THE CLASSIFICATION PROBLEM FOR p-LOCAL TORSION-FREE ABELIAN GROUPS OF FINITE RANK

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Abstract. Let $n \geq 3$. We prove that if $p \neq q$ are distinct primes, then the classification problems for p-local and q-local torsion-free abelian groups of rank n are incomparable with respect to Borel reducibility.

1. Introduction

This paper is a contribution to the project [10, 9, 2, 12, 19, 20] of trying to understand the complexity of the classification problem for the torsion-free abelian groups of finite rank. Recall that, up to isomorphism, the torsion-free abelian groups of rank n are exactly the additive subgroups of the n-dimensional vector space $\mathbb{Q}^n$ which contain n linearly independent elements. In 1937, Baer [3] solved the classification problem for the torsion-free abelian groups of rank 1. Since then, despite the efforts of such mathematicians as Kurosh [13] and Malcev [14], no satisfactory solution of the classification problem has been found for the torsion-free abelian groups of rank $n \geq 2$. Thus it was natural to ask whether the classification problem was genuinely more difficult for the groups of rank $n \geq 2$. In 1999, Hjorth [9] proved that the classification problem for the rank 2 groups was strictly harder than that for the rank 1 groups. A little later, building on work of Adams-Kechris [2], Thomas [20] proved that the complexity of the classification problem increases strictly with the rank n.

In this paper, we shall consider the complexity of the classification problem for the p-local torsion-free abelian groups of finite rank. (Recall that an abelian group A is said to be p-local iff A is q-divisible for every prime $q \neq p$.) In Thomas [20], it was shown that the complexity of the classification problem for the p-local torsion-free abelian groups also increases strictly with the rank n. However, this left open the more natural question of whether the classification problem for the

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p-local torsion-free abelian groups of a *fixed* rank n was strictly easier than the classification problem for arbitrary torsion-free abelian groups of rank n. In this paper, we shall use Zimmer's superrigidity theorem [22], together with Ratner's measure classification theorem [17], to prove that if $n \geq 3$ and $p \neq q$ are distinct primes, then the classification problems for p-local and q-local torsion-free abelian groups of rank n have incomparable complexities. This easily implies that the classification problem for the p-local torsion-free abelian groups of rank n is indeed strictly easier than that for arbitrary torsion-free abelian groups of rank n. In order to give a precise formulation of our results, we need to make use of the notion of Borel reducibility. (Here we follow the example of Friedman-Stanley [5] and Hjorth-Kechris [10].)

Let X be a standard Borel space; i.e. a Polish space equipped with its associated σ-algebra of Borel subsets. Then a *Borel equivalence relation* on X is an equivalence relation $E \subseteq X^2$ which is a Borel subset of X^2. If E, F are Borel equivalence relations on the standard Borel spaces X, Y respectively, then we say that E is *Borel reducible* to F and write $E \leq_B F$ if there exists a Borel function $f : X \to Y$ such that xEy iff $f(x)Ff(y)$. We say that E and F are *Borel bireducible* and write $E \sim_B F$ if both $E \leq_B F$ and $F \leq_B E$. Finally we write $E <_B F$ if both $E \leq_B F$ and $F \nleq_B E$. Most of the Borel equivalence relations that we shall consider in this paper arise from group actions as follows. Let G be a *lcsc* group; i.e. a locally compact second countable group. Then a *standard Borel G-space* is a standard Borel space X equipped with a Borel action $(g, x) \mapsto g.x$ of G on X. The corresponding G-orbit equivalence relation on X, which we shall denote by E_G^X, is a Borel equivalence relation. In fact, by Kechris [11], E_G^X is Borel bireducible with a countable Borel equivalence relation; i.e. a Borel equivalence relation E such that every E-class is countable.

Throughout this paper, we shall identify the class of torsion-free abelian groups of rank n with the set $R(\mathbb{Q}^n)$ of subgroups of $\mathbb{Q}^n$ of rank n. Notice that $R(\mathbb{Q}^n)$ is a Borel subset of the Polish space $\mathcal{P}(\mathbb{Q}^n)$ of all subsets of $\mathbb{Q}^n$ and hence $R(\mathbb{Q}^n)$ can be regarded as a standard Borel space. (Here we are identifying $\mathcal{P}(\mathbb{Q}^n)$ with the space $2^{\mathbb{Q}^n}$ of all functions $h : \mathbb{Q}^n \to \{0, 1\}$ equipped with the product topology.)

Furthermore, the natural action of $GL_n(\mathbb{Q})$ on the vector space $\mathbb{Q}^n$ induces a corresponding Borel action on $R(\mathbb{Q}^n)$; and it is easily checked that if A, $B \in R(\mathbb{Q}^n)$, then $A \cong B$ iff there exists an element $\varphi \in GL_n(\mathbb{Q})$ such that $\varphi(A) = B$. Similarly, we shall identify the class of p-local torsion-free abelian groups of rank n with the standard Borel space $R^{(p)}(\mathbb{Q}^n)$ of p-local subgroups of $\mathbb{Q}^n$ of rank n.

Definition 1.1. For each $n \geq 1$ and prime p, the isomorphism relations on $R(\mathbb{Q}^n)$ and $R^{(p)}(\mathbb{Q}^n)$ will be denoted by $\cong_n$ and $\cong_n^{(p)}$ respectively.

With this notation, the main results of Thomas [20] say that $(\cong_n) <_B (\cong_{n+1})$ and $(\cong_n^{(p)}) <_B (\cong_{n+1}^{(p)})$ for each $n \geq 1$ and each prime p. We are now also able to state the main results of this paper.

Theorem 1.2. *If $n \geq 3$ and $p \neq q$ are distinct primes, then $\cong_n^{(p)}$ and $\cong_n^{(q)}$ are incomparable with respect to Borel reducibility.*

Corollary 1.3. *If $n \geq 3$ and p is a primes, then $(\cong_n^{(p)}) <_B (\cong_n)$.*

Proof. If $q \neq p$ is another prime, then $(\cong_n^{(q)}) \leq_B (\cong_n)$ and $(\cong_n^{(q)}) \not\leq_B (\cong_n^{(p)})$. □

Of course, it is trivially the case that $(\cong_1^{(p)}) <_B (\cong_1)$ for each prime p, since there only exist two p-local torsion-free abelian groups of rank 1 up to isomorphism. However, the following problem remains open.

Conjecture 1.4. $(\cong_2^{(p)}) <_B (\cong_2)$ for each prime p.

This paper is organised as follows. In Section 2, we shall recall some basic notions and results from ergodic theory. In Section 4, we shall use the Kurosh-Malcev p-adic localisation technique [13, 14] to show that the quasi-isomorphism relation on $R^{(p)}(\mathbb{Q}^n)$ is essentially identical to the orbit equivalence relation induced by the action of $GL_n(\mathbb{Q})$ on the standard Borel space $\mathcal{S}_n(\mathbb{Q}_p)$ of vector subspaces of the n-dimensional vector space $\mathbb{Q}_p^n$ over the p-adic field. (Recall that two groups A, $B \in R^{(p)}(\mathbb{Q}^n)$ are said to be *quasi-isomorphic* iff there exists $\varphi \in GL_n(\mathbb{Q})$ such that $\varphi(A) \cap B$ has finite index in both $\varphi(A)$ and B.) In Section 3, we shall prove two basic results concerning the dynamical properties of the action of $PSL_n(\mathbb{Z})$ on $\mathcal{S}_n(\mathbb{Q}_p)$. In Section 5, we shall use the results of Sections 3 and 4, together with Zimmer's superrigidity theorem and Ratner's measure classification theorem, to complete the proof of Theorem 1.2. The argument in Section 5 makes essential use of ideas which were itroduced by Adams [1]. (These ideas can be regarded as constituting a Borel version of a weak form of Furman's superrigidity theorem [8]. For example, see the discussion in Thomas [21].)

We shall assume that the reader is familiar with the basic machinery of Zimmer's superrigidity theory [22], including the notions of Borel cocycles and induced actions. Clear accounts of this material can be found in Zimmer [22] and Adams-Kechris [2]. In particular, Adams-Kechris [2, Section 2] provides a convenient introduction to the basic techniques and results in this area, written for the non-expert in the ergodic theory of groups.

Throughout this paper, $\mathbb{Z}_p$ denotes the ring of p-adic integers and $\mathbb{F}_p$ denotes the field with p elements.

2. Preliminaries

In this section, we shall recall some basic notions and results from ergodic theory. Let G be a lcsc group and let X be a standard Borel G-space. Throughout this paper, a probability measure on X will always mean a Borel probability measure; i.e. a measure which is defined on the collection of Borel subsets of X. The probability measure μ on X is *G-invariant* iff $\mu(g(A)) = \mu(A)$ for every $g \in G$ and Borel subset $A \subseteq X$. If μ is G-invariant, then the action of G on (X, μ) is said to be *ergodic* iff for every G-invariant Borel subset $A \subseteq X$, either $\mu(A) = 0$ or $\mu(A) = 1$. In this case, we shall also say that μ is an ergodic probability measure. The following characterization of ergodicity is well-known.

Proposition 2.1. *If μ is a G-invariant probability measure on the standard Borel G-space X, then the following statements are equivalent.*

(i) *The action of G on (X, μ) is ergodic.*

(ii) *If Y is a standard Borel space and $f : X \to Y$ is a G-invariant Borel function, then there exists a G-invariant Borel subset $M \subseteq X$ with $\mu(M) = 1$ such that $f \restriction M$ is a constant function.*

The action of G on X is said to be *uniquely ergodic* iff there exists a unique G-invariant probability measure μ on X. In this case, it is well-known that μ must be ergodic. (For example, see [4, Section I.3].)

Suppose that Y is a standard Borel space and that $f : X \to Y$ is a Borel function. Then for every probability measure ν on X, we can define a corresponding probability measure $f_*\nu$ on Y by

$$(f_*\nu)(A) = \nu(f^{-1}(A))$$

for every Borel subset $A \subseteq Y$. Now suppose that ν is G-invariant and that Y is also a standard Borel G-space. Then the Borel function $f : X \to Y$ is said to be *almost G-equivariant* iff for all $g \in G$, $g \cdot f(x) = f(g \cdot x)$ for ν-a.e. $x \in X$. In this case, it is easily checked that $f_*\nu$ is also G-invariant; and we say that $(Y, f_*\nu)$ is a *factor* of (X, ν). It is also easily checked that if ν is ergodic, then $f_*\nu$ is ergodic. If f is a finite-to-one function, then we say that (X, ν) is a *finite extension* of $(Y, f_*\nu)$.

Suppose that Γ is a countable group and that X is a standard Borel Γ-space with an invariant ergodic probability measure μ. Let $\Lambda \leqslant \Gamma$ be a subgroup such that $[\Gamma : \Lambda] < \infty$. Then a Λ-invariant Borel subset $Z \subseteq X$ is said to be an *ergodic component* for the action of Λ on X iff

- $\mu(Z) > 0$; and
- Λ acts ergodically on (Z, μ_Z), where μ_Z is the probability measure defined on Z by $\mu_Z(A) = \mu(A)/\mu(Z)$.

It is easily checked that there exists a partition $Z_1 \sqcup \cdots \sqcup Z_d$ of X into finitely many ergodic components and that the collection of ergodic components is uniquely determined up to μ-null sets. Furthermore, if the action of Γ on X is uniquely ergodic, then the action of Λ on each ergodic component $Z \subseteq X$ is also uniquely ergodic.

Finally suppose that K is a compact second countable group and that L is a closed subgroup. Then there exists a unique K-invariant probability measure μ on the standard Borel K-space K/L. (For example, see [16, Theorem 3.17].) The measure μ is called the Haar probability measure on K/L and can be described explicitly as follows. Suppose that ν is the Haar probability measure on K and let $\pi : K \to K/L$ be the canonical surjection. Then $\mu = \pi_*\nu$. In the remaining sections, we shall make repeated use of the following easy observation. (A proof can be found in Thomas [21, 2.2].)

Lemma 2.2. *Let K be a compact second countable group, let $L \leqslant K$ be a closed subgroup and let μ be the Haar probability measure on K/L. If Γ is a countable dense subgroup of K, then the following properties hold.*

(a) *The action of Γ on K/L is uniquely ergodic; i.e. μ is the unique Γ-invariant probability measure on K/L.*

(b) *Suppose that $\Lambda \leqslant \Gamma$ is a subgroup such that $[\Gamma : \Lambda] < \infty$ and that $H = \overline{\Lambda}$ is the closure of Λ in K. Then the ergodic components for the action of Λ on K/L are precisely the H-orbits $\{H \cdot x \mid x \in K/L\}$.*

(c) *Suppose that $\Lambda \trianglelefteq \Gamma$ is a normal subgroup such that $[\Gamma : \Lambda] < \infty$ and let $\mathcal{C}$ be the set of ergodic components for the action of Λ on K/L. Then Γ acts as a transitive permutation group on $\mathcal{C}$.*

3. Groups acting on p-adic spaces

In this section, we shall prove two basic results concerning the dynamical properties of the action of $PSL_n(\mathbb{Z})$ on the standard Borel space $\mathcal{S}_n(\mathbb{Q}_p)$ of nontrivial proper subspaces of the n-dimensional vector space $\mathbb{Q}_p^n$ over the p-adic field. For the rest of this paper, we shall fix some integer $n \geq 3$ and let $\Gamma = PSL_n(\mathbb{Z})$.

Definition 3.1. If $0 \leq k \leq n$, then $V^{(k)}(n, \mathbb{Q}_p)$ denotes the standard Borel space consisting of the k-dimensional vector subspaces of $\mathbb{Q}_p^n$.

It is easily checked that the compact group $PSL_n(\mathbb{Z}_p)$ acts transitively on $V^{(k)}(n, \mathbb{Q}_p)$. (For example, see Thomas [21, 6.1].) Thus we can identify $V^{(k)}(n, \mathbb{Q}_p)$ with the coset space $PSL_n(\mathbb{Z}_p)/L$, where L is a suitably chosen closed subgroup of $PSL_n(\mathbb{Z}_p)$. Let μ_p be the corresponding Haar probability measure on $V^{(k)}(n, \mathbb{Q}_p)$. Since Γ is a dense subgroup of $PSL_n(\mathbb{Z}_p)$, Lemma 2.2 applies to the action of Γ on $V^{(k)}(n, \mathbb{Q}_p)$.

The following entropy argument is a slight variant of the proof of Corollary B(2) of Furman [8].

Proposition 3.2. *Suppose that $(\widetilde{X}, \widetilde{\mu}_p)$ is a finite ergodic extension of the Γ-space $(V^{(n-1)}(n, \mathbb{Q}_p), \mu_p)$. Then $(\widetilde{X}, \widetilde{\mu}_p)$ does not have any factors of the form*

$$(PSL_n(\mathbb{R})/\Delta, m)\,,$$

where Δ is a lattice in $PSL_n(\mathbb{R})$ and m is the Haar probability measure.

Proof. Suppose that $(PSL_n(\mathbb{R})/\Delta, m)$ is a factor of $(\widetilde{X}, \widetilde{\mu}_p)$. Then for all $\gamma \in \Gamma$, the corresponding entropies satisfy $h(PSL_n(\mathbb{R})/\Delta, \gamma) \leq h(\widetilde{X}, \gamma)$. However, $h(\widetilde{X}, \gamma) = 0$ for all $\gamma \in \Gamma$; while if $\gamma \in \Gamma$ corresponds to an element of $SL_n(\mathbb{Z})$ which has at least one eigenvalue off the unit circle, then $h(PSL_n(\mathbb{R})/\Delta, \gamma) > 0$. □

Next suppose that X_1, X_2 are standard Borel Γ-spaces with invariant ergodic probability measures μ_1, μ_2 respectively. Then (X_2, μ_2) is said to be a *virtual quotient* of (X_1, μ_1) iff there exist:

(i) a subgroup $\Gamma_0 \leqslant \Gamma$ with $[\Gamma : \Gamma_0] < \infty$,
(ii) an embedding $\varphi : \Gamma_0 \to \Gamma$,
(iii) ergodic components Z_1, Z_2 for the actions of Γ_0, $\varphi(\Gamma_0)$ on X_1, X_2 respectively, and
(iv) a Borel function $f : Z_1 \to Z_2$

such that the following conditions are satisfied:

(a) $f_*(\mu_1)_{Z_1} = (\mu_2)_{Z_2}$; and
(b) $f(\gamma \cdot x) = \varphi(\gamma) \cdot f(x)$ for all $\gamma \in \Gamma_0$ and $x \in Z_1$.

By the Margulis Superrigidity Theorem [15], the embedding $\varphi : \Gamma_0 \to \Gamma$ extends to a Lie group automorphism of $PSL_n(\mathbb{R})$ and hence we necessarily also have that $[\Gamma : \varphi(\Gamma_0)] < \infty$. (We have used the term "virtual quotient" rather than "virtual factor" because of the slight twisting permitted in clause (b).)

Theorem 3.3. *Suppose that $(\widetilde{X}, \widetilde{\mu}_p)$ is a finite ergodic extension of the Γ-space $(V^{(n-1)}(n, \mathbb{Q}_p), \mu_p)$. If $q \neq p$ and $1 \leq k \leq n-1$, then $(V^{(k)}(n, \mathbb{Q}_q), \mu_q)$ is not a virtual quotient of $(\widetilde{X}, \widetilde{\mu}_p)$.*

The following result was proved in Thomas [21, Section 6] for the special case when $k = 1$. The proof for arbitrary k is essentially identical. For the sake of completeness, we shall sketch the main points of the proof. In the following argument, $K_t = \ker \psi_t$ denotes the congruence subgroup of $PSL_n(\mathbb{Z}_q)$ arising from the canonical surjection

$$\psi_t : PSL_n(\mathbb{Z}_q) \to PSL_n(\mathbb{Z}_q/q^t\mathbb{Z}_q).$$

Lemma 3.4. *Suppose that $1 \leq k \leq n-1$ and that $\Lambda \leqslant \Gamma$ is a subgroup such that $[\Gamma : \Lambda] < \infty$. Let Z be an ergodic component for the action of Λ on $V^{(k)}(n, \mathbb{Q}_q)$ and* *for each subgroup* $\Delta \leqslant \Lambda$ *with* $[\Lambda : \Delta] < \infty$*, let* $e(\Delta, Z)$ *be the number of ergodic components for the action of* Δ *on* Z*.*

(a) *If* $\Delta \trianglelefteq \Lambda$ *is a normal subgroup with* $[\Lambda : \Delta] < \infty$*, then* $e(\Delta, Z) = bq^r$ *for some* $r \geq 0$ *and some divisor* b *of* $|PSL_n(\mathbb{F}_q)|$*.*

(b) *For each* $N \geq 0$*, there exists a normal subgroup* $\Delta \trianglelefteq \Lambda$ *with* $[\Lambda : \Delta] < \infty$ *such that* $e(\Delta, Z) \geq N$*.*

Proof. To see that 3.4(a) holds, let $H = \overline{\Lambda}$ and $N = \overline{\Delta}$ be the closures of Λ, Δ in $PSL_n(\mathbb{Z}_q)$ respectively. Then we can suppose that Z is an orbit of H on $V^{(k)}(n, \mathbb{Q}_q)$ and that the set $\mathcal{C}$ of ergodic components for the action of Δ on Z consists of the orbits of N on Z. Clearly H acts transitively on $\mathcal{C}$. Hence 3.4(a) follows from the fact that K_1 is a pro-p group. Finally 3.4(b) is an easy consequence of the fact that if a_t is the number of orbits of K_t on $V^{(k)}(n, \mathbb{Q}_q)$, then $a_t \to \infty$ as $t \to \infty$. □

Lemma 3.5. *Suppose that* $(\widetilde{X}, \widetilde{\mu}_p)$ *be a finite ergodic extension of the* Γ*-space* $(V^{(n-1)}(n, \mathbb{Q}_p), \mu_p)$ *and that* $\Lambda \leqslant \Gamma$ *is a subgroup such that* $[\Gamma : \Lambda] < \infty$*. Let* $\widetilde{Z}$ *be an ergodic component for the action of* Λ *on* $\widetilde{X}$ *and for each subgroup* $\Delta \leqslant \Lambda$ *with* $[\Lambda : \Delta] < \infty$*, let* $e(\Delta, \widetilde{Z})$ *be the number of ergodic components for the action of* Δ *on* $\widetilde{Z}$*. Then there exists a constant* c *such that whenever* $\Delta \trianglelefteq \Lambda$ *is a normal subgroup with* $[\Lambda : \Delta] < \infty$*, then* $e(\Delta, \widetilde{Z}) = bp^r$ *for some* $r \geq 0$ *and* $b \leq c$*.*

Proof. Let $\pi : \widetilde{X} \to V^{(n-1)}(n, \mathbb{Q}_p)$ be the factor map and let $Z = \pi(\widetilde{Z})$. Then Z is an ergodic component for the action of Λ on $V^{(n-1)}(n, \mathbb{Q}_p)$. Furthermore, by ergodicity, we can suppose that there exists a constant ℓ such that $|\pi^{-1}(z)| = \ell$ for all $z \in Z$. Let $\Delta \trianglelefteq \Lambda$ be a normal subgroup with $[\Lambda : \Delta] < \infty$ and let $\{A_i \mid 1 \leq i \leq e\}$ be the ergodic components for the action of Δ on Z. By Lemma 3.4, $e = bp^r$ for some $r \geq 0$ and some divisor b of $|PSL_n(\mathbb{F}_p)|$. For each $1 \leq i \leq e$, let $\widetilde{A}_i = \pi^{-1}(A_i)$. Then each $\widetilde{A}_i$ is the union of at most ℓ ergodic components for the action of Δ on $\widetilde{Z}$. Since $\Delta \trianglelefteq \Lambda$, it follows that Λ acts transitively on $\{A_i \mid 1 \leq i \leq e\}$ and hence each $\widetilde{A}_i$ contains the same number of ergodic components. □

Proof of Theorem 3.3. Suppose that $(V^{(k)}(n, \mathbb{Q}_q), \mu_q)$ is a virtual quotient of $(\widetilde{X}, \widetilde{\mu}_p)$. Thus there exist:

(i) a subgroup $\Gamma_0 \leqslant \Gamma$ with $[\Gamma : \Gamma_0] < \infty$,

(ii) an embedding $\varphi : \Gamma_0 \to \Gamma$,

(iii) ergodic components $\widetilde{Z}$, Z for the actions of Γ_0, $\varphi(\Gamma_0)$ on $\widetilde{X}$, $V^{(k)}(n, \mathbb{Q}_q)$ respectively, and

(iv) a Borel function $f : \widetilde{Z} \to Z$

such that the following conditions are satisfied:

(a) $f_*(\widetilde{\mu}_p)_{\widetilde{Z}} = (\mu_q)_Z$; and

(b) $f(\gamma \cdot x) = \varphi(\gamma) \cdot f(x)$ for all $\gamma \in \Gamma_0$ and $x \in \widetilde{Z}$.

Let $t \geq 0$ be arbitrary. By Lemma 3.4, there exists a normal subgroup $\Delta \trianglelefteq \varphi(\Gamma_0)$ with $[\varphi(\Gamma_0) : \Delta] < \infty$ such that q^t divides $e = e(\Delta, Z)$. Let $\{A_i \mid 1 \leq i \leq e\}$ be the ergodic components for the action of Δ on Z; and for each $1 \leq i \leq e$, let $\widetilde{A}_i = f^{-1}(A_i)$. Then each $\widetilde{A}_i$ is the union of a finite number of ergodic components for the action of $\varphi^{-1}(\Delta)$ on $\widetilde{Z}$. Arguing as in the proof of Lemma 3.5, we see that each $\widetilde{A}_i$ contains the same number of ergodic components and so e divides $e(\varphi^{-1}(\Delta), \widetilde{Z})$. But if we choose t sufficiently large, this clearly contradicts Lemma 3.5. □

4. The Kurosh-Malcev p-adic localisation technique

In this section, we shall first use the Kurosh-Malcev p-adic localisation technique [13, 14] to relate the classification problem for p-local torsion-free abelian groups to the orbit equivalence relation induced by the action of $GL_n(\mathbb{Q})$ on the standard Borel space $\mathcal{S}_n(\mathbb{Q}_p)$ of vector subspaces of the n-dimensional vector space $\mathbb{Q}_p^n$ over the p-adic field. Then we shall complete the proof of Theorem 1.2, modulo a superrigidity result which we shall prove in Section 5.

Suppose that A, $B \in R(\mathbb{Q}^n)$. Then A and B are said to be *quasi-equal*, written $A \approx B$, iff $A \cap B$ has finite index in both A and B. We say that A and B are *quasi-isomorphic*, written $A \sim B$, if there exists $\varphi \in GL_n(\mathbb{Q})$ such that $\varphi(A) \approx B$. By Thomas [20, Section 3], $\approx$ and $\sim$ are both countable Borel equivalence relations on $R(\mathbb{Q}^n)$.

Definition 4.1. For each $A \in R^{(p)}(\mathbb{Q}^n)$, let $\widehat{A} = \mathbb{Z}_p \otimes A$.

We shall regard each $\widehat{A}$ as a subgroup of $\mathbb{Q}_p^n$ in the usual way; i.e. $\widehat{A}$ is the subgroup consisting of all finite sums

$$\gamma_1 a_1 + \gamma_2 a_2 + \cdots + \gamma_t a_t,$$

where $\gamma_i \in \mathbb{Z}_p$ and $a_i \in A$ for $1 \leq i \leq t$. By Lemma 93.3 [6], there exist integers $0 \leq k, \ell \leq n$ with $k + \ell = n$ and elements v_i, $w_j \in \widehat{A}$ such that

$$\widehat{A} = \bigoplus_{i=1}^{k} \mathbb{Q}_p v_i \oplus \bigoplus_{j=1}^{\ell} \mathbb{Z}_p w_j.$$

Definition 4.2. For each $A \in R^{(p)}(\mathbb{Q}^n)$, let $V_A = \bigoplus_{i=1}^{k} \mathbb{Q}_p v_i$.

Theorem 4.3. *If A, $B \in R^{(p)}(\mathbb{Q}^n)$, then*

(a) *$A \approx B$ iff $V_A = V_B$;*

(b) *$A \sim B$ iff there exists $\pi \in GL_n(\mathbb{Q})$ such that $\pi(V_A) = V_B$.*

Proof. 4.3(a) was proved in Thomas [20, 4.7]. To see that 4.3(b) holds, consider the canonical extension of the action of $GL_n(\mathbb{Q})$ on $\mathbb{Q}^n$ to an action on $\mathbb{Q}_p^n$. Suppose that $\pi \in GL_n(\mathbb{Q})$ and let $\pi(A) = C$. Then it is clear that $\pi(\widehat{A}) = \widehat{C}$ and this implies that $\pi(V_A) = V_C$. Thus 4.3(b) follows from 4.3(a). □

Theorem 4.4. *Suppose that $A \in R^{(p)}(\mathbb{Q}^n)$ and that $\dim V_A = n - 1$. Then for each $B \in R^{(p)}(\mathbb{Q}^n)$, we have that $A \sim B$ iff $A \cong B$.*

Proof. By Exercises 32.5 and 93.1 [6], for every group $C \in R^{(p)}(\mathbb{Q}^n)$, we have that

$$\dim_{\mathbb{Q}_p} V_C = n - \dim_{\mathbb{F}_p} C/pC.$$

In particular, $\dim_{\mathbb{F}_p} A/pA = 1$. It follows that $|A/qA| \leq q$ for every prime q; and so the result follows from Proposition 92.1 [6]. □

Definition 4.5. Let $\mathbf{e}_1, \dots, \mathbf{e}_n$ be the standard basis of $\mathbb{Q}_p^n$. Suppose that S is a $\mathbb{Q}_p$-subspace of $\mathbb{Q}_p^n$ of dimension $0 \leq k \leq n$. Then

$$\sigma(S) = \left(S \oplus \mathbb{Z}_p \mathbf{e}_{i_1} \oplus \cdots \oplus \mathbb{Z}_p \mathbf{e}_{i_{n-k}}\right) \cap \mathbb{Q}^n,$$

where $i_1 < \cdots < i_{n-k}$ is the lexicographically least sequence such that

$$\mathbb{Q}_p^n = \langle S, \mathbf{e}_{i_1}, \dots, \mathbf{e}_{i_{n-k}} \rangle.$$

Theorem 4.6. *If S is a $\mathbb{Q}_p$-subspace of $\mathbb{Q}_p^n$ of dimension $0 \leq k \leq n$, then*

(a) *$\sigma(S) \in R^{(p)}(\mathbb{Q}^n)$;*

(b) *$V_{\sigma(S)} = S$.*

Proof. Arguing as in the proof of Fuchs [6, 93.5], we easily obtain that

$$\mathbb{Z}_p \otimes \sigma(S) = S \oplus \mathbb{Z}_p \mathbf{e}_{i_1} \oplus \cdots \oplus \mathbb{Z}_p \mathbf{e}_{i_{n-k}}.$$

The result follows. □

We are now ready to begin the proof of Theorem 1.2. Suppose $p \neq q$ are distinct primes and that $h : R^{(p)}(\mathbb{Q}^n) \to R^{(q)}(\mathbb{Q}^n)$ is a Borel map such that $A \cong B$ iff $h(A) \cong h(B)$. Let

$$f : V^{(n-1)}(n, \mathbb{Q}_p) \to \mathcal{S}_n(\mathbb{Q}_p) = \bigcup_{k=0}^{n-1} V^{(n)}(n, \mathbb{Q}_q)$$

be the Borel map defined by $f(S) = V_{(h \circ \sigma)(S)}$. Applying Theorems 4.3, 4.4 and 4.6, we see that if S, $T \in V^{(n-1)}(n, \mathbb{Q}_p)$, then the following three statements are equivalent:

(1) S and T lie in the same $GL_n(\mathbb{Q})$-orbit;
(2) $\sigma(S) \cong \sigma(T)$;
(3) $(h \circ \sigma)(S) \cong (h \circ \sigma)(T)$.

Applying Theorem 4.3 once again, we also see that the following two statements are equivalent:

(i) $f(S)$ and $f(T)$ lie in the same $GL_n(\mathbb{Q})$-orbit;
(ii) $(h \circ \sigma)(S) \sim (h \circ \sigma)(T)$.

It follows that if S, $T \in V^{(n-1)}(n, \mathbb{Q}_p)$ lie in the same $GL_n(\mathbb{Q})$-orbit, then $f(S)$ and $f(T)$ lie in the same $GL_n(\mathbb{Q})$-orbit; and also that if Δ is a $GL_n(\mathbb{Q})$-orbit on $V^{(k)}(n, \mathbb{Q}_q)$ for some $1 \leq k \leq n-1$, then $f^{-1}(\Delta)$ is a countable subset of $V^{(n-1)}(n, \mathbb{Q}_p)$.

Now consider the measure-preserving action of $SL_n(\mathbb{Z})$ on $\big(V^{(n-1)}(n, \mathbb{Q}_p), \mu_p\big)$. Since $SL_n(\mathbb{Z})$ acts ergodically on $V^{(n-1)}(n, \mathbb{Q}_p)$, it follows that there exists a fixed $1 \leq k \leq n-1$ and an $SL_n(\mathbb{Z})$-invariant Borel subset $X \subseteq V^{(n-1)}(n, \mathbb{Q}_p)$ with $\mu_p(X) = 1$ such that $f(S) \in V^{(k)}(n, \mathbb{Q}_q)$ for all $S \in X$. However, this clearly contradicts the following theorem, which will be proved in Section 5. This completes the proof of Theorem 1.2.

Theorem 4.7. *Suppose that $n \geq 3$ and that $1 \leq k \leq n-1$. Let $p \neq q$ be distinct primes and suppose that $f : V^{(n-1)}(n, \mathbb{Q}_p) \to V^{(k)}(n, \mathbb{Q}_q)$ is a Borel map such that for all x, $y \in V^{(n-1)}(n, \mathbb{Q}_p)$,*

- *if $SL_n(\mathbb{Z}) \cdot x = SL_n(\mathbb{Z}) \cdot y$, then $GL_n(\mathbb{Q}) \cdot f(x) = GL_n(\mathbb{Q}) \cdot f(y)$.*

Then there exists a Borel subset $Z \subseteq V^{(n-1)}(n, \mathbb{Q}_p)$ with $\mu_p(Z) = 1$ such that f maps Z into a single $GL_n(\mathbb{Q})$-orbit.

5. The proof of Theorem 4.7

In this section, we shall prove Theorem 4.7. (Our argument will make essential use of the techniques introduced by Adams [1]. As we mentioned in Section 1, these techniques can be regarded as constituting a Borel version of a weak form of Furman's superrigidity theorem [8].) Suppose that $n \geq 3$ and that $p \neq q$ are distinct primes. Let $1 \leq k \leq n$ and let $f : V^{(n-1)}(n, \mathbb{Q}_p) \to V^{(k)}(n, \mathbb{Q}_q)$ be a Borel map such that for all x, $y \in V^{(n-1)}(n, \mathbb{Q}_p)$,

- if $SL_n(\mathbb{Z}) \cdot x = SL_n(\mathbb{Z}) \cdot y$, then $GL_n(\mathbb{Q}) \cdot f(x) = GL_n(\mathbb{Q}) \cdot f(y)$.

Suppose that there does not exist a Borel subset $Z \subseteq V^{(n-1)}(n, \mathbb{Q}_p)$ with $\mu_p(Z) = 1$ such that f maps Z into a single $GL_n(\mathbb{Q})$-orbit. For the remainder of this section, we shall work with the corresponding actions of $\Gamma = PSL_n(\mathbb{Z})$ and $PGL_n(\mathbb{Q})$ on $V^{(n-1)}(n, \mathbb{Q}_p)$, $V^{(k)}(n, \mathbb{Q}_q)$ respectively. Let

$$Y = \{y \in V^{(k)}(n, \mathbb{Q}_q) \mid g \cdot y \neq y \text{ for all } 1 \neq g \in PGL_n(\mathbb{Q})\}.$$

In other words, Y is the Borel subset of $V^{(k)}(n, \mathbb{Q}_q)$ where $PGL_n(\mathbb{Q})$ acts freely.

Lemma 5.1. *There exists a Γ-invariant Borel subset $X \subseteq V^{(n-1)}(n, \mathbb{Q}_p)$ with $\mu_p(X) = 1$ such that $f(x) \in Y$ for all $x \in X$.*

In the proof of Lemma 5.1, we shall make use of the following cocycle reduction result. (Throughout this section, $\overline{\mathbb{Q}}$ denotes the algebraic closure of $\mathbb{Q}$. If we strengthen the hypotheses by assuming that G is an algebraic $\mathbb{Q}$-group and that $H \leqslant G(\mathbb{Q})$, then Theorem 5.2 is an easy consequence of [20, Theorem 2.3]. However, an examination of its proof shows that [20, Theorem 2.3] also holds when $\mathbb{Q}$ is replaced by $\overline{\mathbb{Q}}$.)

Theorem 5.2. *Let $n \geq 3$ and let Ω be a standard Borel $PSL_n(\mathbb{Z})$-space with an invariant ergodic probability measure μ. Suppose that $H \leqslant G(\overline{\mathbb{Q}})$, where G is an algebraic $\overline{\mathbb{Q}}$-group such that* $\dim G < n^2 - 1$, *and that Z is a standard Borel H-space on which H acts freely. If $f : \Omega \to Z$ is a Borel function such that for all x, $y \in \Omega$,*

$$PSL_n(\mathbb{Z}) \cdot x = PSL_n(\mathbb{Z}) \cdot y \quad \text{implies} \quad H \cdot f(x) = H \cdot f(y),$$

then there exists an $PSL_n(\mathbb{Z})$-invariant Borel subset $M \subseteq \Omega$ with $\mu(M) = 1$ such that f maps M into a single H-orbit.

Suppose that Lemma 5.1 is false. There exists a Γ-invariant Borel subset X of $V^{(n-1)}(n, \mathbb{Q}_p)$ with $\mu_p(X) = 1$ such that $f(x) \notin Y$ for all $x \in X$. We shall consider the induced action of $GL_n(\mathbb{Q})$ on the exterior power $V = \bigwedge^k(\mathbb{Q}_q^n)$. Let $\mathbf{e}_1, \dots, \mathbf{e}_n$ be the standard basis of $\mathbb{Q}_q^n$. Let $d = \binom{n}{k}$ and let $\mathcal{B} = \{\mathbf{b}_j \mid 1 \leq j \leq d\}$ be the corresponding "standard basis" of V; i.e. $\mathcal{B}$ consists of the vectors $\mathbf{e}_{i_1} \wedge \dots \wedge \mathbf{e}_{i_k}$, where $i_1 < \dots < i_k$. Then $\overline{\mathbb{Q}}^d \cap V$ denotes the collection of vectors $\mathbf{v} \in V$ of the form

$$\mathbf{v} = a_1\mathbf{b}_1 + \dots + a_d\mathbf{b}_d,$$

where each $a_j \in \overline{\mathbb{Q}} \cap \mathbb{Q}_q$. The subspace $E \leqslant V$ is said to be a $\overline{\mathbb{Q}}$-subspace iff there exists a (possibly empty) collection of vectors $\mathbf{w}_1, \dots, \mathbf{w}_t \in \overline{\mathbb{Q}}^d \cap V$ such that $E = \langle \mathbf{w}_1, \dots, \mathbf{w}_t \rangle$. Clearly if E, $F \leqslant V$ are $\overline{\mathbb{Q}}$-subspaces, then $E \cap F$ is also a $\overline{\mathbb{Q}}$-subspace. In particular, for each 1-dimensional subspace $\langle \mathbf{v} \rangle$ of V, there exists a unique minimal $\overline{\mathbb{Q}}$-subspace E such that $\langle \mathbf{v} \rangle \leqslant E$.

For each k-dimensional subspace $S = \langle \mathbf{s}_1, \dots, \mathbf{s}_k \rangle$ of $\mathbb{Q}_q^n$, let

$$[S] = \langle \mathbf{s}_1 \wedge \dots \wedge \mathbf{s}_k \rangle$$

be the corresponding 1-dimensional subspace of V.

Claim 5.3. *For each $x \in X$, there exists a proper $\overline{\mathbb{Q}}$-subspace E of V such that $[f(x)] \leqslant E$.*

Proof. For each $x \in X$, there exists a noncentral element $g \in GL_n(\mathbb{Q})$ and an eigenspace E for the induced action of g on V such that $[f(x)] \leqslant E$. Clearly E is a proper $\overline{\mathbb{Q}}$-subspace of V. □

Recall that the *Effros Borel space* on V is the standard Borel space consisting of the set

$$F(V) = \{Z \subseteq V \mid Z \text{ is a closed subset of } V\}$$

equipped with the σ-algebra generated by the sets of the form

$$\{Z \in F(V) \mid Z \cap U \neq \emptyset\},$$

where U varies over the open subsets of V. For each $x \in X$, let $E_x \in F(V)$ be the unique minimal $\overline{\mathbb{Q}}$-subspace such that $[f(x)] \leqslant E_x$; and let $m : X \to F(V)$ be the Borel map defined by $m(x) = E_x$. Since there are only countably many possibilities for E_x, there exists a Borel subset $X_0 \subseteq X$ with $\mu_p(X_0) > 0$ and a fixed $\overline{\mathbb{Q}}$-subspace E such that $E_x = E$ for all $x \in X_0$. Let $X_1 = \Gamma \cdot X_0$. Since μ_p is ergodic, it follows that $\mu_p(X_1) = 1$. After slightly adjusting f if necessary, we can suppose that $E_x = E$ for all $x \in X_1$. (More precisely, let $c : X_1 \to X_1$ be a Borel function such that $c(x) \in \Gamma \cdot x \cap X_0$ for all $x \in X_1$. Then we can replace f with $f' = f \circ c$.)

Now suppose that x, $y \in X_1$ and that $y \in \Gamma \cdot x$. Then there exists $g \in GL_n(\mathbb{Q})$ such that $g \cdot [f(x)] = [f(y)]$. We claim that

$$g \in R = \{\varphi \in GL_n(\mathbb{Q} \mid \varphi(E) = E\}.$$

To see this, note that $g(E)$ is also a $\overline{\mathbb{Q}}$-subspace and that $[f(y)] \leqslant E \cap g(E)$. Hence, by the minimality of E, we must have that $g(E) = E$.

Next let $x \in X_1$ and suppose that $\varphi \in R$ satisfies $\varphi \cdot [f(x)] = [f(x)]$. Then $[f(x)]$ is contained in the eigenspace W of φ corresponding to some eigenvalue $\lambda \in \overline{\mathbb{Q}}$. By the minimality of E, we must have that $E \leqslant W$ and so $\varphi(\mathbf{v}) = \lambda \mathbf{v}$ for all $\mathbf{v} \in E$. Let $H \leqslant PGL(E)$ be the group of projective linear transformations induced by R on the set of 1-dimensional subspaces of E. Then we have just shown that

- H acts freely on the standard Borel space $Z = \{h \cdot [f(x)] \mid h \in H \text{ and } x \in X_1\}$;
- if x, $y \in X_1$ and $y \in \Gamma \cdot x$, then there exists an element $h \in H$ such that $h \cdot [f(x)] = [f(y)]$.

Claim 5.4. *There exists an algebraic $\overline{\mathbb{Q}}$-group G with $\dim G < n^2 - 1$ such that $H \leqslant G(\overline{\mathbb{Q}})$.*

Proof. Clearly it is enough to show that the $\overline{\mathbb{Q}}$-subspace E is not $GL_n(\mathbb{Q})$-invariant. To see this, note that $SL_n(\mathbb{Z}_p)$ acts transitively on the subset $\{[S] \mid S \in V^{(k)}(n, \mathbb{Q}_q)\}$. In particular, for each $\mathbf{b}_j \in \mathcal{B}$ and $x \in X_1$, there exists $g \in SL_n(\mathbb{Z}_p)$ such that $g \cdot [f(x)] = \mathbf{b}_j$. Since E is a proper subspace of V, it follows that E is not $SL_n(\mathbb{Z}_p)$-invariant. Because $SL_n(\mathbb{Z})$ is a dense subgroup of $SL_n(\mathbb{Z}_p)$, it follows that E is not $SL_n(\mathbb{Z})$-invariant. □

By Theorem 5.2, there exists a Γ-invariant Borel subset $M \subseteq X_1$ with $\mu_p(M) = 1$ such that $\{[f(x)] \mid x \in M\}$ is contained in a single H-orbit on Z. But this means that f maps M into a single $GL_n(\mathbb{Q})$-orbit on $V^{(k)}(n, \mathbb{Q}_q)$, which is a contradiction. This completes the proof of Lemma 5.1.

Let $\alpha : \Gamma \times X \to PGL_n(\mathbb{Q})$ be the Borel cocycle defined by

$$\alpha(\gamma, x) \cdot f(x) = f(\gamma \cdot x)$$

for all $\gamma \in \Gamma$ and $x \in X$. For each matrix $M \in GL_n(\mathbb{Q})$, let $[M]$ be the corresponding element of $PGL_n(\mathbb{Q})$. Let $A = \mathbb{Q}^*/(\mathbb{Q}^*)^n$ and let $\delta : PGL_n(\mathbb{Q}) \to A$ be the homomorphism defined by

$$\delta(g) = \det(M_g)(\mathbb{Q}^*)^n,$$

where $M_g \in GL_n(\mathbb{Q})$ satisfies $[M_g] = g$. Then $\ker \delta = PSL_n(\mathbb{Q})$. Consider the cocycle $\delta \circ \alpha : \Gamma \times X \to A$. By Zimmer [22, 9.1.1], since Γ is a Kazhdan group and A is an amenable group, $\delta \circ \alpha$ is equivalent to a cocycle taking values in a finite subgroup $F \leqslant A$. It easily follows that α is equivalent to a cocycle α' taking values in $L = \delta^{-1}(F)$. (For example, see the proof of Adams-Kechris [2, 6.1].) Clearly $[L : PSL_n(\mathbb{Q})] < \infty$. Hence by Adams-Kechris [2, 2.5], there exists a finite ergodic extension $(\widetilde{X}, \widetilde{\mu}_p)$ of (X, μ_p) such that the lift $\widetilde{\alpha} : \Gamma \times \widetilde{X} \to L$ of α' is equivalent to a cocycle β taking values in $PSL_n(\mathbb{Q})$. Let $\widetilde{f} : \widetilde{X} \to Y$ be the corresponding Borel function such that for all $\gamma \in \Gamma$,

$$\beta(\gamma, x) \cdot \widetilde{f}(x) = \widetilde{f}(\gamma \cdot x) \quad \text{ for } \widetilde{\mu}_p\text{-a.e. } x \in \widetilde{X}.$$

Then it is clear that there does not exist a Borel subset $\widetilde{Z} \subseteq \widetilde{X}$ with $\widetilde{\mu}_p(\widetilde{Z}) = 1$ such that $\widetilde{f}$ maps $\widetilde{Z}$ into a single $PGL_n(\mathbb{Q})$-orbit. By Zimmer [23, 2.2], since Γ is a Kazhdan group, β is equivalent to a cocycle β' taking values in a finitely generated subgroup of $PSL_n(\mathbb{Q})$. To simplify the notation, we shall suppose that

$\beta' = \beta$. So there exists a finite set of primes $q_1, \dots, q_t$ such that β takes values in $\Lambda = PSL_n(\mathbb{Z}[1/q_1, \dots, 1/q_t])$. By Zimmer [22, 10.1.1], if we identify Λ with its image under the diagonal embedding into

$$H = PSL_n(\mathbb{R}) \times PSL_n(\mathbb{Q}_{q_1}) \times \cdots \times PSL_n(\mathbb{Q}_{q_t}),$$

then Λ is a lattice in H. Let $i : \Lambda \to H$ denote the diagonal embedding.

Now consider the induced Borel action of $PSL_n(\mathbb{R})$ on

$$\widehat{X} = \widetilde{X} \times (PSL_n(\mathbb{R})/PSL_n(\mathbb{Z}))$$

and let $\widehat{\beta} : PSL_n(\mathbb{R}) \times \widehat{X} \to \Lambda$ be the cocycle induced from β. Suppose that $i \circ \widehat{\beta} : PSL_n(\mathbb{R}) \times \widehat{X} \to H$ is equivalent to a cocycle taking values in a compact subgroup of H. Then Adams-Kechris [2, 2.4] implies that $\widehat{\beta}$ is equivalent to a cocycle taking values in a finite subgroup F of Λ; and hence, by Adams-Kechris [2, 2.3], β is also equivalent to a cocycle taking values in F. But then a standard argument shows that there exists a Borel subset $\widetilde{Z} \subseteq \widetilde{X}$ with $\widetilde{\mu}_p(\widetilde{Z}) = 1$ such that $\widetilde{f}$ maps $\widetilde{Z}$ into a single $PGL_n(\mathbb{Q})$-orbit, which is a contradiction. Thus $i \circ \widehat{\beta}$ is not equivalent to a cocycle taking values in a compact subgroup of H. For each $1 \leq \ell \leq t$, let $\pi_\ell : H \to PSL_n(\mathbb{Q}_{q_\ell})$ be the canonical projection and let $\widehat{\beta}_\ell : PSL_n(\mathbb{R}) \times \widehat{X} \to PSL_n(\mathbb{Q}_{q_\ell})$ be the cocycle defined by $\widehat{\beta}_\ell = \pi_\ell \circ i \circ \widehat{\beta}$. By Zimmer [22, 5.2.5], together with the methods of Adams-Kechris [2], $\widehat{\beta}_\ell$ is equivalent to a cocycle taking values in a compact subgroup K_ℓ of $PSL_n(\mathbb{Q}_{q_\ell})$. (More precisely, arguing as in the proof of Adams-Kechris [2, 3.5], we can easily reduce to the case when the range of the cocycle $\widehat{\beta}_\ell$ is Zariski dense in $PSL_n(\mathbb{Q}_{q_\ell})$.) It follows that if $\pi_0 : H \to PSL_n(\mathbb{R})$ is the canonical projection, then the cocycle

$$\widehat{\beta}_0 = \pi_0 \circ i \circ \widehat{\beta} : PSL_n(\mathbb{R}) \times \widehat{X} \to PSL_n(\mathbb{R})$$

is not equivalent to a cocycle taking values in a compact subgroup of $PSL_n(\mathbb{R})$. Hence, by Adams [1, 5.3], there exists a Lie group automorphism φ of $PSL_n(\mathbb{R})$ such that $\widehat{\beta}_0$ is equivalent to the cocycle $\widehat{\beta}_\varphi : PSL_n(\mathbb{R}) \times \widehat{X} \to PSL_n(\mathbb{R})$ defined by $\widehat{\beta}_\varphi(g, x) = \varphi(g)$. Arguing as in the proof of Adams [1, 5.4], $i \circ \beta$ is equivalent to a cocycle

$$\bar{\beta} : \Gamma \times \widetilde{X} \to H = PSL_n(\mathbb{R}) \times PSL_n(\mathbb{Q}_{q_1}) \times \cdots \times PSL_n(\mathbb{Q}_{q_t})$$

given by

$$\bar{\beta}(\gamma, x) = \langle \varphi(\gamma), \beta_1(\gamma, x), \ldots, \beta_t(\gamma, x) \rangle,$$

where each $\beta_\ell : \Gamma \times \widetilde{X} \to PSL_n(\mathbb{Q}_{q_\ell})$ is a cocycle taking values in the compact subgroup K_ℓ of $PSL_n(\mathbb{Q}_{q_\ell})$. Since $PSL_n(\mathbb{Z}_{q_\ell})$ is an open subgroup of $PSL_n(\mathbb{Q}_{q_\ell})$, it follows that $PSL_n(\mathbb{Z}_{q_\ell}) \cap K_\ell$ has finite index in K_ℓ. Applying Adams-Kechris [2, 2.5], after replacing $(\widetilde{X}, \widetilde{\mu}_p)$ by a finite ergodic extension if necessary, we can suppose that β_ℓ takes values in the compact subgroup $PSL_n(\mathbb{Z}_{q_\ell})$ for each $1 \leq \ell \leq t$. Let $K = PSL_n(\mathbb{Z}_{q_1}) \times \cdots \times PSL_n(\mathbb{Z}_{q_t})$ and $G = PSL_n(\mathbb{Q}_{q_1}) \times \cdots \times PSL_n(\mathbb{Q}_{q_t})$. From now on, we shall identify G and $PSL_n(\mathbb{R})$ with the corresponding subgroups of H. In particular, this allows us to consider the commuting actions of K and $PSL_n(\mathbb{R})$ on H/Λ.

Let $S \subseteq H$ be a Borel transversal for H/Λ chosen so that $G \subseteq S$ and identify S with H/Λ by identifying each $s \in S$ with $s\Lambda$. Then the action of H on H/Λ induces a corresponding Borel action of H on S, defined by

$$h \cdot s = \text{ the unique element in } S \cap hs\Lambda.$$

The associated cocycle $\rho : H \times S \to \Lambda$ is defined by

$$\begin{aligned} \rho(h, s) &= \text{ the unique } \lambda \in \Lambda \text{ such that } (h \cdot s)\lambda = hs \\ &= (h \cdot s)^{-1} hs \end{aligned}$$

Recall that the induced action of H on

$$\widehat{Y} = Y \times S = Y \times (H/\Lambda)$$

is defined by

$$h \cdot (y, s) = (\rho(h, s) \cdot y, h \cdot s).$$

Let $j : Y \to \widehat{Y}$ be the Λ-equivariant map defined by $j(y) = (y, 1)$ and let $\widehat{f} : \widetilde{X} \to \widehat{Y}$ be the map defined by $\widehat{f} = j \circ \widetilde{f}$. Then for all $\gamma \in \Gamma$,

$$(i \circ \beta)(\gamma, x) \cdot \widehat{f}(x) = \widehat{f}(\gamma \cdot x) \quad \text{ for } \widetilde{\mu}_p\text{-a.e. } x \in \widetilde{X}.$$

Let $b : \widetilde{X} \to H$ be a Borel map such that for all $\gamma \in \Gamma$,

$$\bar{\beta}(\gamma, x) = b(\gamma \cdot x)(i \circ \beta)(\gamma, x) b(x)^{-1} \quad \text{ for } \widetilde{\mu}_p\text{-a.e. } x \in \widetilde{X};$$

and define $\bar{f} : \widetilde{X} \to \widehat{Y}$ by $\bar{f}(x) = b(x) \cdot \widehat{f}(x)$. Then for all $\gamma \in \Gamma$,

$$\bar{\beta}(\gamma, x) \cdot \bar{f}(x) = \bar{f}(\gamma \cdot x) \quad \text{for } \widetilde{\mu}_p\text{-a.e. } x \in \widetilde{X}.$$

Next we shall study the distribution of $\bar{f}(\widetilde{X})$ within $\widehat{Y}$. Since K is a compact group, it follows that $K\backslash H/\Lambda$ is a standard Borel space. Let $\eta : \widehat{Y} \to K\backslash H/\Lambda$ be the map defined by $\eta(y, s\Lambda) = Ks\Lambda$ and let $\omega = (\eta \circ \bar{f})_* \widetilde{\mu}_p$. Then ω is a $\varphi(\Gamma)$-invariant ergodic probability measure on $K\backslash H/\Lambda$. Furthermore, by definition, for any Borel subset $A \subseteq K\backslash H/\Lambda$,

$$\omega(A) = \widetilde{\mu}_p(\{x \in \widetilde{X} \mid (\eta \circ \bar{f})(x) \in A\}).$$

Since K has countable index in G, it follows that $PSL_n(\mathbb{R})$ has only countably many orbits on $K\backslash H/\Lambda$. Hence, since $\varphi(\Gamma)$ acts ergodically on $K\backslash H/\Lambda$, it follows that ω is concentrated on a single $PSL_n(\mathbb{R})$-orbit Ω on $K\backslash H/\Lambda$. (The following proof is based on an unpublished argument of Dave Witte, which appeared in an early version of Adams [1]. It is very closely related to Lemma 4.6 of Furman [7].)

Lemma 5.5. *ω is supported on a finite subset $\Omega_0 \subset \Omega$.*

Proof. By Shah [18, 1.4], since ω is a $\varphi(\Gamma)$-invariant ergodic probability measure on the homogeneous $PSL_n(\mathbb{R})$-space Ω and $\varphi(\Gamma)$ is a lattice in $PSL_n(\mathbb{R})$, there exists a (topologically) closed subgroup C of $PSL_n(\mathbb{R})$ containing $\varphi(\Gamma)$ such that ω is C-invariant and concentrated on a C-orbit. Because C contains $\varphi(\Gamma)$, it follows that $PSL_n(\mathbb{R})/C$ has finite volume. Hence, by the Borel Density Theorem, one of the following two possibilities holds:

- $C = PSL_n(\mathbb{R})$; or
- C is a discrete subgroup of $PSL_n(\mathbb{R})$.

First suppose that $C = PSL_n(\mathbb{R})$. Then there exists a lattice Δ of $PSL_n(\mathbb{R})$ such that the $\varphi(\Gamma)$-space (Ω, ω) is isomorphic to $(PSL_n(\mathbb{R})/\Delta, m)$, where m is the Haar probability measure. But this means that $\big(PSL_n(\mathbb{R})/\varphi^{-1}(\Delta), m\big)$ is a factor of the Γ-space $(\widetilde{X}, \widetilde{\mu}_p)$, which contradicts Proposition 3.2. Hence C must be a discrete subgroup of $PSL_n(\mathbb{R})$. In particular, C is a countable group and so ω is concentrated on a countable subset Ω_0 of Ω. Since ω is a C-invariant probability measure, this implies that Ω_0 is actually a finite set. $\square$

Clearly $\Omega_0 \subset K\backslash H/\Lambda$ must be $\varphi(\Gamma)$-invariant and so there exists a subgroup $\Gamma_0 \leqslant \Gamma$ such that $[\Gamma : \Gamma_0] < \infty$ and $\varphi(\Gamma_0)$ acts trivially on Ω_0. Fix some element $Ks\Lambda \in \Omega_0$. After replacing $\bar{f} : \widetilde{X} \to \widehat{Y}$ by the map $\bar{f}'(x) = \pi_0(s)^{-1}\bar{f}(x)$ and $\varphi \in \mathrm{Aut}(PSL_n(\mathbb{R}))$ by the automorphism $\varphi'(g) = \pi_0(s)^{-1}\varphi(g)\pi_0(s)$ if necessary, we can suppose that $s \in G$. (Recall that $\pi_0 : H \to PSL_n(\mathbb{R})$ denotes the canonical projection.) Since $\varphi(\Gamma_0)$ fixes $Ks\Lambda$, we have that $\varphi(\gamma)s \in Ks\Lambda$ for each $\gamma \in \Gamma_0$; and hence, applying the projection $\pi_0 : H \to PSL_n(\mathbb{R})$, we obtain that

$$\varphi(\gamma) \in \pi_0(\Lambda) = PSL_n(\mathbb{Z}[1/q_1, \dots, 1/q_t]) \leqslant PSL_n(\mathbb{R}).$$

In particular, the lattice $\varphi(\Gamma_0)$ of $PSL_n(\mathbb{R})$ satisfies $\varphi(\Gamma_0) \leqslant PSL_n(\mathbb{Q})$. Hence, by Margulis [15, IX.4.14], $\varphi(\Gamma_0)$ is commensurable with $PSL_n(\mathbb{Z})$. After replacing Γ_0 by a subgroup of finite index if necessary, we can suppose that $\varphi(\Gamma_0) \leqslant PSL_n(\mathbb{Z})$.

Let $\widetilde{X}_0 = \{x \in \widetilde{X} \mid (\eta \circ \bar{f})(x) = Ks\Lambda\}$. Clearly $\omega(\{Ks\Lambda\}) = 1/|\Omega_0|$ and so $\widetilde{\mu}_p(\widetilde{X}_0) = 1/|\Omega_0| > 0$. Recall that G is contained in the distinguished Borel transversal S of H/Λ. Consequently, for each $x \in \widetilde{X}_0$, there exist $\bar{f}_1(x) \in Y$ and $k_x \in K$ such that $\bar{f}(x) = (\bar{f}_1(x), k_x s)$. Now suppose that $x \in \widetilde{X}_0$ and $\gamma \in \Gamma_0$ satisfy

$$\bar{\beta}(\gamma, x) \cdot \bar{f}(x) = \bar{f}(\gamma \cdot x).$$

Define c, $d \in K$ by $\bar{\beta}(\gamma, x) = \varphi(\gamma)c$ and $d = \langle 1, \varphi(\gamma)^{-1}, \dots, \varphi(\gamma)^{-1}\rangle$. Note that

$$\varphi(\gamma)ck_x s = ck_x sd\lambda,$$

where $\lambda = \langle \varphi(\gamma), \dots, \varphi(\gamma)\rangle \in \Lambda$. It follows that

$$(\bar{f}_1(\gamma \cdot x), k_{\gamma \cdot x} s) = (\varphi(\gamma) \cdot \bar{f}_1(x), ck_x sd).$$

Hence the Borel map $\bar{f}_1 : \widetilde{X}_0 \to Y$ has the property that for all $\gamma \in \Gamma_0$,

$$\bar{f}_1(\gamma \cdot x) = \varphi(\gamma) \cdot \bar{f}_1(x) \quad \text{for } \widetilde{\mu}_p\text{-a.e. } x \in \widetilde{X}_0.$$

Since $\widetilde{\mu}_p(\widetilde{X}_0) > 0$, there exists an ergodic component $\widetilde{X}_1$ for the action of Γ_0 on $\widetilde{X}_0$ such that $\widetilde{X}_1 \subseteq \widetilde{X}_0$; and clearly we can suppose that

$$\bar{f}_1(\gamma \cdot x) = \varphi(\gamma) \cdot \bar{f}_1(x)$$

for all $\gamma \in \Gamma_0$ and $x \in \widetilde{X}_1$. Furthermore, by the ergodicity of the action of Γ_0 on $\widetilde{X}_1$, we can suppose that there exists an ergodic component Y_1 for the action of $\varphi(\Gamma_0)$ on Y such that $\bar{f}_1(\widetilde{X}_1) \subseteq Y_1$. Since Γ_0 preserves the probability measure

$(\widetilde{\mu}_p)_{\widetilde{X}_1}$ on $\widetilde{X}_1$, it follows that $\varphi(\Gamma_0)$ preserves the probability measure $(\bar{f}_1)_*(\widetilde{\mu}_p)_{\widetilde{X}_1}$ on Y_1; and since the action of $\varphi(\Gamma_0)$ on Y_1 is uniquely ergodic, this implies that $(\bar{f}_1)_*(\widetilde{\mu}_p)_{\widetilde{X}_1} = (\mu_q)_{Y_1}$. But this means that $(V^{(k)}(n, \mathbb{Q}_q), \mu_q)$ is a virtual quotient of $(\widetilde{X}, \widetilde{\mu}_p)$, which contradicts Theorem 3.3. This completes the proof of Theorem 4.7.

References

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