, suppose that $w \in \langle\, a, b\,\rangle$ is a nontrivial cyclically reduced word. If s is a subword of some symmetrized relator $r \in \mathcal{R}_S^*$ with $|s| > 1/2\,|r|$, then s must contain either d or d^{-1}. In particular, s is not a subword of a cyclically reduced conjugate of w, and it follows that $w \neq 1$. □

Lemma 4.11. *Suppose that S, $T \in \mathcal{P}^{\infty}(\mathbb{Z}^2)$ and that $\theta : G_S \to G_T$ is a surjective homomorphism such that $\theta[\langle\, a, b\,\rangle] \leqslant \langle\, a, b\,\rangle$ and $\theta[\langle\, c, d\,\rangle] \leqslant \langle\, c, d\,\rangle$. Then it follows that $\theta[\langle\, a, b\,\rangle] = \langle\, a, b\,\rangle$ and $\theta[\langle\, c, d\,\rangle] = \langle\, c, d\,\rangle$.*

Proof. Suppose that $z \in \langle\, a, b\,\rangle \smallsetminus \theta[\langle\, a, b\,\rangle]$. Choose an element $z' \in G_S$ such that $\theta(z') = z$ and express

$$z' = x_1\, y_1 \cdots x_n\, y_n\, x_{n+1},$$

where each $x_i \in \langle\, a, b\,\rangle$ and each $y_j \in \langle\, c, d\,\rangle$. Furthermore, suppose that z' has been chosen so that n is minimalized. Consider the identity

$$w = \theta(x_1)\,\theta(y_1)\cdots\theta(x_n)\,\theta(y_n)\,\theta(x_{n+1})\,z^{-1} = 1.$$

By the minimality of n, we have that:

(i) $\theta(x_i) \neq 1$ for $1 < i \leq n$; and
(ii) $\theta(y_j) \neq 1$ for $1 \leq j \leq n$.

Since $z \notin \theta[\langle\, a, b\,\rangle]$, we also have that:

(iii) $\theta(x_{n+1})\,z^{-1}\,\theta(x_1) \neq 1$.

Hence, except possibly for some cancellation within $\theta(x_{n+1})\,z^{-1}\,\theta(x_1)$, the word w is cyclically reduced. Applying Theorem 3.6, it follows easily that there exists an integer i and a pair $(k, \ell) \in T$ such that one of the following two possibilites occurs:

(a) $\theta(x_i)\,\theta(y_i)\,\theta(x_{i+1})\,\theta(y_{i+1}) = \varphi^k(a)\,\psi^\ell(c)\,b\,d$; or
(b) $\theta(y_i)\,\theta(x_{i+1})\,\theta(y_{i+1})\,\theta(x_{i+2}) = d^{-1}\,b^{-1}\,\psi^\ell(c^{-1})\,\varphi^k(a^{-1})$.

If (a) holds, then $\theta(x_i) = \varphi^k(a)$ and $\theta(x_{i+1}) = b$. Clearly φ^k induces an automorphism of $\langle\, a, b\,\rangle$ and so $\{\, \varphi^k(a), b\,\} = \{\, \varphi^k(a), \varphi^k(b)\,\}$ generates $\langle\, a, b\,\rangle$. But this means that $z \in \theta[\langle\, a, b\,\rangle]$, which is a contradiction. A similar argument deals with the case when (b) holds. This completes the proof that $\theta[\langle a,b\rangle] = \langle a,b\rangle$; and a similar argument shows that $\theta[\langle c,d\rangle] = \langle c,d\rangle$. □

Proof of Theorem 4.8. We have already noted that (i) implies (ii). So suppose that $\theta : G_S \to G_T$ is a surjective homomorphism such that $\theta[\langle a,b\rangle] \leqslant \langle a,b\rangle$ and $\theta[\langle c,d\rangle] \leqslant \langle c,d\rangle$. Applying Lemma 4.11, it follows that $\theta[\langle a,b\rangle] = \langle a,b\rangle$ and $\theta[\langle c,d\rangle] = \langle c,d\rangle$. Recall that if $\mathbb{F}$ is a finitely generated free group, then every surjective homomorphism $f : \mathbb{F} \to \mathbb{F}$ is an automorphism. (For example, see Lyndon-Schupp [11, Proposition 3.5].) Hence, by Lemma 4.10, there exist automorphisms $\pi \in \mathrm{Aut}(\langle a,b\rangle)$ and $\tau \in \mathrm{Aut}(\langle c,d\rangle)$ such that $\theta \restriction \langle a,b\rangle = \pi$ and $\theta \restriction \langle c,d\rangle = \tau$. If $(k,\ell) \in S$, then applying θ to the identity

$$(\,\varphi^k(a)\,\psi^\ell(c)\,b\,d\,)^{17} = 1$$

in G_S, we obtain that

$$(\,\pi\varphi^k(a)\,\tau\psi^\ell(c)\,\pi(b)\,\tau(d)\,)^{17} = 1$$

in G_T. Note that $\pi\varphi^k(a)$, $\pi(b) \in \langle a,b\rangle$ and that $\tau\psi^\ell(c)$, $\tau(d) \in \langle c,d\rangle$; and so $(\,\pi\varphi^k(a)\,\tau\psi^\ell(c)\,\pi(b)\,\tau(d)\,)^{17}$ is cyclically reduced. Hence, applying Theorem 3.6, it follows easily that there exists a pair $(k',\ell') \in T$ such that one of the following four possibilities occurs:

(i) $\pi\varphi^k(a)\,\tau\psi^\ell(c)\,\pi(b)\,\tau(d) = \varphi^{k'}(a)\,\psi^{\ell'}(c)\,b\,d$;

(ii) $\pi\varphi^k(a)\,\tau\psi^\ell(c)\,\pi(b)\,\tau(d) = b\,d\,\varphi^{k'}(a)\,\psi^{\ell'}(c)$;

(iii) $\pi\varphi^k(a)\,\tau\psi^\ell(c)\,\pi(b)\,\tau(d) = b^{-1}\,\psi^{\ell'}(c^{-1})\,\varphi^{k'}(a^{-1})\,d^{-1}$;

(iv) $\pi\varphi^k(a)\,\tau\psi^\ell(c)\,\pi(b)\,\tau(d) = \varphi^{k'}(a^{-1})\,d^{-1}\,b^{-1}\,\psi^{\ell'}(c^{-1})$.

Since $S \in \mathcal{P}^\infty(\mathbb{Z}^2)$, we can choose $(k,\ell) \in S$ such that $\pi\varphi^k(a) \neq b$, b^{-1} and $\tau\psi^\ell(c) \neq d$, d^{-1}; and it then follows that

$$\pi\varphi^k(a) = \varphi^{k'}(a) \qquad \tau\psi^\ell(c) = \psi^{\ell'}(c)$$

$$\pi(b) = b \qquad \tau(d) = d$$

Hence $\pi = \varphi^m$ and $\tau = \psi^n$, where $m = k' - k$ and $n = \ell' - \ell$. It follows that for each $r \in \mathcal{R}_{(m,n)+S}$, we have that $r = 1$ in G_T. Since the presentation

$$G_{\mathbb{Z}^2} = \langle\, a,b,c,d \mid \mathcal{R}_{\mathbb{Z}^2}\,\rangle$$

satisfies the $C'(1/6)$ cancellation condition, it follows that $\mathcal{R}_{(m,n)+S} \subseteq \mathcal{R}_T$ and hence $(m,n) + S \subseteq T$. This completes the proof of Theorem 4.8. □

We are now ready to prove that the map $S \mapsto K_S$ is a Borel reduction from $\subseteq^{\mathbb{Z}^2}$ to $\succeq_s$. We will begin by proving the easier implication.

Proposition 4.12. *If $S \subseteq^{\mathbb{Z}^2} T$, then $K_S \succeq_s K_T$.*

Proof. Suppose that $S \subseteq^{\mathbb{Z}^2} T$. Then, combining Theorem 4.8 and Lemma 4.11, it follows that there exists a surjective homomorphism $\theta : G_S \to G_T$ such that $\theta[\langle a, b \rangle] = \langle a, b \rangle$ and $\theta[\langle c, d \rangle] = \langle c, d \rangle$. Clearly θ extends canonically to a surjective homomorphism $\theta' : H_S \to H_T$ such that $\theta'[V] = V$; and θ' extends canonically to a surjective homomorphism $\theta'' : K_S \to K_T$ such that $\theta''[W] = W$. □

The proof of the converse implication makes use of the following two lemmas.

Lemma 4.13. *If $1 \neq w \in W$, then $C_{K_S}(w) = D \times W$.*

Proof. Suppose that $z \in C_{K_S}(w) \setminus D$. Then z can be written as

$$z = dy_1 \cdots y_r,$$

where $d \in D$, $r \geq 1$, each $y_i \notin D$ and successive pairs y_i, y_{i+1} lie in different factors H_S or $D \times W$ of the free product with amalgamation $K_S = H_S *_D (D \times W)$. Note that

$$dy_1 \cdots y_r w y_r^{-1} \cdots y_1^{-1} d^{-1}$$

is equal to w. This implies that $y_r \in D \times W$ and hence $y_r w y_r^{-1} \in D \times W$. Since $y_r w y_r^{-1}$ has order 3, it follows that $y_r w y_r^{-1} \notin D$ and this implies that $r = 1$. Thus $z = dy_1 \in D \times W$. □

A similar argument yields the following result.

Lemma 4.14. *If $1 \neq v \in V$, then $C_{H_S}(v) = A \times V$.*

□

Proof of Theorem 4.6. It only remains to prove that if $K_S \succeq_s K_T$, then $S \subseteq^{\mathbb{Z}^2} T$. Suppose that $f : K_S \to K_T$ is a surjective homomorphism. Then Remark 4.5 implies that $f[W] \neq 1$. Fix some element $w \in W$ with $f(w) \neq 1$. Then $f(w) \in K_T$

has order 3. Applying the Torsion Theorem [4] for $C'(1/6)$ presentations, it follows that G_T does not contain any elements of order 3; and by Corollary 4.4.5 [12], the same is also true of H_T. Applying Corollary 4.4.5 [12] once more, it follows that the element $f(w)$ lies in a conjugate of $D \times W$. Hence, after adjusting f by an inner automorphism of K_T, we can suppose that $f(w) \in W$. Using Lemma 4.13, it follows easily that $f[D \times W] \leqslant D \times W$ and this implies that $f[W] \leqslant W$. Let $\pi : K_T \to H_T$ be the canonical surjection such that $\pi[W] = 1$. Then the map

$$K_S \xrightarrow{f} K_T \xrightarrow{\pi} H_T$$

induces a surjective homomorphism $\bar{f} : H_S \to H_T$ such that $\bar{f}[D] \leqslant D$. Once again, Remark 4.5 implies that $\bar{f}[V] \neq 1$. Fix some element $v \in V$ such that $\bar{f}(v) \neq 1$. Then $\bar{f}(v) \in D = \langle c, d \rangle * V$ has order 2 and so $\bar{f}(v)$ is conjugate in D to an element of V. Hence after conjugating by a suitable element of D, we can suppose that $\bar{f}(v) \in V$. (Of course, even after this adjustment to $\bar{f}$, we still have that $\bar{f}[D] \leqslant D$.) Arguing as above, this implies that $\bar{f}[A \times V] \leqslant A \times V$ and that $\bar{f}[V] \leqslant V$. Let $\bar{\pi} : H_T \to G_T$ be the canonical surjection such that $\bar{\pi}[V] = 1$. Then the map

$$H_S \xrightarrow{\bar{f}} H_T \xrightarrow{\bar{\pi}} G_T$$

induces a surjective homomorphism $\theta : G_S \to G_T$ such that $\theta[\langle a, b \rangle] \leqslant \langle a, b \rangle$ and $\theta[\langle c, d \rangle] \leqslant \langle c, d \rangle$. Hence, applying Theorem 4.8, we obtain that $S \subseteq^{\mathbb{Z}^2} T$, as required. This completes the proof of Theorem 4.6. □

Finally, the following result will play an important role in the next section.

Proposition 4.15. *Suppose that* $1 \neq x \in K_S$ *and let* $N = \langle x^{K_S} \rangle$ *be the corresponding normal closure. Then* $C_{K_S}(N) = 1$.

Proof. Suppose that $1 \neq y \in C_{K_S}(N)$. Then it follows that $y^{K_S} \subseteq C_{K_S}(N)$ and hence $M = \langle y^{K_S} \rangle$ satisfies $[M, N] = 1$. It is easily checked that if L is any nontrivial normal subgroup of K_S, then L contains an element which is not conjugate to an element of either of the factors H_S or $D \times W$. In particular, there exist such elements $u \in N$ and $v \in M$, which can be written as $u = u_1 \cdots u_n$ and $v = v_1 \cdots v_m$, where:

(1) $n, m \geq 2$;

(2) u_i, $v_j \notin D$;

(3) the u_i are alternately from different factors H_S or $D \times W$; and

(4) the v_j are alternately from different factors H_S or $D \times W$.

Furthermore, after conjugating by suitably chosen elements of H_S or $D \times W$, we can suppose that:

(5) u_1, $u_n \in H_S$ and v_1, $v_m \in D \times W$.

But then both

$$uv = u_1 \cdots u_n v_1 \cdots v_m \quad \text{and} \quad vu = v_1 \cdots v_m u_1 \cdots u_n$$

are in reduced form and it follows that $uv \neq vu$, which is a contradiction. □

5. Pride's Quasi-order

In this section, we will prove that Pride's quasi-order $\succeq_p$ is a universal $\mathbf{K}_\sigma$ quasi-order on the space $\mathcal{G}$ of finitely generated groups. As explained in Section 2, this implies that the associated equivalence relation $\approx_p$ is a universal $\mathbf{K}_\sigma$ equivalence relation. To prove that $\succeq_p$ is a universal $\mathbf{K}_\sigma$ quasi-order, it is enough to show that $\succeq_p$ is a $\mathbf{K}_\sigma$ relation and that there exists a Borel reduction from $\subseteq^{\mathbb{Z}^2}$ to $\succeq_p$.

Lemma 5.1. *$\succeq_p$ is a $\mathbf{K}_\sigma$ quasi-order on the space $\mathcal{G}$ of finitely generated groups.*

Proof. First notice that if G, $H \in \mathcal{G}$, then $G \succeq_p H$ if and only if there exist groups G', $H' \in \mathcal{G}$ such that $G' \approx_{VI} G$, $H' \approx_{VI} H$ and $G' \succeq_s H'$. Applying Thomas [19, 6.4] and Lemma 4.1, the virtual isomorphism relation $\approx_{VI}$ and the surjectivity relation $\succeq_s$ are both $\mathbf{K}_\sigma$ subsets of $\mathcal{G}^2$. It follows easily that each of the following is a $\mathbf{K}_\sigma$ subset of $\mathcal{G}^4$:

- $R_1 = \{ (G, G', H', H) \in \mathcal{G}^4 \mid G \approx_{VI} G' \}$,
- $R_2 = \{ (G, G', H', H) \in \mathcal{G}^4 \mid G' \succeq_s H' \}$,
- $R_3 = \{ (G, G', H', H) \in \mathcal{G}^4 \mid H' \approx_{VI} H \}$;

and hence $R = \bigcap_{i=1}^3 R_i$ is a $\mathbf{K}_\sigma$ subset of $\mathcal{G}^4$. Letting $\pi : \mathcal{G}^4 \to \mathcal{G}^2$ be the projection defined by $(G, G', H', H) \mapsto (G, H)$, we have that $\succeq_p$ is equal to $\pi[R]$ and it follows that $\succeq_p$ is a $\mathbf{K}_\sigma$ subset of $\mathcal{G}^2$. □

In the remainder of this section, adapting the ideas of Thomas [18, Section 2], we will define a Borel reduction $S \mapsto W_S$ from $\subseteq^{\mathbb{Z}^2}$ to $\succeq_p$. Throughout this section, Γ

will denote an infinite finitely generated simple group, which satisfies the following two additional properties:

- Every proper subgroup of Γ is finite.
- Every automorphism of Γ is inner.

For the existence of such a group Γ, see Obraztsov [15].

Definition 5.2. For each $S \in \mathcal{P}^{\infty}(\mathbb{Z}^2)$, let

$$W_S = K_S \operatorname{wr} \Gamma,$$

where K_S is the finitely generated group given by Definition 4.4.

Theorem 5.3. *The map $S \mapsto W_S$ is a Borel reduction from $\subseteq^{\mathbb{Z}^2}$ to $\succeq_p$.*

Here $K_S \operatorname{wr} \Gamma$ denotes the *(restricted) wreath product* of K_S by Γ, which is defined as follows. For each function $b : \Gamma \to K_S$, the *support* supp(b) is defined to be

$$\operatorname{supp}(b) = \{ \alpha \in \Gamma \mid b(\alpha) \neq 1 \};$$

and the *base group* B_S of $K_S \operatorname{wr} \Gamma$ is defined to be

$$B_S = \{ b : \Gamma \to K_S \mid \operatorname{supp}(b) \text{ is finite } \},$$

equipped with pointwise multiplication; i.e. if $b,\, c \in B_S$, then

$$(bc)(\alpha) = b(\alpha)\, c(\alpha)$$

for all $\alpha \in \Gamma$. There is a natural action of Γ on B_S defined by

$$b^{\gamma}(\alpha) = b(\alpha\, \gamma^{-1});$$

and $K_S \operatorname{wr} \Gamma$ is defined to be the corresponding semidirect product

$$K_S \operatorname{wr} \Gamma = \{ (\, \gamma, b\,) \mid \gamma \in \Gamma, b \in B_S \}$$

with multiplication defined by

$$(\, \gamma, b\,)\, (\, \delta, c\,) = (\, \gamma\, \delta, b^{\delta} c\,).$$

As usual, we identify Γ and B_S with the corresponding subgroups of $K_S \operatorname{wr} \Gamma$ and we write γb instead of $(\, \gamma, b\,)$.

Notation 5.4. For each finite subset $F \subseteq \Gamma$, we define the corresponding subgroup $B_S^{(F)}$ of the base group B_S by

$$B_S^{(F)} = \{ \, b \in B_S \mid \operatorname{supp}(b) \subseteq F \, \}.$$

We will begin by proving the easier direction of Theorem 5.3.

Lemma 5.5. *If $S \subseteq^{\mathbb{Z}^2} T$, then $W_S \succeq_p W_T$.*

Proof. By Proposition 4.12, if $S \subseteq^{\mathbb{Z}^2} T$, then there exists a surjective homomorphism $\theta : K_S \to K_T$; and it is clear that θ can be extended to a surjective homomorphism $\theta' : W_S \to W_T$. □

The next two lemmas explain how to recognize the group K_S within any group L_S such that $L_S \approx_{VI} K_S \operatorname{wr} \Gamma$.

Lemma 5.6. *If $S \in \mathcal{P}^\infty(\mathbb{Z}^2)$ and $L_S \leqslant W_S$ is a subgroup of finite index, then L_S has no nontrivial finite normal subgroups.*

Proof. This is an immediate consequence of Thomas [18, Lemma 2.2]. □

Lemma 5.7. *Suppose that $S \in \mathcal{P}^\infty(\mathbb{Z}^2)$ and that $L_S \leqslant W_S$ is a subgroup of finite index. Let $F \subseteq \Gamma$ be a finite subset with $|F| \geq 2$ and let $\gamma \in F$. Then for each $g \in K_S$, there exists an element $b \in B_S^{(F)} \cap L_S$ such that $b(\gamma) = g$. Hence*

$$\frac{(\, B_S^{(F)} \cap L_S \,)}{(\, B_S^{(F \smallsetminus \{\gamma\})} \cap L_S \,)} \cong K_S.$$

Proof. Applying Thomas [18, Lemma 2.2], since $[W_S : L_S] < \infty$, it follows that

$$[W_S, W_S] = [L_S, L_S] \leqslant L_S.$$

Furthermore, by Neumann [13, Theorem 4.1 and Corollary 4.5], we have that

$$[W_S, W_S] \cap B_S = [\Gamma, B_S] = \{ \, b \in B_S \mid \prod_{\gamma \in \operatorname{supp}(b)} b(\gamma) \in [K_S, K_S] \, \}.$$

Let $\delta \in F \smallsetminus \{\gamma\}$. Then for each $g \in K_S$, we can define an element $b \in B_S^{(F)} \cap L_S$ with $b(\gamma) = g$ by

$$b(\alpha) = \begin{cases} g, & \text{if } \alpha = \gamma; \\ g^{-1}, & \text{if } \alpha = \delta; \\ 1, & \text{otherwise.} \end{cases}$$

In particular, the homomorphism $\psi : B_S^{(F)} \cap L_S \to K_S$, defined by $\psi(b) = b(\gamma)$, is surjective and clearly $\ker \psi = B_S^{(F \smallsetminus \{\gamma\})} \cap L_S$. □

For the remainder of this section, suppose that $S, T \in \mathcal{P}^\infty(\mathbb{Z}^2)$ and that $W_S \succeq_p W_T$. Then, applying Lemma 5.6, there exist subgroups $L_S \leqslant W_S$ and $L_T \leqslant W_T$ of finite index such that $L_S \succeq_s L_T$. Let $\pi : L_S \to L_T$ be a surjective homomorphism. Since Γ is an infinite simple group and $[\Gamma : \Gamma \cap L_S]$, $[\Gamma : \Gamma \cap L_T] < \infty$, it follows that $\Gamma \leqslant L_S$ and $\Gamma \leqslant L_T$. Hence $L_S = (B_S \cap L_S) \rtimes \Gamma$ and $L_T = (B_T \cap L_T) \rtimes \Gamma$; and, of course, $[B_S : B_S \cap L_S]$, $[B_T : B_T \cap L_T] < \infty$. Let $\rho : L_T \to \Gamma$ be the canonical surjective homomorphism and let $\varphi = \rho \circ \pi : L_S \to \Gamma$.

Lemma 5.8. $\pi[B_S \cap L_S] \leqslant B_T \cap L_T$.

Proof. Suppose not. Since $B_S \cap L_S \trianglelefteq L_S$ and $\varphi[L_S] = \Gamma$, it follows that $\varphi[B_S \cap L_S]$ is a nontrivial normal subgroup of Γ and hence $\varphi[B_S \cap L_S] = \Gamma$. Choose an element $b \in B_S \cap L_S$ such that $\varphi(b) \neq 1$ and let $F = \operatorname{supp}(b)$. Since $B_S^{(F)} \cap L_S \trianglelefteq B_S \cap L_S$, it follows that $\varphi[B_S^{(F)} \cap L_S] = \Gamma$. Choose an element $\gamma \in \Gamma$ such that $F\gamma \cap F = \emptyset$. Since

$$\gamma^{-1}(B_S^{(F)} \cap L_S)\,\gamma = B_S^{(F\gamma)} \cap L_S,$$

it follows that

$$[\,B_S^{(F)} \cap L_S, \gamma^{-1}(B_S^{(F)} \cap L_S)\,\gamma\,] = 1.$$

On the other hand, we also have that

$$\varphi[\gamma^{-1}(B_S^{(F)} \cap L_S)\,\gamma] = \varphi(\gamma)^{-1}\Gamma\varphi(\gamma) = \Gamma$$

and hence $[\,\Gamma, \Gamma\,] = 1$, which is a contradiction. □

Lemma 5.9. $\varphi[\Gamma] = \Gamma$.

Proof. By Lemma 5.8, we have that $B_S \cap L_S \leqslant \ker \varphi$. Hence, since φ is surjective, we must have that $\Gamma \nleqslant \ker \varphi$ and so $\Gamma \cap \ker \varphi = 1$. Thus $\varphi[\Gamma]$ is an infinite subgroup of Γ and this implies that $\varphi[\Gamma] = \Gamma$. □

Since every automorphism of Γ is inner, after adjusting π by an inner automorphism of L_S if necessary, we can suppose that $\varphi(\gamma) = \gamma$ for all $\gamma \in \Gamma$.

Lemma 5.10. $\pi[B_S \cap L_S] = B_T \cap L_T$.

Proof. Let $b \in B_T \cap L_T$. Then there exists an element $c \in L_S$ such that $\pi(c) = b$. Express $c = \gamma d$, where $d \in B_S \cap L_S$ and $\gamma \in \Gamma$. Then

$$1 = \rho(b) = \varphi(\gamma d) = \gamma$$

and hence $c \in B_S \cap L_S$. □

Lemma 5.11. $\pi[\, B_S^{(\{\gamma\})} \cap L_S \,] \neq 1$ *for any* $\gamma \in \Gamma$.

Proof. Suppose that there exists $\gamma \in \Gamma$ such that $\pi[\, B_S^{(\{\gamma\})} \cap L_S \,] = 1$. If $\beta \in \Gamma$ is arbitrary and $\alpha = \gamma^{-1}\beta$, then

$$\pi[\, B_S^{(\{\beta\})} \cap L_S \,] = \pi[\, \alpha^{-1} (\, B_S^{(\{\gamma\})} \cap L_S \,)\, \alpha \,] = \pi(\alpha)^{-1} \pi[\, B_S^{(\{\gamma\})} \cap L_S \,]\, \pi(\alpha) = 1.$$

Applying Lemma 5.10, since K_T is infinite and finitely generated, there exists a finite subset $F \subseteq \Gamma$ such that $\pi[\, B_S^{(F)} \cap L_S \,]$ is infinite. But notice that

$$[\, B_S^{(\{\beta\})} : B_S^{(\{\beta\})} \cap L_S \,] < \infty$$

for each $\beta \in F$ and hence

$$[\, B_S^{(F)} \cap L_S : \bigoplus_{\beta \in F} (\, B_S^{(\{\beta\})} \cap L_S \,) \,] < \infty.$$

But this implies that $\pi[\, B_S^{(F)} \cap L_S \,]$ is finite, which is a contradiction. □

Lemma 5.12. *There exists a* fixed $\gamma_0 \in \Gamma$ *such that*

$$\pi[\, B_S^{(F)} \cap L_S \,] \leqslant B_T^{(\gamma_0 F)} \cap L_T$$

for each nonempty finite subset $\emptyset \neq F \subseteq \Gamma$.

Proof. First we will consider the case when $F = \{1\}$. Suppose that $b \in B_S^{(\{1\})} \cap L_S$ satisfies $\pi(b) \neq 1$ and let $F_b = \mathrm{supp}(\pi(b))$. We claim that $|F_b| = 1$. Suppose not and let $\alpha \neq \beta \in F_b$. Let $N = \langle\, b^{B_S \cap L_S} \,\rangle \leqslant B_S^{(\{1\})} \cap L_S$ and let $\gamma = \beta^{-1}\alpha$. Then

$$\gamma^{-1} N \,\gamma \leqslant \gamma^{-1} (\, B_S^{(\{1\})} \cap L_S \,)\, \gamma = B_S^{(\{\gamma\})} \cap L_S$$

and so $[\, N, \gamma^{-1} N \gamma \,] = 1$. On the other hand, by Lemma 5.10, we have that

$$\pi[N] = \langle\, \pi(b)^{B_T \cap L_T} \,\rangle \leqslant B_T^{(F_b)} \cap L_T.$$

Let $M = \{\, c(\alpha) \mid c \in \pi[N] \,\}$. Then Lemma 5.7 implies that M is the normal closure in K_T of the nonidentity element $\pi(b)(\alpha)$. Similarly, since $\alpha \in F_b \cap F_b\gamma$ and

$\pi(\gamma) = f\gamma$ for some $f \in B_T \cap L_T$, we see that $M' = \{ d(\alpha) \mid d \in \pi[\gamma^{-1}N\gamma] \}$ is the normal closure in K_T of the nonidentity element $\pi(\gamma^{-1}b\gamma)(\alpha)$. However, since $[N, \gamma^{-1}N\gamma] = 1$, it follows that $[M, M'] = 1$ and this contradicts Proposition 4.15. Thus $|\operatorname{supp}(\pi(b))| = 1$ for every $b \in B_S^{(\{1\})} \cap L_S$ such that $\pi(b) \neq 1$. A similar argument shows that if $b, c \in B_S^{(\{1\})} \cap L_S$ are such that $\pi(b)$, $\pi(c) \neq 1$, then $\operatorname{supp}(\pi(b)) = \operatorname{supp}(\pi(c))$. Let $\gamma_0 \in \Gamma$ denote the element such that

$$\pi[B_S^{(\{1\})} \cap L_S] \leqslant B_T^{(\{\gamma_0\})} \cap L_T.$$

If $\gamma \in \Gamma$ is arbitrary, then $\pi(\gamma) = f\gamma$ for some $f \in B_T \cap L_T$ and so

$$\pi[B_S^{(\{\gamma\})} \cap L_S] = \pi[\gamma^{-1}(B_S^{(\{1\})} \cap L_S)\gamma] \leqslant \gamma^{-1}(B_T^{(\{\gamma_0\})} \cap L_T)\gamma = B_T^{(\{\gamma_0\gamma\})} \cap L_T.$$

Finally, suppose that $b \in B_S \cap L_S$ is any element such that $\pi(b) \neq 1$. Let $\gamma \in \Gamma \smallsetminus \operatorname{supp}(b)$ and let $c \in B_S^{(\{\gamma\})} \cap L_S$ with $\pi(c) \neq 1$. Then, letting $N = \langle c^{B_S \cap L_S} \rangle$, we have that $[b, N] = 1$ and hence $[\pi(b), \pi[N]] = 1$. Arguing as above, this implies that $\gamma_0\gamma \notin \operatorname{supp}(\pi(b))$. Hence $\operatorname{supp}(\pi(b)) \subseteq \gamma_0 \operatorname{supp}(b)$, as required. □

We are now ready to complete the proof of Theorem 5.3. Let $\sigma : B_T \cap L_T \to K_T$ be the homomorphism defined by $\sigma(b) = b(\gamma_0)$; and let $\psi : B_S \cap L_S \to K_T$ be the homomorphism defined by $\psi = \sigma \circ \pi$. Applying Lemmas 5.7 and 5.10, it follows that ψ is surjective. Since K_T is finitely generated, there exists a finite subset $F \subseteq \Gamma$ such that $\psi[B_S^{(F)} \cap L_S] = K_T$. By Lemma 5.12, we have that $1 \in F$ and that $B_S^{(F \smallsetminus \{1\})} \cap L_S \leqslant \ker \psi$. Hence, by Lemma 5.7, ψ induces a surjective homomorphism $\theta : K_S \to K_T$; and by Theorem 4.6, this implies that $S \subseteq^{\mathbb{Z}^2} T$. This completes the proof of Theorem 2.1.

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