UNIFORMLY RECURRENT SUBGROUPS OF SIMPLE LOCALLY FINITE GROUPS

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Abstract. We study the uniformly recurrent subgroups of simple locally finite groups.

1. Introduction

Let G be a countably infinite group and let Sub_G be the compact space of subgroups $H \leqslant G$. Then G acts as a group of homeomorphisms of Sub_G via the conjugation action, $H \overset{g}{\mapsto} gHg^{-1}$. Following Glasner-Weiss [5], a subset $X \subseteq \mathrm{Sub}_G$ is said to be a *uniformly recurrent subgroup* or *URS* if X is a minimal G-invariant closed subset of Sub_G. For example, if $N \trianglelefteq G$ is a normal subgroup, then the singleton set $\{ N \}$ is a URS of G. Throughout this paper, these singleton URSs will be regarded as *trivial URSs*. More interesting examples of URSs arise as the stabilizer URSs of minimal actions. For example, suppose that Δ is a compact space and that $G \curvearrowright \Delta$ is a minimal G-action. Let $f : \Delta \to \mathrm{Sub}_G$ be the G-equivariant stabilizer map defined by

$$x \mapsto G_x = \{ g \in G \mid g \cdot x = x \}.$$

and let $X_\Delta = f(\Delta)$. If f is continuous, then it follows easily that X_Δ is a URS of G; and, as expected, X_Δ is called the *stabilizer URS* of the minimal action $G \curvearrowright \Delta$. (It is well-known and easily checked that the map $f : \Delta \to \mathrm{Sub}_G$ is continuous if and only $\mathrm{Fix}_\Delta(g) = \{ x \in \Delta \mid g \cdot x = x \}$ is clopen for every $g \in G$.) By Matte-Bon-Tsankov [16], if $X \subseteq \mathrm{Sub}_G$ is any URS of G, then there exists a minimal action $G \curvearrowright \Delta$ such that the stabilizer map $f : \Delta \to \mathrm{Sub}_G$ is continuous and $f(\Delta) = X$. (In fact, Matte-Bon-Tsankov [16] have proved this realization theorem in the wider setting of locally compact groups.)

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In this paper, we will study the URSs of the countably infinite simple locally finite groups. *From now on, in order to slightly simplify the terminology, we will use the expression "locally finite group" as an abbreviation for "countably infinite locally finite group".*

We will begin by discussing two representative examples of nontrivial URSs of simple locally finite groups.

Example 1.1. If $\mathrm{Alt}(\Omega_1)$, $\mathrm{Alt}(\Omega_2)$ are finite alternating groups, then a proper embedding $\varphi : \mathrm{Alt}(\Omega_1) \to \mathrm{Alt}(\Omega_2)$ is said to be *strictly diagonal* if $\varphi(\mathrm{Alt}(\Omega_1))$ acts via its natural permutation representation on each of its orbits in Ω_2. The simple locally finite group $G = \bigcup_{i \in \mathbb{N}} G_i$ is said to be the *strictly diagonal limit* of the finite alternating groups $G_i = \mathrm{Alt}(\Delta_i)$ if every embedding $G_i \hookrightarrow G_{i+1}$ is strictly diagonal. In this case, let $s_0 = |\Delta_0|$ and let $s_{i+1} = |\Delta_{i+1}|/|\Delta_i|$ be the number of G_i-orbits on Δ_{i+1}. Then each $s_i > 1$ and we can suppose that

$$\Delta_i = s_0 \times s_1 \times \cdots \times s_i,$$

where the embedding $G_i \hookrightarrow G_{i+1}$ is given by

$$g \cdot (\, \ell_0, \cdots, \ell_i, \ell_{i+1} \,) = (\, g \cdot (\ell_0, \cdots, \ell_i), \ell_{i+1} \,).$$

Equip the infinite product $\Delta = \prod_{i \geq 0} s_i$ with its usual product topology. Then G acts as a group of homeomorphisms of the compact space Δ via

$$g \cdot (\, \ell_0, \cdots, \ell_i, \ell_{i+1}, \ell_{i+2}, \cdots \,) = (\, g \cdot (\ell_0, \cdots, \ell_i), \ell_{i+1}, \ell_{i+2}, \cdots), \quad g \in G_i,$$

and it is clear that every G-orbit is dense in Δ. Thus $G \curvearrowright \Delta$ is a minimal G-action. Since the stabilizer map $x \overset{f}{\mapsto} G_x$ is continuous and $1 \subsetneq G_x \subsetneq G$ for each $x \in \Delta$, it follows that $X_\Delta = f(\Delta)$ is a nontrivial URS of G. We will refer to $G \curvearrowright \Delta$ as the *canonical minimal action* of G.

Recall that a simple locally finite group G is said to be an $L(\mathrm{Alt})$-group if we can express $G = \bigcup_{i \in \mathbb{N}} G_i$ as the union of a strictly increasing chain of finite alternating groups $G_i = \mathrm{Alt}(\Delta_i)$. (Here we allow arbitrary embeddings $G_i \hookrightarrow G_{i+1}$.) In [21], Thomas-Tucker-Drob proved the following classification theorem as a corollary of their classification of the ergodic invariant random subgroups of the $L(\mathrm{Alt})$-groups.

Theorem 1.2 (Thomas-Tucker-Drob [21]). *If G is an $L(\mathrm{Alt})$-group and $X \subseteq Sub_G$ is a nontrivial URS, then G can be expressed as a strictly diagonal limit of finite alternating groups and X is the stabilizer URS of the corresponding canonical minimal action $G \curvearrowright \Delta$.*

For later use, we record the following consequence of Theorem 1.2.

Corollary 1.3 (Thomas-Tucker-Drob [21]). *The infinite alternating group* $\mathrm{Alt}(\mathbb{N})$ *has no nontrivial URSs.*

Proof. Clearly $\mathrm{Alt}(\mathbb{N})$ is an $L(\mathrm{Alt})$-group, and it is easily seen that $\mathrm{Alt}(\mathbb{N})$ cannot be expressed as a strictly diagonal limit of finite alternating groups. □

In Section 2, we will extend Theorem 1.2 to a classification of the nontrivial URSs of the simple locally finite groups which can be expressed as an increasing union $G = \bigcup_{i \in \mathbb{N}} G_i$ of products of finite alternating groups $G_i = \mathrm{Alt}(\Delta_{i1}) \times \cdots \times \mathrm{Alt}(\Delta_{ir_i})$. More precisely, we will show that every nontrivial URS of such a group G arises as the stabilizer URS of the minimal action of G on the compact path space $\mathcal{P}(B)$ of an associated unital Bratteli diagram B.

Example 1.4. Let K be a finite field and let σ be the nondegenerate symplectic form on the infinite dimensional vector space $V = \bigoplus_{\ell \in \mathbb{N}^+} K e_\ell$ such that

$$\sigma(e_{2\ell+1}, e_{2\ell+2}) = -\sigma(e_{2\ell+2}, e_{2\ell+1}) = 1$$

for all $\ell \in \mathbb{N}$; and otherwise $\sigma(e_i, e_j) = 0$. Let $Sp(K)$ be the corresponding symplectic group; and for each $n \geq 1$, let $Sp(2n, K)$ be the subgroup consisting of the elements $g \in Sp(K)$ such that $g(e_\ell) = e_\ell$ for all $\ell \geq 2n+1$. (Thus $Sp(2n, K)$ can be identified with the finite symplectic group on $V_{2n} = \bigoplus_{1 \leq \ell \leq 2n} K e_\ell$.) Then the *stable symplectic group* $FSp(K) = \bigcup_{n \in \mathbb{N}^+} Sp(2n, K)$ is a simple locally finite group.

Now suppose that K has characteristic 2 and let $\mathcal{Q}$ be the set of quadratic forms q on V such that for all $x, y \in V$,

$$\sigma(x, y) = q(x+y) + q(x) + q(y);$$

and for each $q \in \mathcal{Q}$, let $GO(q, K)$ be the corresponding orthogonal group and let $FGO(q, K) = GO(q, K) \cap FSp(K)$.

Clearly $\mathcal{Q}$ is a closed subset of the compact space K^V of functions $f : V \to K$. Consider the action of the stable symplectic group $G = FSp(K)$ on $\mathcal{Q}$ defined by $(g \cdot q)(x) = q(g^{-1}(x))$. Then, as we will explain in Section 5, every G-orbit is dense in $\mathcal{Q}$ and thus $G \curvearrowright \mathcal{Q}$ is a minimal G-action. Since the stabilizer map $q \overset{f}{\mapsto} G_q = FGO(q, K)$ is continuous and each orthogonal group $FGO(q, K)$ satisfies $1 \subsetneqq FGO(q, K) \subsetneqq G$, it follows that

$$X = \{\, FGO(q, K) \mid q \in \mathcal{Q}\,\}$$

is a nontrivial URS of G.

We can also define a closely related second nontrivial URS of G as follows. For each $q \in \mathcal{Q}$, let $F\Omega(q, K) = FGO(q, K)'$ be the commutator subgroup of $FGO(q, K)$. (Alternatively, $F\Omega(q, K)$ can be characterized as the subgroup of $FGO(q, K)$ consisting of the elements that can be expressed as a product of an even number of orthogonal transvections. For example, see Wilson [23] for a clear account of the corresponding orthogonal subgroups of the finite symplectic groups $Sp(2n, K)$.) Then $[\, FGO(q, K) : F\Omega(q, K)\,] = 2$. Clearly the G-equivariant map $FGO(q, K) \mapsto F\Omega(q, K)$ is continuous, it follows that

$$X' = \{\, F\Omega(q, K) \mid q \in \mathcal{Q}\,\}$$

is also a nontrivial URS of G.

In Section 5, we will prove the following classification theorem.

Theorem 1.5. *If G is a finitary linear simple locally finite group and $X \subseteq Sub_G$ is a nontrivial URS, then G is the stable symplectic group $FSp(K)$ over a finite field K of characteristic 2, and either*

(i) $X = \{\, FGO(q, K) \mid q \in \mathcal{Q}\,\}$, *or*
(ii) $X = \{\, F\Omega(q, K) \mid q \in \mathcal{Q}\,\}$.

Here a group G is said to be *finitary linear* if G has a faithful representation as a group of linear transformations of an infinite dimensional vector space V over a field K such that the fixed-point subspace of every element $g \in G$ has finite codimension in V. (In particular, the class of finitary linear simple locally finite groups includes the class of linear simple locally finite groups.)

In the remainder of this introduction, we will discuss the problem of classifying the URS of *arbitrary* simple locally finite groups. We will begin by making the following easy but useful observation.

Proposition 1.6. *If G is a simple locally finite group, then G has a nontrivial URS if and only if there exist subgroups $H,\, F \leqslant G$ such that:*

(i) *F is finite;*

(ii) *$1 \subsetneq gHg^{-1} \cap F \subsetneq F$ for all $g \in G$.*

Proof. First suppose that $X \subseteq \mathrm{Sub}_G$ is a nontrivial URS of G. Then $1,\, G \notin X$. Let $H \leqslant G$ be any subgroup such that $H \in X$. Then we claim that there exists a finite subgroup F such that $1 \subsetneq gHg^{-1} \cap F \subsetneq F$ for all $g \in G$. Suppose not. Then it follows easily that either:

(a) for each finite subgroup $F \leqslant G$, there exists an element $g \in G$ such that $gHg^{-1} \cap F = 1$; or

(b) for each finite subgroup $F \leqslant G$, there exists an element $g \in G$ such that $gHg^{-1} \cap F = F$.

However, if (a) holds, then 1 lies in the closure of $\{\, gHg^{-1} \mid g \in G \,\}$; while if (b) holds, then G lies in the closure of $\{\, gHg^{-1} \mid g \in G \,\}$. Hence such a finite subgroup $F \leqslant G$ does indeed exist.

Next suppose that there exists a subgroups $H,\, F \leqslant G$ such that conditions (i) and (ii) hold. Let Z be the closure of $\{\, gHg^{-1} \mid g \in G \,\}$ in Sub_G. Then $1,\, G \notin Z$. Let $X \subseteq Z$ be a minimal G-invariant closed subset of Sub_G. Then X is a nontrivial URS of G. □

A weaker version of the above condition on the subgroup $H \leqslant G$ has been extensively studied in connection with the problem of the determining the ideal lattices of the complex group algebras $\mathbb{C}G$ of simple locally finite groups G.

Definition 1.7. A subgroup H of a locally finite group G is said to be *confined* if there exists a finite subgroup $F \leqslant G$ such that $gHg^{-1} \cap F \neq 1$ for all $g \in G$.

In particular, if $H,\, F \leqslant G$ satisfy conditions 1.6(i) and 1.6(ii), then it follows that H is a proper confined subgroup of G. (However, as we will see later, the existence of a proper confined subgroup is not enough to guarantee the existence of a nontrivial URS.) The following classification theorem summarizes the relevant results of Hartley-Zalesskii [9] and Leinen-Puglisi [13, 14]. (It should be stressed that Hall's classification [8] of the nonlinear finitary linear simple locally finite groups also plays an essential role in the proof of Theorem 1.8.)

Theorem 1.8. *If G is a simple locally finite group, then G has a proper confined subgroup if and only if G satisfies one of the following conditions:*

(i) *G is isomorphic to a nonlinear finitary linear group over a finite field K;*
(ii) *G is of 1-type.*

Thus, in view of Theorem 1.5 and Theorem 1.8, the classification problem for the nontrivial URSs of simple locally finite groups has been reduced to the case when G is of 1-type. The definition of "1-type" makes use of the notion of a Kegel sequence. (By Kegel-Wehrfritz [11, Lemma 4.5], every simple locally finite group has a Kegel sequence.)

Definition 1.9. If G is a simple locally finite group, then a *Kegel sequence* for G is a sequence $\mathcal{K} = \{ (G_i, M_i) \mid i \in \mathbb{N} \}$ of pairs of finite subgroups of G such that:

- $G_0 \leqslant G_1 \leqslant \cdots \leqslant G_i \leqslant G_{i+1} \leqslant \cdots$
- $G = \bigcup_{i \in \mathbb{N}} G_i$
- M_i is a maximal proper normal subgroup of G_i
- $G_i \cap M_{i+1} = 1$.

The finite simple groups $\{ G_i/M_i \mid i \in \mathbb{N} \}$ are called the *factors* of the Kegel sequence $\mathcal{K}$.

Definition 1.10. Suppose that G is a simple locally finite group which is not finitary linear. Then G is of 1-*type* if every Kegel sequence of G has a factor which is isomorphic to an alternating group.

Of course, it follows that if G is a simple locally finite group of 1-type and $\mathcal{K} = \{ (G_i, M_i) \mid i \in \mathbb{N} \}$ is a Kegel sequence, then G_i/M_i is isomorphic to an alternating group for all but finitely many $i \in \mathbb{N}$. The assumption that G is of 1-type has strong structural consequences. (For the general theory of groups of 1-type, see Delcroix-Meierfrankenfeld [4].)

For example, suppose that G is a simple locally finite group and that G can be expressed as an increasing union $G = \bigcup_{i \in \mathbb{N}} G_i$ of products of finite alternating groups $G_i = \mathrm{Alt}(\Delta_{i1}) \times \cdots \times \mathrm{Alt}(\Delta_{ir_i})$; and for each $i \in \mathbb{N}$ and $1 \leq j \leq r_i$, let

$$N_{ij} = \prod \{ \mathrm{Alt}(\Delta_{i\ell}) \mid 1 \leq \ell \leq r_i, \ell \neq j \}.$$

The union $G = \bigcup_{i \in \mathbb{N}} G_i$ is said to be *strongly diagonal* if whenever $i < k$ and Σ is a nontrivial orbit of G_i on some Δ_{kt}, then there exists j such that N_{ij} acts trivially on Σ and $\mathrm{Alt}(\Delta_{ij})$ acts naturally on Σ.

Theorem 1.11 (Hartley-Zalesskii [10])**.** *Suppose that G is a simple locally finite group which can be expressed as an increasing union $G = \bigcup_{i \in \mathbb{N}} G_i$ of products of finite alternating groups. If G is of 1-type, then there exists i_0 such that the union $G = \bigcup_{i \geq i_0} G_i$ is strongly diagonal.*

Definition 1.12. The simple locally finite group G is an *LDA-group* if G can be expressed as a strongly diagonal union $G = \bigcup_{i \in \mathbb{N}} G_i$ of products of finite alternating groups.

On the other hand, suppose that G is a finitary linear simple locally finite group which can be expressed as an increasing union of products of finite alternating groups. Then G has a Kegel sequence, each of whose factors is isomorphic to a finite alternating group. (For example, this follows easily from Meierfrankenfeld [17, Lemma 2.15].) Thus the following result is an immediate consequence of Hall [6, Theorem 5.2].

Theorem 1.13. *If G is a finitary linear simple locally finite group which can be expressed as an increasing union of products of finite alternating groups, then $G \cong \mathrm{Alt}(\mathbb{N})$.*

Once again, suppose that G is a simple locally finite group which can be expressed as an increasing union of products of finite alternating groups. Then, applying Theorems 1.8, 1.11, 1.13 and Corollary 1.3, it follows that if G has a nontrivial URS, then G is an LDA-group.

In Sections 2, 3 and 4, we will classify the URSs of the LDA-groups; and in Section 5, we will classify the URSs of the finitary linear simple locally finite groups. As we will see in Section 5, the classification of the URSs of the finitary linear simple locally finite groups is an easy consequence of Leinen-Puglisi's classification [13] of the confined subgroups of the classical finitary linear simple locally finite groups. The classification of the URSs of the LDA-groups is more interesting and makes use of an observation that is potentially useful in the setting of arbitary countable amenable groups; namely, that if G is a countable amenable group and $X \subseteq \mathrm{Sub}_G$ is a URS, then there exists a G-invariant ergodic Borel probability measure ν on Sub_G which concentrates on X. Consequently, measure-theoretic techniques (such as the Pointwise Ergodic Theorem for countable amenable groups [15]) can be employed in the study of the URSs of countable amenable groups.

Recall that a Borel probability measure ν on Sub_G which is invariant under the conjugation action $G \curvearrowright \mathrm{Sub}_G$ is called an *invariant random subgroup* or *IRS*. For example, suppose that G acts via measure-preserving maps on the Borel probability space (Z, μ) and let $f : Z \to \mathrm{Sub}_G$ be the G-equivariant stabilizer map defined by

$$z \mapsto G_z = \{ g \in G \mid g \cdot z = z \}.$$

Then the corresponding *stabilizer distribution* $\nu = f_* \mu$ is an IRS of G. In fact, by a result of Abért-Glasner-Virag [2], every IRS of G can be realized as the stabilizer distribution of a suitably chosen measure-preserving action. Moreover, by Creutz-Peterson [3], if ν is an ergodic IRS of G, then ν is the stabilizer distribution of an ergodic action $G \curvearrowright (Z, \mu)$.

This remaining sections of this paper are organized as follows. In Section 2, following Lavrenyuk-Nekrashevych [12], we will first describe how to realize each LDA-group G as the group $A(B)$ associated with a suitable Bratteli diagram B; and then we will state the classification theorem for the URSs of the LDA-groups. In Section 3, we will discuss the ergodic theory of countably infinite locally finite groups and present some results on the normalized permutation characters of finite groups. In Section 4, we will prove the classification theorem for the URSs of the LDA-groups; and in Section 5, we will prove the classification theorem for the URSs of the finitary linear simple locally finite groups.

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2. LDA-groups

Suppose that G is a simple locally finite group which can be expressed as an increasing union $G = \bigcup_{i \in \mathbb{N}} G_i$ of products of finite alternating groups. Then, as we explained in Section 1, if G has a nontrivial URS, then it follows that G is an LDA-group. In this section, we will state the classification theorem for the URSs of the LDA-groups. First, following Lavrenyuk-Nekrashevych [12], we will describe how to realize each LDA-group G as the group $A(B)$ associated with a suitable (labelled) Bratteli diagram B.

Definition 2.1. A *Bratteli diagram* $B = (\, \{V_i\}, \{E_i\}, \mathbf{s}, \mathbf{r}, d \,)$ consists of:

- a set $V(B) = \bigsqcup_{i \in \mathbb{N}} V_i$ of *vertices*, where each V_i is a finite nonempty set;
- a set $E(B) = \bigsqcup_{i \in \mathbb{N}^+} E_i$ of *edges*, where each E_i is a finite nonempty set;
- *source maps* $\mathbf{s} : E_i \to V_{i-1}$ with the property that for each vertex $v \in V_{i-1}$, there exists at least one edge $e \in E_i$ such that $\mathbf{s}(e) = v$;
- *range maps* $\mathbf{r} : E_i \to V_i$;
- a *labelling* $d : V(B) \to \mathbb{N}^+$ such that for every vertex $v \in V(B)$

$$d(v) \geq \sum_{\mathbf{r}(e)=v} d(\mathbf{s}(e)). \tag{2.1}$$

The Bratteli diagram B is said to be *unital* if there is an equality in (2.1) for all but finitely many vertices $v \in V(B)$. (This terminology comes from the fact that B is unital if and only if an associated approximately finite-dimensional C^*-algebra is unital.)

To each Bratelli diagram $B = (\, \{V_i\}, \{E_i\}, \mathbf{s}, \mathbf{r}, d \,)$, we associate a corresponding LDA-group, $A(B) = \bigcup_{i \in \mathbb{N}} G_i$, where each $G_i = \prod_{v \in V_i} \operatorname{Alt}(\Delta_v)$ for some set Δ_v of cardinality $d(v)$ as follows. First we define the sets Δ_v inductively by setting $\Delta_v = \{\, \alpha^v_\ell \mid 0 \leq \ell < d(v) \,\}$ if $v \in V_0$, and

$$\Delta_v = \{\, \sigma^\frown e \mid e \in E_{i+1}, r(e) = v, \sigma \in \Delta_{s(e)} \,\} \cup \{\, \alpha^v_\ell \mid k(v) \leq \ell < d(v) \,\}$$

if $v \in V_{i+1}$, where $k(v) = d(v) - \sum_{\mathbf{r}(e)=v} d(\mathbf{s}(e))$. Next for each $u \in V_i$, we define the embedding $\mathrm{Alt}(\Delta_u) \hookrightarrow G_{i+1}$ by specifying the action of each $g \in \mathrm{Alt}(\Delta_u)$ on each $z \in \bigsqcup_{v \in V_{i+1}} \Delta_v$ as follows:

- if $z = \sigma \frown e$ for some $\sigma \in \Delta_u$ and some edge e with $s(e) = u$, then $g(\sigma \frown e) = g(\sigma) \frown e$;
- otherwise, $g(z) = z$.

Clearly these embeddings extend to an embedding $G_i \hookrightarrow G_{i+1}$, and we define $A(B) = \bigcup_{i \in \mathbb{N}} G_i$.

Remark 2.2. If $(n_i \mid \in \mathbb{N})$ is a strictly increasing sequence of natural numbers, then we can define an associated Bratelli diagram $B' = (\{V_i'\}, \{E_i'\}, \mathbf{s}', \mathbf{r}', d')$ as follows:

- $V_i' = V_{n_i}$ and $d' = d \restriction \bigsqcup_{i \in \mathbb{N}} V_i'$.
- E_i' is the set of paths $p = e_1 \cdots e_\ell$ in B from $V_{n_{i-1}}$ to V_{n_i}, $\mathbf{s}'(p) = \mathbf{s}(e_1)$ and $\mathbf{r}'(p) = \mathbf{r}(e_\ell)$.

The Bratteli diagram B' is said to be a *telescoping* of B. Clearly we can identify $A(B')$ with $\bigcup_{i \in \mathbb{N}} G_{n_i}$ and so $A(B') \cong A(B)$.

The set $\mathcal{P}(B)$ of paths through B consists of the infinite sequences γ of the form

$$\gamma = \alpha_\ell^v \frown e_{i+1} \frown e_{i+2} \frown \cdots \frown e_k \frown e_{k+1} \frown \cdots ,$$

where $v \in V_i$ and $k(v) \leq \ell < d(v)$, $s(e_{i+1}) = \alpha_\ell^v$, and $\mathbf{r}(e_k) = \mathbf{s}(e_{k+1})$ for all $k \geq i+1$. Let $A(B) \curvearrowright \mathcal{P}(B)$ be the action defined by

$$g(\gamma) = g(\alpha_\ell^v \frown e_{i+1} \frown e_{i+2} \frown \cdots \frown e_k) \frown e_{k+1} \frown \cdots , \qquad g \in G_k.$$

For each $i \in \mathbb{N}$, $v \in V_i$ and $\sigma \in \Delta_v$, let $P(\sigma)$ be the set of paths $\gamma \in \mathcal{P}(B)$ of the form

$$\gamma = \sigma \frown e_{i+1} \frown e_{i+2} \frown \cdots$$

Then the sets $P(\sigma)$ form a clopen basis for a locally compact topology on $\mathcal{P}(B)$. Clearly $\mathcal{P}(B)$ is compact if and only if B is unital.

Remark 2.3. If B' is a telescoping of B, then $A(B') \curvearrowright \mathcal{P}(B')$ is canonically isomorphic to $A(B) \curvearrowright \mathcal{P}(B)$.

Definition 2.4. Let $B = (\{V_i\}, \{E_i\}, \mathbf{s}, \mathbf{r}, d)$ be a Bratelli diagram.

(a) B is *thick* if for every vertex $u \in V_i$, there exists a vertex $v \in V_k$ for some $k > i$ such that u and v are connected by more than one path.

(b) B is *simple* if for every path $\gamma \in \mathcal{P}(B)$ and vertex $v \in V(B)$, there exists a path starting at v and ending in a vertex on γ.

Remark 2.5. It is easily seen that if B is a simple unital Bratteli diagram, then for each $i \in \mathbb{N}$, there exists $j > i$ such that for every pair of vertices $u \in V_i$ and $v \in V_j$, there exists a path from u to v.

Theorem 2.6 (Lavrenyuk-Nekrashevych [12])**.** *If G is a simple locally finite LDA-group, then exactly one of the following holds:*

(i) $G \cong \mathrm{Alt}(\mathbb{N})$.

(ii) $G \cong A(B)$ *for some simple thick Bratelli diagram* B.

By Corollary 1.3, the infinite alternating group $\mathrm{Alt}(\mathbb{N})$ has no nontrivial URSs. Thus we can restrict our attention to the LDA-groups which are isomorphic to $A(B)$ for some simple thick Bratelli diagram B.

Example 2.7. Suppose that B is a simple thick unital Bratelli diagram and that $G = A(B)$. Then $\mathcal{P}(B)$ compact. Also, since B is simple, it follows easily that every G-orbit is dense in $\mathcal{P}(B)$. Thus $G \curvearrowright \mathcal{P}(B)$ is a minimal G-action. Since the stabilizer map $\gamma \overset{f}{\mapsto} G_\gamma$ is continuous and $1 \subsetneq G_\gamma \subsetneq G$ for each $\gamma \in \mathcal{P}(B)$, it follows that $X_{\mathcal{P}(B)} = f(\mathcal{P}(B))$ is a nontrivial URS of G. We will refer to $G \curvearrowright \mathcal{P}(B)$ as the *canonical minimal action* of G.

In Section 4, we will prove the following classification theorem.

Theorem 2.8. *Suppose that B is a simple thick Bratelli diagram.*

(i) *If B is unital, then the only nontrivial URS of $A(B)$ is the stabilizer URS of the canonical minimal action $A(B) \curvearrowright \mathcal{P}(B)$.*

(ii) *$A(B)$ has a nontrivial URS if and only if B is unital.*

3. Ergodic theory and normalized permutation characters

In this section, in preparation for the proof of Theorem 2.8, we will discuss the ergodic theory of countably infinite locally finite groups and present some results on the normalized permutation characters of finite groups.

Let $G = \bigcup_{i \in \mathbb{N}} G_i$ be the union of the strictly increasing chain of finite subgroups G_i and let $G \curvearrowright (Z, \mu)$ be an ergodic action on a Borel probability space. Then the following is a special case of more general results of Vershik [22, Theorem 1] and Lindenstrauss [15, Theorem 1.3].

The Pointwise Ergodic Theorem. *With the above hypotheses, if $B \subseteq Z$ is a μ-measurable subset, then for μ-a.e. $z \in Z$,*

$$\mu(B) = \lim_{i \to \infty} \frac{1}{|G_i|} |\{ g \in G_i \mid g \cdot z \in B \}|.$$

In particular, the Pointwise Ergodic Theorem applies when B is the μ-measurable subset $\mathrm{Fix}_Z(g) = \{ z \in Z \mid g \cdot z = z \}$ for some $g \in G$. For each $z \in Z$ and $i \in \mathbb{N}$, let $\Omega_i(z) = \{ g \cdot z \mid g \in G_i \}$ be the corresponding G_i-orbit. Then, as pointed out in Thomas-Tucker-Drob [21, Theorem 2.1], the following result is an easy consequence of the Pointwise Ergodic Theorem.

Theorem 3.1. *With the above hypotheses, for μ-a.e. $z \in Z$, for all $g \in G$,*

$$\mu(\mathrm{Fix}_Z(g)) = \lim_{i \to \infty} |\mathrm{Fix}_{\Omega_i(z)}(g)| / |\Omega_i(z)|.$$

Of course, the permutation group $G_i \curvearrowright \Omega_i(z)$ is isomorphic to $G_i \curvearrowright G_i/H_i$, where G_i/H_i is the set of cosets of $H_i = \{ h \in G_i \mid h \cdot z = z \}$ in G_i. The following simple observation will be used repeatedly in our later applications of Theorem 3.1. (For example, see Thomas-Tucker-Drob [21, Proposition 2.2].)

Definition 3.2. If $K \curvearrowright \Omega$ is an action of a finite group K on a finite set Ω, then the corresponding *normalized permutation character* θ is defined by

$$\theta(g) = |\mathrm{Fix}_\Omega(g)| / |\Omega|, \qquad g \in K.$$

Proposition 3.3. *If $H \leqslant K$ are finite groups and θ is the normalized permutation character corresponding to the action $K \curvearrowright K/H$, then*

$$\theta(g) = \frac{|g^K \cap H|}{|g^K|} = \frac{|\{s \in K \mid sgs^{-1} \in H\}|}{|K|}.$$

The following consequence of Proposition 3.3 implies that when computing upper bounds for the normalized permutation characters of actions $A \curvearrowright A/H$, we can restrict our attention to those coming from maximal subgroups $H < A$.

Corollary 3.4. *If $H \leqslant H' \leqslant A$ are finite groups and θ, θ' are the normalized permutation characters corresponding to the actions $A \curvearrowright A/H$ and $A \curvearrowright A/H'$, then $\theta(g) \leq \theta'(g)$ for all $g \in A$.*

We will next present two easy but useful results on the normalized permutation characters of finite products of finite groups. Suppose that $K = \prod_{1 \leq \ell \leq m} K_\ell$ is a finite product of finite groups. Let $H \leqslant K$ and let θ be the normalized permutation character corresponding to the action $K \curvearrowright K/H$. For each $g \in K$ and $1 \leq \ell \leq m$, let $g_\ell \in K_\ell$ be the element such that $g = g_1 \cdots g_\ell \cdots g_m$. Let $H_\ell = \{ h_\ell \mid h \in H \}$ and let θ_ℓ be the normalized permutation character corresponding to the action $K_\ell \curvearrowright K_\ell/H_\ell$.

Proposition 3.5. *With the above hypotheses, $\theta(g) \leq \prod_{1 \leq \ell \leq m} \theta_\ell(g_\ell)$ for all $g \in K$.*

Proof. Let $P = \prod_{1 \leq \ell \leq m} H_\ell$. Then $H \leqslant P \leqslant K$; and by repeated applications of Proposition 3.3, we obtain that

$$\begin{aligned}
\theta(g) &= \frac{|\{ s \in K \mid sgs^{-1} \in H \}|}{|K|} \\
&\leq \frac{|\{ s \in K \mid sgs^{-1} \in P \}|}{|K|} \\
&= \frac{|\{ s \in K \mid s_\ell g_\ell s_\ell^{-1} \in H_\ell \text{ for all } 1 \leq \ell \leq m \}|}{|K|} \\
&= \prod_{1 \leq \ell \leq m} \frac{|\{ t \in K_\ell \mid t g_\ell t^{-1} \in H_\ell \}|}{|K_\ell|} \\
&= \prod_{1 \leq \ell \leq m} \theta_\ell(g_\ell).
\end{aligned}$$

□

Next suppose that $S_1, \cdots, S_r$ are pairwise isomorphic groups and that $r \geq 2$. Then $H \leqslant \prod_{1 \leq \ell \leq r} S_\ell$ is said to be a *diagonal subgroup* if there exist isomorphisms $\pi_\ell : S_1 \to S_\ell$ for $2 \leq \ell \leq r$ such that $H = \{ (h, \pi_2(h), \cdots, \pi_r(h)) \mid h \in S_1 \}$. In the degenerate case when $r = 1$, we will take S_1 to be the only diagonal subgroup of S_1. Slightly abusing notation, since none of our arguments depend on the particular isomorphisms π_ℓ, we will write

$$H = \mathrm{Diag}(\prod_{1 \leq \ell \leq r} S_\ell).$$

Lemma 3.6. *Let* $K = \underbrace{S \times \cdots \times S}_{r \text{ times}}$*, where* S *is a finite group and* $r \geq 2$*, and let* $H = \operatorname{Diag}(K) = \{ (g, \cdots, g) \mid g \in S \}$*. If* θ *is the normalized permutation character corresponding to the action* $K \curvearrowright K/H$ *and* $h = (g, \cdots, g) \in H$*, then* $\theta(h) = \left(\frac{1}{|g^S|} \right)^{r-1}$.

Proof. Note that

$$\theta(h) = |h^K \cap H| / |h^K| = \frac{|g^S|}{|g^S|^r} = \left(\frac{1}{|g^S|} \right)^{r-1}.$$

□

In the proof of Theorem 2.8, we will also make use of the following result on the subgroups of finite products of simple nonabelian groups.

Proposition 3.7. *Suppose that* $S_1, \cdots, S_m$ *are simple nonabelian groups and that* $H \leqslant S_1 \times \cdots \times S_m$ *is a subgroup such that each of the projections* $H \to S_\ell$ *is surjective. Then there exists a partition* $A_1 \sqcup \cdots \sqcup A_r$ *of* $\{ 1, \cdots, m \}$ *such that*

$$H = \prod_{k=1}^{r} \operatorname{Diag}(\prod_{\ell \in A_k} S_\ell).$$

Sketch proof. For each $h \in H$, let $h = h_1 h_2 \cdots h_m$, where each $h_\ell \in S_\ell$, and let $s(h) = \{ \ell \mid h_\ell \neq 1 \}$. Let $\mathcal{P} = \{ A_1, \cdots, A_r \}$ be the collection of minimal subsets $A \subseteq \{ 1, \cdots, m \}$ such that there exists $1 \neq h \in H$ with $s(h) = A$. Then it is easily checked that $\mathcal{P}$ is a partition of $\{ 1, \cdots, m \}$ and that

$$H = \prod_{k=1}^{r} \operatorname{Diag}(\prod_{\ell \in A_k} S_\ell).$$

□

Finally we record two results from Thomas-Tucker-Drob [21, Section 6] on the normalized permutation characters of finite alternating groups. In the statements of both of the following lemmas, Δ denotes a finite set of cardinalty $|\Delta| = n$.

Lemma 3.8. *For any* $\varepsilon > 0$ *and* $0 < a \leq 1$*, there exists an integer* $r_{a,\varepsilon}$ *such that if* $r_{a,\varepsilon} \leq r \leq n/2$ *and* $H < \operatorname{Alt}(\Delta)$ *is a subgroup with an* H*-invariant set* $\Sigma \subseteq \Delta$ *of cardinality* $|\Sigma| = r$*, then for any element* $g \in \operatorname{Alt}(\Delta)$ *satisfying* $|\operatorname{supp}(g)| \geq an$*,*

$$\frac{|\{ s \in \operatorname{Alt}(\Delta) \mid sgs^{-1} \in H \}|}{|\operatorname{Alt}(\Delta)|} < \varepsilon.$$

Lemma 3.9. *For any $\varepsilon > 0$ and $0 < a \leq 1$ and $r \geq 0$, there exists an integer $d_{a,r,\varepsilon}$ such that if $d_{a,r,\varepsilon} \leq d \leq (n-r)/2$ and $H < \mathrm{Alt}(\Delta)$ is any subgroup such that*

(i) *there exists an H-invariant subset $\Sigma \subseteq \Delta$ of cardinality r, and*

(ii) *H acts imprimitively on $\Delta \smallsetminus \Sigma$ with a proper system of imprimitivity $\mathcal{B}$ of blocksize d,*

then for any element $g \in \mathrm{Alt}(\Delta)$ satisfying $|\operatorname{supp}(g)| \geq an$,

$$\frac{|\{\, s \in \mathrm{Alt}(\Delta) \mid sgs^{-1} \in H \,\}|}{|\,\mathrm{Alt}(\Delta)|} < \varepsilon.$$

4. The proof of Theorem 2.8

In this section, we will present the proof of Theorem 2.8. We will first prove that if B is a unital simple thick Bratelli diagram, then the only nontrivial URS of $A(B)$ is the stabilizer URS of the canonical minimal action $A(B) \curvearrowright \mathcal{P}(B)$. So suppose that B is a unital simple thick Bratelli diagram. Then, after replacing B by a suitable telescoping if necessary, we can suppose that

$$k(v) = d(v) - \sum_{\mathbf{r}(e)=v} d(\mathbf{s}(e)) = 0$$

for every vertex $v \in V(B) \smallsetminus V_0$. Similarly, by Remark 2.5, we can suppose that for every $i \in \mathbb{N}$, every vertex $u \in V_i$ is joined by an edge to every vertex $v \in V_{i+1}$. Let $G = A(B) = \bigcup_{i \in \mathbb{N}} G_i$ be the corresponding LDA-group, where each $G_i = \prod_{v \in V_i} \mathrm{Alt}(\Delta_v)$.

Lemma 4.1. $\lim \min\{\, |\Delta_v| : v \in V_i \,\} = \infty$.

Proof. Let $v \in V_{i+1}$. Since every $u \in V_i$ is joined to v by an edge, we have that

$$|\Delta_v| \geq \sum_{u \in V_i} |\Delta_u|,$$

and the result follows. □

Lemma 4.2. $\liminf \min\{\, |\operatorname{supp}_{\Delta_v}(g)|/|\Delta_v| : v \in V_i \,\} > 0$ *for all $1 \neq g \in G$.*

Proof. Suppose that $(\, k_i \mid i \in \mathbb{N} \,)$ is a strictly increasing sequence of natural numbers and that $v_i \in V_{k_i}$ for each $i \in \mathbb{N}$. Then it is easily checked that

$$N = \{\, g \in G \mid \lim_{i \to \infty} |\operatorname{supp}_{\Delta_{v_i}}(g)|/|\Delta_{v_i}| = 0 \,\}$$

is a normal subgroup of G. On the other hand, let $h \in G_0$ be such that $h \upharpoonright \Delta_u$ is fixed-point-free for all $u \in V_0$. Then for all $i \geq 0$, regarding h as an element of G_i, we have that $h \upharpoonright \Delta_v$ is fixed-point-free for all $v \in V_i$ and so $h \notin N$. Since G is simple, the result follows. □

Suppose that $X \subseteq \mathrm{Sub}_G$ is a nontrivial URS. Since G is amenable, there exists a G-invariant ergodic Borel probability measure ν on Sub_G which concentrates on X. Applying Creutz-Peterson [3, Proposition 3.3.1], we can suppose that ν is the stabilizer distribution of an ergodic action $G \curvearrowright (Z, \mu)$. Let $\chi(g) = \mu(\mathrm{Fix}_Z(g))$; and for each $z \in Z$ and $i \in \mathbb{N}$, let $\Omega_i(z) = \{ g \cdot z \mid g \in G_i \}$. Then, by Theorem 3.1, for μ-a.e. $z \in Z$, for all $g \in G$, we have that

$$\chi(g) = \mu(\mathrm{Fix}_Z(g)) = \lim_{i \to \infty} |\mathrm{Fix}_{\Omega_i(z)}(g)| / |\Omega_i(z)|.$$

Fix such an element $z \in Z$ and let $H = \{ g \in G \mid g \cdot z = z \}$ be the corresponding point stabilizer. Then we can suppose that $\chi(g) > 0$ for all $g \in H$. For each $i \in \mathbb{N}$, let $H_i = H \cap G_i$. Then $G_i \curvearrowright \Omega_i(z)$ is isomorphic to $G_i \curvearrowright G_i/H_i$. Let θ_i be the normalized permutation character corresponding to the action $G_i \curvearrowright G_i/H_i$.

For each $i \in \mathbb{N}$, $v \in V_i$ and $g \in G_i$, let $g_v \in \mathrm{Alt}(\Delta_v)$ be such that $g = \prod_{v \in V_i} g_v$; and for each $v \in V_i$, let $H_v = \{ h_v \mid h \in H_i \}$ and let θ_v be the normalized permutation character corresponding to the action $\mathrm{Alt}(\Delta_v) \curvearrowright \mathrm{Alt}(\Delta_v)/H_v$. Then, applying Proposition 3.5, we obtain:

Lemma 4.3. $\theta_i(g) \leq \prod_{v \in V_i} \theta_v(g_v)$ *for all* $g \in G_i$.

For each $i \in \mathbb{N}$ and $v \in V_i$, let $n_v = |\Delta_v|$ and let $n_i = \sum_{v \in V_i} n_v$. (Of course, using the notation introduced earlier, we have that $n_v = |\Delta_v| = d(v)$, where d is the labelling function of the Bratelli diagram B. We have introduced the extra notation n_v in order to make the calculations in this section a little more readable.)

Lemma 4.4. *There exist constants* $s \geq 0$, $d \geq 1$ *such that for all but finitely many* $i \in \mathbb{N}$, *for all* $v \in V_i$, *there exists a unique* H_v*-invariant subset* $\Sigma_v \subseteq \Delta_v$ *of cardinality* $0 \leq |\Sigma_v| \leq s$ *such that:*

- H_v *acts transitively on* $\Delta_v \smallsetminus \Sigma_v$; *and*
- *there exists a maximal system of imprimitivity* $\mathcal{B}_v$ *of blocksize* $1 \leq d_v \leq d$ *for the action* $H_v \curvearrowright \Delta_v \smallsetminus \Sigma_v$ *and* H_v *induces at least* $\mathrm{Alt}(\mathcal{B}_v)$ *on* $\mathcal{B}_v$.

Remark 4.5. Note that we allow the possibility that $d_v = 1$, in which case H_v induces at least $\mathrm{Alt}(\Delta_v \smallsetminus \Sigma_v)$ on $\Delta_v \smallsetminus \Sigma_v$.

Proof of Lemma 4.4. Let $1 \neq g \in H$ be an element of prime order p; say, $g \in H_{i_0}$. Then, applying Lemma 4.3, since

$$0 < \chi(g) = \lim_{i\to\infty} |\,\mathrm{Fix}_{\Omega_i(z)}(g)\,|/|\,\Omega_i(z)\,| = \lim_{i\to\infty} \theta_i(g),$$

it follows that

$$\liminf \min\{\, \theta_v(g_v) : v \in V_i \,\} > 0.$$

Also, by Lemma 4.2, there exists a constant $0 < a \leq 1$ such that for all $i \geq i_0$ and $v \in V_i$, we have that $|\,\mathrm{supp}_{\Delta_v}(g_v)| \geq apn_v$. For each $i \geq i_0$ and $v \in V_i$, let

$$r_v = \max\{\, |\Sigma| : \Sigma \subseteq \Delta_v \text{ is } H_v\text{-invariant and } |\Sigma| \leq \tfrac{1}{2}|\Delta_v|\,\}.$$

Then, applying Lemma 3.8, we see that there exists s such that $0 \leq r_v \leq s$ for all $v \in V_i$ with $i \geq i_0$. Furthermore, choosing i_0 so that $|\Delta_v| \geq 4s$ for all $v \in V_{i_0}$, it follows that for all $i \geq i_0$ and $v \in V_i$, there exists a unique H_v-invariant subset $\Sigma_v \subseteq \Delta_v$ of cardinality r_v and that H_v acts transitively on $\Delta_v \smallsetminus \Sigma_v$. Similarly, applying Lemma 3.9, there exists an integer $d \geq 1$ such that for all $i \geq i_0$ and $v \in V_i$, there exists a maximal system of imprimitivity $\mathcal{B}_v$ of blocksize $1 \leq d_v \leq d$ for the transitive action of $H_v \curvearrowright \Delta_v \smallsetminus \Sigma_v$.

Clearly we can also suppose that i_0 has been chosen so that $an_v \geq 2$ for $v \in V_i$ with $i \geq i_0$. Fix such a vertex v and let $n = n_v$. Suppose that $g_v \in \mathrm{Alt}(\Delta_v)$ is a product of b p-cycles. Since $b \geq an \geq 2$, it follows that the conjugacy classes of g_v in $\mathrm{Alt}(\Delta_v)$ and $\mathrm{Sym}(\Delta_v)$ coincide and hence

$$|g_v^{\mathrm{Alt}(\Delta_v)}| = \frac{n!}{p^b b!(n-bp)!}.$$

Applying Stirling's Approximation and the fact that $b \geq a\,n$, it follows that there exist constants $c,\, k > 0$ such that

$$|g_v^{\mathrm{Alt}(\Delta_v)}| > c\,k^n \frac{n^n}{b^b(n-bp)^{n-bp}} > c\,k^n \frac{n^n}{n^b n^{n-bp}} \geq c\,k^n (n^n)^{(p-1)a}.$$

Let $m = |\mathcal{B}_v|$ and let $K_v \leqslant \mathrm{Sym}(\mathcal{B}_v)$ be the group induced by the action $H_v \curvearrowright \mathcal{B}_v$. Suppose that $\mathrm{Alt}(\mathcal{B}_v) \nleqslant K_v$ Then, by Praeger-Saxl [19], it follows that

$$|K_v| < 4^m \leq 4^n;$$

and since H_v is isomorphic to a subgroup of $\mathrm{Sym}(\Sigma_v) \times (\,\mathrm{Sym}(d_v) \operatorname{wr} K_v\,)$, it follows that $|H_v| \leq s!(d!)^n 4^n$. Thus

$$\theta_v(g_v) = \frac{|g_v^{\mathrm{Alt}(\Delta_v)} \cap H_v|}{|g_v^{\mathrm{Alt}(\Delta_v)}|} < \frac{|H_v|}{|g_v^{\mathrm{Alt}(\Delta_v)}|} < \frac{s!(d!)^n 4^n}{c\, k^n (n^n)^{(p-1)a}}.$$

It follows that there exist only finitely many $i \geq i_0$ such that $\mathrm{Alt}(\mathcal{B}_v) \not\leq K_v$ for some $v \in V_i$. □

Next we prove the following strengthening of Lemma 4.4.

Lemma 4.6. *There exist constants s, $t \geq 1$ such that for all but finitely many $i \in \mathbb{N}$, there exists a subset $T_i \subseteq V_i$ of cardinality $0 \leq |T_i| \leq t$ such that for all $v \in V_i$,*

- *if $v \notin T_i$, then $H_v = \mathrm{Alt}(\Delta_v)$;*
- *if $v \in T_i$, then there exists a unique H_v-invariant subset $\emptyset \neq \Sigma_v \subseteq \Delta_v$ of cardinality at most s such that H_v induces at least $\mathrm{Alt}(\Delta_v \smallsetminus \Sigma_v)$ on $\Delta_v \smallsetminus \Sigma_v$.*

Proof. Throughout, let p be an odd prime such that $p > \max\{\,5, s, d\,\}$, where s, d are the constants given by Lemma 4.4. We will first show that we can take $d = 1$ in the statement of Lemma 4.4.

Since $\liminf\{\,|\Delta_v| : v \in V_i\,\} = \infty$ and p, d are fixed, there exists i_0 such that $|\Delta_v| \geq p(p+1)(d+1)$ and $|\mathcal{B}_v| \geq 7$ for all $v \in V_{i_0}$. Furthermore, we can suppose that i_0 has been chosen so that H_{i_0} satisfies the conclusion of Lemma 4.4. Let

$$\varphi : H_{i_0} \to \prod_{v \in V_{i_0}} \mathrm{Sym}(\mathcal{B}_v)$$

be the homomorphism induced by the actions $H_v \curvearrowright \mathcal{B}_v$ and let $K = \varphi(H_{i_0})$. Then, by Lemma 4.4, K projects onto either $\mathrm{Alt}(\mathcal{B}_v)$ or $\mathrm{Sym}(\mathcal{B}_v)$ for every $v \in V_{i_0}$. It follows easily that $K_0 = K \cap \prod_{v \in V_{i_0}} \mathrm{Alt}(\mathcal{B}_v)$ projects onto $\mathrm{Alt}(\mathcal{B}_v)$ for every $v \in V_{i_0}$. Hence, applying Proposition 3.7, there exists a partition $\mathcal{P}$ of V_{i_0} such that

$$K_0 = \prod_{S \in \mathcal{P}} \mathrm{Diag}(\prod_{v \in S} \mathrm{Alt}(\mathcal{B}_v)).$$

Notice that since each $|\mathcal{B}_v| \geq 7$, if u, $v \in S \in \mathcal{P}$ and $\pi : \mathrm{Alt}(\mathcal{B}_u) \to \mathrm{Alt}(\mathcal{B}_v)$ is the isomorphism associated with the corresponding diagonal factor of K_0, then π preserves the cycle structure of the permutations. Consequently, there exists an element $k = \prod_{v \in V_{i_0}} k_v \in K_0$ such that each k_v is an element of order p which fixes at most $p-1$ blocks of $\mathcal{B}_v$. Since $p > \max\{\, s, d\,\}$, it follows that $\ker\varphi$ has no elements of order p, and this means that there exists an element $g \in H_{i_0}$ of order p such that $\varphi(g) = k$. Let $v \in V_{i_0}$ and let $a_v \in [\,0,1\,]$ be such that g_v be a product of $a_v n_v$ p-cycles. Since

$$|\operatorname{supp}(g_v)| \geq n_v - [s + (p-1)d] > n_v - (d+1)p,$$

it follows that

$$a_v = \frac{|\operatorname{supp}(g_v)|}{p n_v} > \frac{1}{p} - \frac{(d+1)}{n_v} \geq \frac{1}{(p+1)}.$$

Now suppose that $i \geq i_0$ and $u \in V_i$. Then, regarding g as an element of H_i, it follows that g_u is a product of more than $n_u/(p+1)$ p-cycles. Arguing as in the proof of Lemma 4.4, it follows that there exists constants c, $k > 0$ such that

$$|g_u^{\mathrm{Alt}(\Delta_u)}| > c k^{n_u} (n_u^{n_u})^{(p-1)/(p+1)}.$$

On the other hand, by another application of Stirling's Approximation, there also exist constants C, $K > 0$ such that

$$|H_u| \leq |\operatorname{Sym}(\Sigma_u) \times (\operatorname{Sym}(d_u) \operatorname{wr} \operatorname{Sym}(n/d_u))| < C\, K^{n_u} n_u^{n_u/d_u}.$$

If $d_u \geq 2$, then $(p-1)/(p+1) \geq 4/6 > 1/2 \geq 1/d_u$. Since

$$\theta_u(g_u) = \frac{|g_u^{\mathrm{Alt}(\Delta_u)} \cap H_u|}{|g_u^{\mathrm{Alt}(\Delta_u)}|} < \frac{|H_u|}{|g_u^{\mathrm{Alt}(\Delta_u)}|} < \frac{C\, K^{n_u} (n_u^{n_u})^{1/d_u}}{c k^{n_u} (n_u^{n_u})^{(p-1)/(p+1)}},$$

it follows that there exist only finitely many $i \geq i_0$ such that $d_u \geq 2$ for some $u \in V_i$. Thus we can assume that $d = 1$.

For each $i \geq i_0$, let $T_i = \{\, v \in V_i \mid \Sigma_v \neq \emptyset \,\}$. Let p be a prime such that $p > s$. Then, arguing as with the element g above, we can suppose that there exists an element $h \in H_{i_0}$ of order p such that $|\operatorname{Fix}_{\Delta_v}(h_v)| \leq n_v/2$ for all $v \in V_{i_0}$. It follows that if $i \geq i_0$ and we regard h as an element of H_i, then $|\operatorname{Fix}_{\Delta_u}(h_u)| \leq n_u/2$ for all $u \in V_i$. Fix an $i \geq i_0$ and suppose that $u \in T_i$. Let K_u be the setwise stabilizer of Σ_u in $\mathrm{Alt}(\Delta_u)$ and let θ'_u be the normalized permuation character of the action $\mathrm{Alt}(\Delta_u) \curvearrowright \mathrm{Alt}(\Delta_u)/K_u$. Applying Corollary 3.4, since $H_u \leqslant K_u$, it follows that

$$\theta_u(h_u) \leq \theta'_u(h_u) = \frac{|\{\, s \in \mathrm{Alt}(\Delta_u) \mid s h_u s^{-1} \in K_u \,\}|}{|\mathrm{Alt}(\Delta_u)|}.$$

Let $x \in \Sigma_u$. Since $p > |\Sigma_u|$, it follows that if $sh_us^{-1} \in K_u$, then sh_us^{-1} acts trivially on Σ_u and so $sh_us^{-1}(x) = x$. Thus

$$\theta_u(h_u) \leq \frac{|\{\, s \in \operatorname{Alt}(\Delta_u) \mid s^{-1}(x) \in \operatorname{Fix}_{\Delta_u}(h_u)\,\}|}{|\operatorname{Alt}(\Delta_u)|} = \frac{|\operatorname{Fix}_{\Delta_u}(h_u)|}{|\Delta_u|} \leq \frac{1}{2}.$$

Applying Lemma 4.3, we see that $\theta_i(h) \leq (1/2)^{t_i}$, where $t_i = |T_i|$. Since

$$\chi(h) = \lim_{i\to\infty} \theta_i(h) > 0,$$

it follows that there exists a constant t such that $|T_i| \leq t$ for all $i \geq i_0$. □

In order to simplify notation, we will suppose that the conclusion of Lemma 4.6 holds for all $i \in \mathbb{N}$. For those $i \in \mathbb{N}$ such that $T_i \neq \emptyset$, let $\pi_i : H_i \to \prod_{v\in T_i} \operatorname{Sym}(\Sigma_v)$ be the homomorphism such that $g \mapsto \prod_{v\in T_i} g \restriction \Sigma_v$ and let $K_i = \ker \pi_i$; and for those $i \in \mathbb{N}$ such that $T_i = \emptyset$, let $K_i = H_i$. For each $v \in V_i$, let $K_v = \{\, g_v \mid g \in K_i \,\}$. Then $K_v \leqslant \operatorname{Alt}(\Delta_v \smallsetminus \Sigma_v)$ if $v \in T_i$; and, of course, $K_v \leqslant \operatorname{Alt}(\Delta_v)$ if $v \notin T_i$. Since there is a constant c such that each K_i is a normal subgroup of H_i of index at most c, it follows that there exists i_0 such that for all $i \geq i_0$,

- if $v \notin T_i$, then $K_v = \operatorname{Alt}(\Delta_v)$;
- if $v \in T_i$, then $K_v = \operatorname{Alt}(\Delta_v \smallsetminus \Sigma_v)$.

Applying Proposition 3.7, for each $i \geq i_0$, there exists a partition $\mathcal{P}_i$ of V_i such that

$$K_i = \prod_{S\in\mathcal{P}_i} \operatorname{Diag}(\prod_{v\in S} \operatorname{Alt}(\Omega_v)),$$

where

$$\Omega_v = \begin{cases} \Delta_v \smallsetminus \Sigma_v & \text{if } v \in T_i; \\ \Delta_v & \text{if } v \notin T_i. \end{cases}$$

In particular, it follows that each K_i is isomorphic to a product of alternating groups. Since $[H_{i+1} : K_{i+1}] \leq c$, it follows that $K_i \cap K_{i+1}$ is a normal subgroup of K_i of index at most c, and this implies that $K_i \leqslant K_{i+1}$. Let $K = \bigcup_{i\geq i_0} K_i$.

Claim 4.7. *There exists $i_1 \geq i_0$ such that $\mathcal{P}_i$ is the partition of V_i into singleton sets for all $i \geq i_1$.*

Proof. Let I be the set of $i \geq i_0$ such that there exists $S_i \in \mathcal{P}_i$ with $|S_i| \geq 2$. Suppose that I is infinite. Let $1 \neq g \in K_{i_0}$ be an element of prime order p, where $p > s$. For each $i > i_0$ and $v \in V_i$, let $g_v \in \operatorname{Alt}(\Delta_v)$ be the corresponding element when g is regarded as an element of G_i. Then, since each vertex in the Bratelli diagram B is joined by an edge to every vertex on the next level, it follows that each such $g_v \neq 1$; and since $\liminf\{ |\Delta_v| : v \in V_i \} = \infty$, it follows that $\liminf\{ |g_v^{\mathrm{Alt}(\Delta_v)}| : v \in V_i \} = \infty$. Also note that since g has prime order $p > s$, it follows that $g^{G_i} \cap H_i = g^{G_i} \cap K_i$ for all $i \geq i_0$.

For each $i \in I$, choose some $v \in S_i$. Applying Proposition 3.5 and Lemma 3.6, it follows that

$$\theta_i(g) = |g^{G_i} \cap H_i|/|g^{G_i}| = |g^{G_i} \cap K_i|/|g^{G_i}| \leq \left(1/|g_v^{\mathrm{Alt}(\Delta_v)}|\right)^{|S_i|-1}.$$

But this means that $\chi(g) = \lim_{i\to\infty} \theta_i(g) = 0$, which is a contradiction. □

Thus we obtain that for all $i \geq i_1$,

$$K_i = \prod_{v\in T_i} \mathrm{Alt}(\Delta_v \smallsetminus \Sigma_v) \times \prod_{v\in V_i \smallsetminus T_i} \mathrm{Alt}(\Delta_v).$$

Since $K \leqslant H$ and $H \neq G$, we can suppose that i_1 has been chosen so that $T_i \neq \emptyset$ for all $i \geq i_1$. Note that for all $i \geq i_1$,

$$\sum_{v\in T_i} |\Sigma_v| \leq \sum_{u\in T_{i+1}} |\Sigma_u|.$$

Hence there exist an integer $r \geq 1$ such that $\sum_{v\in T_i} |\Sigma_v| = r$ for all but finitely many $i \in \mathbb{N}$.

Claim 4.8. $X = \{ G_\gamma \mid \gamma \in \mathcal{P}(B) \}$.

Proof. Let $\gamma = \alpha ^\frown e_1 ^\frown e_2 ^\frown \cdots ^\frown e_i ^\frown e_{i+1} ^\frown \cdots$ be any path through B, where $\alpha \in \Delta_x$ for some $x \in V_0$. Fix some $i \geq i_1$ and let $\gamma_i = \alpha ^\frown e_1 ^\frown e_2 ^\frown \cdots ^\frown e_i$. Then $\gamma_i \in \Delta_u$ for some $u \in V_i$; and there exists $j > i$ with $\sum_{v\in T_j} |\Sigma_v| = r$ such that for all $v \in V_j$, there are at least r paths between u and v. It follows that there exists $g_i \in G_j$ such that each element $\beta \in g_i(\bigcup_{v\in T_j} \Sigma_v)$ has the form $\gamma_i ^\frown e_{i+1} ^\frown \cdots ^\frown e_j$ for some edges $e_{i+1}, \cdots, e_j$, and this implies that

$$g_i H g_i^{-1} \cap G_i \geqslant g_i K_j g_i^{-1} \cap G_i = \mathrm{Alt}(\Delta_u \smallsetminus \{ \gamma_i \}) \times \prod_{w\in V_i \smallsetminus \{ u \}} \mathrm{Alt}(\Delta_w) = (G_i)_{\gamma_i}.$$

As $(G_i)_{\gamma_i}$ is a maximal proper subgroup of G_i, it follows that $g_i H g_i^{-1} \cap G_i = (G_i)_{\gamma_i}$ for all but finitely many $i \in \mathbb{N}$, and this implies that $G_\gamma \in X$. By the minimality of X, we must have that $X = \{ G_\gamma \mid \gamma \in \mathcal{P}(B) \}$. □

Remark 4.9. The above argument shows that if ν is *any* ergodic IRS of G, then ν concentrates on the stabilizers of the r-subsets $F \in [\,\mathcal{P}(B)\,]^r$ for some $r \geq 1$. Of course, in order to classify the ergodic IRS of G, it is first necessary to classify the G-invariant ergodic measures on $[\,\mathcal{P}(B)\,]^r$.

Conjecture 4.10. If B is simple thick Bratelli unital diagram and ν is an ergodic IRS of $A(B)$, then there exist $A(B)$-invariant ergodic probability measures $\mu_1, \cdots, \mu_r$ on $\mathcal{P}(B)$ such that ν is the stabilizer distribution of

$$A(B) \curvearrowright (\,\mathcal{P}(B)^r, \mu_1 \times \cdots \times \mu_r\,).$$

Finally, in order to complete the proof of Theorem 2.8, it only remains to prove that if $B = (\,\{V_i\}, \{E_i\}, \mathbf{s}, \mathbf{r}, d\,)$ is a non-unital simple thick Bratelli diagram, then $A(B)$ has no nontrivial URSs. After replacing B by a suitable telescoping if necessary, we can suppose that for every $i \in \mathbb{N}^+$, there exists $v \in V_i$ such that

$$k(v) = d(v) - \sum_{\mathbf{r}(e)=v} d(\mathbf{s}(e)) > 0.$$

(Of course, it is possible that there are no edges e with $\mathbf{r}(e) = v$.) Next we will define a sequence of unital simple thick Bratelli diagrams $B_n = (\,\{V_i^n\}, \{E_i^n\}, \mathbf{s}_n, \mathbf{r}_n, d_n\,)$ such that:

- $\mathcal{P}(B) = \bigcup_{n \in \mathbb{N}} \mathcal{P}(B_n)$;
- $A(B) = \bigcup_{n \in \mathbb{N}} A(B_n)$.

Fix some $n \in \mathbb{N}$. First for each $i \leq n$, we define:

- $V_i^n = V_i$ and $d_n \restriction V_i^n = d \restriction V_i^n$;

and then we define V_{j+1}^n and $d_n \restriction V_{j+1}^n$ for $j \geq i$ inductively by:

- V_{j+1}^n is the set of vertices $v \in V_{j+1}$ such that there exists an edge $e \in E_{j+1}$ with $\mathbf{r}(e) = v$ and $\mathbf{s}(e) \in V_j^n$;
- if $v \in V_{j+1}^n$, then $d_n(v) = \sum\{\, d_n(s(e)) \mid \mathbf{s}(e) \in V_j^n, \mathbf{r}(e) = v\,\}$.

Finally, E_i^n is the set of edges $e \in E_i$ such that $\mathbf{s}(e) \in V_{i-1}^n$ and $\mathbf{r}(e) \in V_i^n$; and $\mathbf{s}_n$, $\mathbf{r}_n$ are the restrictions of $\mathbf{s}$, $\mathbf{r}$ to $\bigcup_{i \in \mathbb{N}^+} E_i^n$. It is easily checked that each B_n is a unital simple thick Bratelli diagram and that $\mathcal{P}(B) = \bigcup_{n \in \mathbb{N}} \mathcal{P}(B_n)$. Note that

$$A(B_n) = \bigcup_{i \in \mathbb{N}} \prod_{v \in V_i^n} \mathrm{Alt}(\Delta_v^n),$$

where $\Delta_v^n = \Delta_v$ if $v \in V_i^n = V_i$ for some $i \leq n$, and

$$\Delta_v^n = \{ \sigma^\frown e \mid r(e) = v, \sigma \in \Delta_{s(e)}^n \}$$

if $v \in V_i^n$ for some $i > n$. Thus $A(B) = \bigcup_{n \in \mathbb{N}} A(B_n)$.

Note that if $\gamma = \alpha_\ell^v {}^\frown e_{i+1} {}^\frown e_{i+2} {}^\frown \cdots \in \mathcal{P}(B)$, where $v \in V_i$ and $k(v) \leq \ell < d(v)$, then $\gamma \in \mathcal{P}(B_n)$ if and only if $n \geq i$. Also if $\gamma \notin \mathcal{P}(B_n)$, then $A(B_n) \leqslant A(B)_\gamma$.

Now suppose that $X \subseteq \mathrm{Sub}_{A(B)}$ is a nontrivial URS. As usual, let $A(B) = \bigcup_{i \in \mathbb{N}} G_i$, where each $G_i = \prod_{v \in V_i} \mathrm{Alt}(\Delta_v)$. For each n, let

$$f_n : X \to \mathrm{Sub}_{A(B_n)}$$

be the $A(B_n)$-equivariant continuous map defined by $H \mapsto H \cap A(B_n)$ and let $X_n = f_n(X)$. Then X_n is a compact subspace of $\mathrm{Sub}_{A(B_n)}$ and $A(B_n) \curvearrowright X_n$. Suppose that $1 \in X_n$ for infinitely many $n \in \mathbb{N}$. Then for infinitely many $n \in \mathbb{N}$, there exists $H_n \in X$ such that

$$H_n \cap G_n \leqslant H_n \cap A(B_n) = 1;$$

and it follows that $1 \in X$, which is a contradiction. Thus we can suppose that $1 \notin X_n$ for all but finitely many $n \in \mathbb{N}$. Similary, we can suppose that $A(B_n) \notin X_n$ for all but finitely many $n \in \mathbb{N}$. Let $n_0 \in \mathbb{N}$ be such that $1 \notin X_n$ and $A(B_n) \notin X_n$ for all $n \geq n_0$.

For each $n \geq n_0$, let $Y_n \subseteq \mathrm{Sub}_{A(B_n)}$ be the stabilizer URS of the canonical minimal action $A(B_n) \curvearrowright \mathcal{P}(B_n)$. Then the minimal $A(B_n)$-invariant closed subsets of $\mathrm{Sub}_{A(B_n)}$ are precisely $\{1\}$, $\{A(B_n)\}$ and Y_n. In particular, Y_n is the unique minimal $A(B_n)$-invariant closed subset of X_n. Let $\varphi_n : \mathrm{Sub}_{A(B_{n+1})} \to \mathrm{Sub}_{A(B_n)}$ be the $A(B_n)$-equivariant continuous map defined by $K \mapsto K \cap A(B_n)$. Then $\varphi_n(Y_{n+1})$ is an $A(B_n)$-invariant closed subset of X_n and thus $Y_n \subseteq \varphi_n(Y_{n+1})$. Hence we can inductively define subgroups $H_n \in Y_n$ for $n \geq n_0$ such that $H_{n+1} \cap A(B_n) = H_n$. It follows easily that $H = \bigcup_{n \geq n_0} H_n \in X$. Furthermore, by Theorem 2.8(i), if $n \geq n_0$, then H_n is the stabilizer of a path $\gamma_n \in \mathcal{P}(B_n)$. Recall that if $\gamma_n \notin \mathcal{P}(B_{n_0})$, then $A(B_{n_0}) \leqslant A(B)_\gamma$. It follows easily that there exists a fixed path $\gamma \in \mathcal{P}(B_{n_0})$ such that $\gamma_n = \gamma$ for all $n \geq n_0$ and hence H is the stabilizer of γ with respect to the action $A(B) \curvearrowright \mathcal{P}(B)$. However, there exists $g \in A(B)$ such that $g(\gamma) \notin \mathcal{P}(B_{n_0})$, and this means that

$$A(B_{n_0}) \leqslant A(B)_{g(\gamma)} = gA(B)_\gamma g^{-1} = gHg^{-1},$$

which contradicts the fact that $A(B_{n_0}) \notin X_{n_0}$.

5. The nonlinear finitary linear groups

In this section, we will classify the URSs of the nonlinear finitary linear simple locally finite groups. These groups have been explicitly classified by Hall as follows. (In [8], Hall classified all of the nonlinear finitary linear simple locally finite groups, including also the uncountable ones. If κ is an uncountable cardinal, then it is not true that there exists a unique group of cardinality κ of each geometric type over each locally finite field K.)

Theorem 5.1 (Hall [8]). *A countable nonlinear simple locally finite group G that has a faithful representation as a finitary linear group is isomorphic to one of:*

(i) *the infinite alternating group* $\mathrm{Alt}(\mathbb{N})$*;*
(ii) *the stable special linear group $SL^0_\infty(K)$ over some locally finite field K;*
(iii) *the stable symplectic group $FSp(K)$ over some locally finite field K;*
(iv) *the stable special unitary group $FSU(K)$ over some locally finite field K;*
(v) *the stable orthogonal group $F\Omega(K)$ over some locally finite field K.*

For a clear introduction to the classical finitary linear groups $SL^0_\infty(K)$, $FSp(K)$, $FSU(K)$, $F\Omega(K)$ and their associated geometries, see Hall [7].

Recall that, by Corollary 1.3, the infinite alternating group $\mathrm{Alt}(\mathbb{N})$ has no nontrivial URSs. Also, Theorem 1.8 implies that if G has a nontrivial URS and G is isomorphic to $SL^0_\infty(K)$, $FSp(K)$, $FSU(K)$ or $F\Omega(K)$, then K is a finite field.

5.1. **The stable special linear group.** For the rest of this section, let $V = \bigoplus_{n\in\mathbb{N}^+} Ke_n$ be an infinite dimensional vector space over the finite field K and let V^* be the corresponding dual space. For each $\varphi \in V^*$ and $x \in V$ such that $\varphi(x) = 0$, let $t_{\varphi,x}$ be the corresponding transvection defined by

$$t_{\varphi,x}(v) = v + \varphi(v)x.$$

Let $\mathcal{B}^* = \{ e_n^* \mid n \in \mathbb{N}^+ \}$ be the set of elements of the dual space V^* corresponding to the basis $\mathcal{B} = \{ e_n \mid n \in \mathbb{N}^+ \}$ and let $T \leqslant V^*$ be the subspace generated by $\mathcal{B}^*$. If $\varphi = \sum_{i=1}^m k_i e_{n_i}^* \in T$, then we will write $\varphi^* = \sum_{i=1}^m k_i e_{n_i}$.

Definition 5.2. The *stable special linear group* $SL_\infty^0(K)$ is the subgroup of $GL(V)$ generated by $\{ t_{\varphi,x} \mid \varphi \in T, x \in V, \varphi(x) = 0 \}$.

For each $n \in \mathbb{N}^+$, let V_n be the subspace of V generated by $\{ e_1, e_2, \cdots, e_n \}$ and let V_n^* be the subspace of V^* generated by $\{ e_1^*, e_2^*, \cdots, e_n^* \}$. Then we can identify $SL(V_n)$ with the subgroup of $SL_\infty^0(K)$ generated by

$$\{ t_{\varphi,x} \mid \varphi \in V_n^*, x \in V_n, \varphi(x) = 0 \};$$

and we have that $SL_\infty^0(K) = \bigcup_{n \in \mathbb{N}^+} SL(V_n)$, where the corresponding embedding $SL(V_n) \hookrightarrow SL(V_{n+1})$ is given by

$$A \longmapsto \left(\begin{array}{c|c} A & 0 \\ \hline 0 & 1 \end{array} \right).$$

In [13], Leinen-Puglisi classified the confined subgroups of $SL_\infty^0(K)$ as follows. Here, by definition, if $S \leqslant T$ and $U \leqslant V$ are subspaces, then

$$\mathrm{ann}_S(U) = \{ \varphi \in S \mid \varphi(u) = 0 \text{ for all } u \in U \},$$

and similarly for $\mathrm{ann}_U(S)$.

Theorem 5.3 (Leinen-Puglisi [13])**.** *If K is a finite field, then a subgroup H of $SL_\infty^0(K)$ is confined if and only if there exist subspaces $S \leqslant T$ and $U \leqslant V$ of finite codimensions such that:*

- $\mathrm{ann}_S(U) = 0$ *and* $\mathrm{ann}_U(S) = 0$*;*
- $T_K(S, U) = \langle t_{\varphi,x} \mid \varphi \in S, x \in U, \varphi(x) = 0 \rangle \leqslant H$*.*

This classification theorem easily implies the following result.

Theorem 5.4. *The stable special linear group $SL_\infty^0(K)$ has no nontrivial URSs.*

Proof. Let $G = SL_\infty^0(K)$ and suppose that $X \subseteq \mathrm{Sub}_G$ is a nontrivial URS. Let $H \leqslant G$ be a subgroup such that $H \in X$. Then H is confined and hence there exist subspaces $S \leqslant T$ and $U \leqslant V$ of finite codimensions such that:

- $\mathrm{ann}_S(U) = 0$ and $\mathrm{ann}_U(S) = 0$;

- $T_K(S,U) = \langle\, t_{\varphi,x} \mid \varphi \in S, x \in U, \varphi(x) = 0 \,\rangle \leqslant H$.

Let $R = \{\, \varphi^* \in V \mid \varphi \in S \,\}$ and let $W = R \cap U$. Then W also has finite codimension; say, $\operatorname{codim} W = d$. For each $n \in \mathbb{N}^+$, let $W_n = W \cap V_n$. Then for each $n \geq d+2$, we have that $\dim W_n \geq n - d$ and that

$$SL(W_n) = \langle\, t_{\varphi,x} \mid \varphi^*, x \in W_n, \varphi(x) = 0 \,\rangle \leqslant SL(V_n) \cap H.$$

But then there exists $g_n \in SL(V_n)$ such that $SL(V_{n-d}) \leqslant g_n\, H\, g_n^{-1}$ and so $G = SL^0_\infty(K)$ lies in the closure of $\{\, gHg^{-1} \mid g \in G \,\}$, which contradicts the assumption that X is a nontrivial URS. □

5.2. **The classical finitary linear groups of isometries.** Once again, let $V = \bigoplus_{n \in \mathbb{N}^+} K e_n$ be an infinite dimensional vector space over the finite field K and let $SL^0_\infty(K)$ be the stable special linear group. Then the classical finitary linear groups of isometries $FSp(K)$, $FSU(K)$ and $F\Omega(K)$ are defined as follows.

First let σ be the canonical nondegenerate symplectic form on V such that

$$\sigma(e_{2\ell+1}, e_{2\ell+2}) = -\sigma(e_{2\ell+2}, e_{2\ell+1}) = 1$$

for all $\ell \in \mathbb{N}$; and otherwise $\sigma(e_i, e_j) = 0$. Let $Sp(K)$ be the corresponding symplectic group; i.e. the group of all σ-preserving elements $g \in GL(V)$.

Definition 5.5. The *stable symplectic group* is $FSp(K) = Sp(K) \cap SL^0_\infty(K)$.

Next suppose that the finite field K admits an automorphism α of order 2. Let σ be the canonical nondegenerate unitary form on V such that $\sigma(e_i, e_j) = \delta_{ij}$; and let $GU(K)$ be the corresponding unitary group; i.e. the group of all σ-preserving elements $g \in GL(V)$.

Definition 5.6. The *stable special unitary group* is $FSU(K) = GU(K) \cap SL^0_\infty(K)$.

Finally let $Q : V \to K$ be the quadratic form such that

$$V = H_1 \perp H_2 \perp \cdots \perp H_n \perp \cdots$$

is the direct sum of the pairwise orthogonal hyperbolic 2-spaces $H_n = \langle\, e_{2n-1}, e_{2n} \,\rangle$, where $Q(\alpha e_{2n-1} + \beta e_{2n}) = \alpha\beta$. Let $GO(K)$ be the corresponding orthogonal group; i.e. the group of all Q-preserving elements $g \in GL(V)$.

Definition 5.7. The *stable orthogonal group* is $F\Omega(K) = (\, GO(K) \cap SL^0_\infty(K)\,)'$; i.e. $F\Omega(K)$ is the commutator subgroup of $GO(K) \cap SL^0_\infty(K)$.

We next need to recall some features of the quadratic geometry associated with each finite field of characteristic 2. So suppose that K has characteristic 2 and that σ is the canonical nondegenerate symplectic form on $V = \bigoplus_{n\in\mathbb{N}^+} Ke_n$. Recall that a quadratic form $q : V \to K$ is said to be associated with σ if for all x, $y \in V$,

$$\sigma(x, y) = q(x + y) + q(x) + q(y).$$

Let $\mathcal{Q}$ be the compact space of the quadratic forms associated with σ. Then we can define an action of the stable symplectic group $FSp(K)$ as a group of homeomorphisms of $\mathcal{Q}$ by $(g \cdot q)(x) = q(g^{-1}(x))$.

Proposition 5.8. *$FSp(K) \curvearrowright \mathcal{Q}$ is a minimal action.*

Proof. For each $m \geq 1$, let σ_m be the restriction of σ to $V_{2m} = \bigoplus_{n=1}^{2m} Ke_n$ and let $\mathcal{Q}_m$ be the set of quadratic forms $q_m : V_{2m} \to K$ associated with σ_m. Then the finite symplectic group $Sp(2m, K)$ has two orbits under its action on $\mathcal{Q}_m$; namely,

$$\mathcal{Q}^\varepsilon_m = \{\, q_m \in \mathcal{Q}_m \mid \langle\, V_{2m}, q_m \,\rangle \text{ is an } O^\varepsilon\text{-geometry } \},$$

where $\varepsilon \in \{\, +, - \,\}$. Here $\langle\, V_{2m}, q_m \,\rangle$ is an O^+-geometry if V_{2m} is an othogonal sum of m hyperbolic 2-spaces; and $\langle\, V_{2m}, q_m \,\rangle$ is an O^--geometry if V_{2m} is an othogonal sum of $m - 1$ hyperbolic 2-spaces and one anisotropic 2-space. Note that the sets

$$\mathcal{Q}(q_m) = \{\, q \in \mathcal{Q} \mid q \restriction V_{2m} = q_m \,\}, \qquad q_m \in \mathcal{Q}_m, m \in \mathbb{N}^+,$$

form a clopen basis for the compact topology on $\mathcal{Q}$.

Let $q \in \mathcal{Q}$; and for each $m \in \mathbb{N}^+$, let $q_m = q \restriction V_{2m}$. Let $\mathcal{Q}(q'_m)$ be any basic clopen subset of $\mathcal{Q}$, where $q'_m \in \mathcal{Q}_m$. Recall that the orthogonal sum of two hyperbolic 2-spaces is isometric to the orthogonal sum of two anisotropic 2-spaces. (For example, see Aschbacher [1, 21.2].) It follows that q'_m can be extended to a quadratic form $q'_{m+1} \in \mathcal{Q}_{m+1}$ such that q'_{m+1}, q_{m+1} lie in the same $Sp(2m+2, K)$-orbit; and hence there exists $g \in FSp(K)$ such that $g \cdot q \in \mathcal{Q}(q'_{m+1}) \subseteq \mathcal{Q}(q'_m)$. $\square$

For later use, we also record the following result, which is an easy consequence of Parker-Rowley [18, Theorem 1.1].

Lemma 5.9. *Suppose that K is a finite field of characteristic 2 and that $m \geq 3$. Let $V(2m,K)$ be a $2m$-dimensional vector space over K, let σ be a nondegenerate symplectic form on $V(2m,K)$ and let q be an associated quadratic form. Let $GO(2m,K) \leqslant Sp(2m,K)$ be the orthogonal group corresponding to q and let $\Omega(2m,K) = GO(2m,K)'$. Then the only subgroups H of $Sp(2m,K)$ satisfying*

$$\Omega(2m,K) \leqslant H \leqslant Sp(2m,K).$$

are $\Omega(2m,K)$, $GO(2m,K)$ and $Sp(2m,K)$.

In the remainder of this section, we will prove that if G is a finitary linear simple locally finite group and $X \subseteq \mathrm{Sub}_G$ is a nontrivial URS, then G is the stable symplectic group $FSp(K)$ over a finite field K of characteristic 2, and either $X = \{ FGO(q,K) \mid q \in \mathcal{Q} \}$ or $X = \{ F\Omega(q,K) \mid q \in \mathcal{Q} \}$. Our proof makes use of the following classification of the confined subgroups of the classical finitary linear groups of isometries.

Notation 5.10. If $q \in \mathcal{Q}$, then $GO(q,K)$ denotes the corresponding orthogonal group and $F\Omega(q,K) = (\, GO(q,K) \cap SL^0_\infty(K)\,)'$.

Notation 5.11. If G is a subgroup of $GL(V)$ and W is a subspace of V, then

$$N_G(W) = \{ g \in G \mid g(W) = W \}$$

is the setwise stabilizer of W in G and

$$C_G(W) = \{ g \in G \mid g(w) = w \text{ for all } w \in W \}$$

is the pointwise stabilizer of W in G.

Theorem 5.12 (Leinen-Puglisi [13])**.** *Suppose that G is a classical finitary linear group of isometries relative to a non-degenerate symplectic, unitary, or quadratic form on the vector-space V over the finite field K and that $H \leqslant G$ is a confined subgroup. Then there exists a unique minimal H-invariant subspace W of finite codimension in V such that one of the following holds:*

(a) *If $\mathrm{char}(K) = 2$ and G is a stable symplectic group, then there exists a quadratic form q associated with the symplectic form σ with the property that $H \cap \Gamma$ has finite index in $N_\Gamma(W)$, where $\Gamma = F\Omega(q,K)$ is the corresponding stable orthogonal subgroup of G.*

(b) *In all other cases, H has finite index in $N_G(W)$.*

Lemma 5.13. *Suppose that G is a classical finitary linear group of isometries relative to a non-degenerate symplectic, unitary, or quadratic form on the vector-space V over the finite field K and that $H \leqslant G$ is a confined subgroup. If there exists an H-invariant subspace W of finite codimension in V such that H has finite index in $N_G(W)$, then G lies in the closure of $\{\, gHg^{-1} \mid g \in G \,\}$.*

Proof. We will just consider the case when $G = FSp(K)$ is the stable symplectic group. (The other cases are very similar.) Suppose that $H \leqslant G$ is a confined subgroup and that there exists an H-invariant subspace W of finite codimension d in V such that H has finite index in $N_G(W)$. Let $[N_G(W) : H] = \ell$ and let $m \geq 3$ be such that $|PSp(2m, K)| > \ell!$. Since $\dim W \cap V_{2(d+m)} \geq 2m+d$, it follows that there exists a nondegenerate subspace $U \leqslant W \cap V_{2(d+m)}$ such that $\dim U = 2m$. Since U is nondegenerate, it follows that $V = U \oplus U^{\perp}$. Let $Sp(U) = N_G(U) \cap C_G(U^{\perp})$. Then, since $Sp(U)$ acts trivially on V/U, it follows that $Sp(U) \leqslant N_G(W)$. Also $[Sp(U) : Sp(U) \cap H] \leq \ell$ and hence $Sp(U) \leqslant H$. Finally, since U, $V_{2m} \leqslant V_{2(d+m)}$ are nondegenerate subspaces of dimension $2m$, it follows that there exists an element $g \in Sp(2(d+m), K)$ such that

$$Sp(2m, K) = g\, Sp(U)\, g^{-1} \leqslant g\, H\, g^{-1}.$$

Thus $G = \bigcup_{m \geq 1} Sp(2m, K)$ lies in the closure of $\{\, gHg^{-1} \mid g \in G \,\}$. □

Proof of Theorem 1.5. Suppose that G is a classical finitary linear group and that $X \subseteq \mathrm{Sub}_G$ is a nontrivial URS. Let $H \in X$. Then H is a confined subgroup of G. Applying Theorem 5.12 and Lemma 5.13, since $G \notin X$, it follows that $G = FSp(K)$ is the stable symplectic group over a finite field K of characteristic 2 and that there exists:

- a unique minimal H-invariant subspace W of finite codimension in V;
- a quadratic form q on V associated with the symplectic form σ;

such that $H \cap \Gamma$ has finite index in $N_\Gamma(W)$, where $\Gamma = F\Omega(q, K)$. For each $m \geq 3$, let $\Gamma_m = F\Omega(q, K) \cap Sp(2m, K)$. Then, arguing as in the proof of Lemma 5.13, for each $m \geq 3$, there exists $g_m \in \Gamma$ such that $\Gamma_m \leqslant g_m\, H\, g_m^{-1}$. Thus

$$\Gamma_m \leqslant g_m\, H\, g_m^{-1} \cap Sp(2m, K) \leqslant Sp(2m, K).$$

Since $G \notin X$, it follows that for all but finitely many m,

$$g_m\, H\, g_m^{-1} \cap Sp(2m, K) \neq Sp(2m, K).$$

Applying Lemma 5.9, it follows that for all but finitely many m, either

$$g_m\, H\, g_m^{-1} \cap Sp(2m, K) = F\Omega(q, K) \cap Sp(2m, K)$$

or else

$$g_m\, H\, g_m^{-1} \cap Sp(2m, K) = FGO(q, K) \cap Sp(2m, K).$$

This implies that either $F\Omega(q, K) \in X$ or $FGO(q, K) \in X$, and the result follows. □

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