# First order corrections to the homogenized eigenvalues of a periodic composite medium. The case of Neumann boundary conditions.

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## 1 Introduction

In this paper we study the behavior of the Neumann eigenvalues of an elliptic operator corresponding to a composite medium with periodic microstructure. Initially we assume that $\Omega \subset R^2$ is a convex, bounded domain and we analyse the following eigenvalue problem

$$-\nabla \cdot a(x/\epsilon)\nabla v_\epsilon = \lambda_\epsilon v_\epsilon \quad \text{in } \Omega \tag{1}$$

$$a(x/\epsilon)\nabla v_\epsilon \cdot \nu = 0 \quad \text{on } \partial\Omega \ ,$$

where $\nu$ is the outward unit normal, $a(y) = (a_{ij}(y))$ is a symmetric positive definite matrix, and each $a_{ij}(y)$ is a smooth ($C^\infty$) periodic function with period $Y$. For simplicity we take $Y = (0,1) \times (0,1)$. Corresponding to (1) is the homogenized eigenvalue problem

$$-\nabla \cdot A\nabla v = \lambda v \quad \text{in } \Omega \tag{2}$$

$$A\nabla v \cdot \nu = 0 \quad \text{on } \partial\Omega$$

where

$$A_{ij} = \int_Y a_{ik}(y)\left(\delta_{kj} - \frac{\partial \chi^j}{\partial y_k}(y)\right) dy$$

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is a constant, positive definite symmetric matrix and $\chi^j$ is the $Y$-periodic (smooth) solution to

$$\nabla \cdot a(y)\nabla\chi^j = \frac{\partial}{\partial y_i} a_{ij}(y) \quad , \tag{3}$$

one of the so called "cell functions". If $\lambda$ is an eigenvalue for (2) then it is well known (see [3] or [14]) that there exists a family of eigenvalues for (1), $\lambda_\epsilon$, such that

$$\lambda_\epsilon \to \lambda \qquad \text{as} \qquad \epsilon \to 0.$$

Formally, we may write an asymptotic expansion for $\lambda_\epsilon$,

$$\lambda_\epsilon = \lambda + \epsilon\lambda_1 + \ldots$$

If the eigenfunctions for (2) corresponding to $\lambda$ are in $H^2(\Omega)$ then

$$|\lambda_\epsilon - \lambda| \leq C\epsilon.$$

The main goal of this paper is to characterize the possible first order corrections $\lambda_1$, that is, to characterize the limit points of the ratio

$$\frac{\lambda_\epsilon - \lambda}{\epsilon} \tag{4}$$

as $\epsilon \to 0$. The value 0 is a trivial eigenvalue for (2) (as well as for (1)), we shall only consider corrections to the nontrivial (positive) eigenvalues.

The first attempt to study the asymptotic magnitude of the term $\lambda_\epsilon - \lambda$ is to the best of our knowledge found in [6] and [7], however this work does not appear to recognize the fact that the ratio $(\lambda_\epsilon - \lambda)/\epsilon$ may possibly have many different limit points (and not just a single limit) as $\epsilon$ tends to zero. The problem of characterizing (all possible) first order eigenvalue corrections of Dirichlet eigenvalues was discussed extensively in [15] and [11]. A particular representation formula was given for the limit points of the ratio (4); this formula is valid for any simple eigenvalue $\lambda$ and for any curvilinear polygonal domain in which the corresponding homogenized eigenvector is in $H^{2+\omega}(\Omega)$ for some $\omega > 0$. In the case when the domain $\Omega$ was a (classical) convex polygon whose sides have rational (or infinite) slopes, it was possible to derive a much more explicit formula for the limit points of (4). This latter formula demonstrates that with Dirichlet boundary conditions there may frequently not just be one correction $\lambda_1$, but a whole continuum of corrections. That is, the limit of (4) may depend on the choice of a particular (sub)sequence $\epsilon_k \to 0$.

In order to obtain the general representation formula for the first order Dirichlet eigenvalue corrections, it was necessary to first establish certain error estimates for the corresponding elliptic source problem. We then used an error estimate for the convergence of the eigenvalues of a sequence of compact operators (due to Osborn [14]) to obtain the representation formula. In case of Neumann boundary conditions, the same procedure yields a general representation formula for the limit points of (4) similar to the one for Dirichlet boundary conditions. Just as in the Dirichlet case, this formula involves certain boundary corrector functions. In the Dirichlet case the boundary corrector functions could be rather explicitly characterized when the domain was a convex polygon with rational (or infinite) slopes. A similar characterization is possible for the Neumann case but it is somewhat more complicated to establish. To simplify matters as much as possible we make extensive use of the duality between the (two dimensional) Dirichlet- and Neumann problems and we make use of the analysis already carried out in [11]. In the end we find a quite explicit formula for the possible $\lambda_1$ when the domain is a (classical) convex polygon whose sides have rational (or infinite) slopes. This formula reveals the same phenomenon as in the Dirichlet case, namely the possibility of a continuum of corrections depending on the interaction between the periodic microstructure and the domain boundary. This may be viewed as somewhat surprising, since in one dimension the Neumann eigenvalues converge with rate $O(\epsilon^2)$; that is, in one dimension the first order Neumann eigenvalue correction is always zero.

The paper is organized as follows. In Section 2 we prove some error estimates for the Neumann source problem. Since we will apply these results to eigenfunctions associated with non-smooth domains, it is necessary to derive these estimates with minimal regularity requirements. In Section 3 we describe Osborn's eigenvalue approximation estimate and apply it to our problem. Using the error estimates from Section 2, we then obtain a general representation formula for the limit points of (4), valid for any convex domain for which the corresponding homogenized eigenfunction is in $H^{2+\omega}(\Omega)$, $\omega > 0$. In Section 4 we compare the Neumann case and Dirichlet case in one dimension. We show that the one dimensional first order Neumann eigenvalue correction is always zero, while we know from [15] that this is not so with Dirichlet boundary conditions. In Section 5 we return to two dimensions and thoroughly discuss case where the domain is a convex polygon whose sides have rational (or infinite) slopes. We use the results from [15] and [11] about boundary corrector functions to find the limits of the expression obtained in Section 3. This yields a formula for the possible limit points of (4) that can be computed rather explicitly. In Section 6 we compute these possible limits for a simple example where the domain is a square and the coefficient depends only on one variable. This example shows how the first order correction $\lambda_1$ may change depending on the interaction of the microstructure with the domain boundary. Section 7 contains a discussion and some concluding remarks about an open, interesting problem.

## 2 Error estimates related to homogenization

In this section we establish certain estimates for the Neumann source problem. In the following section we shall use these estimates in the case when the right hand side (the source) is an eigenfunction for the homogenized problem. To be quite precise we estimate the error between the solution to

$$-\nabla \cdot a(x/\epsilon)\nabla u_\epsilon = f \quad \text{in } \Omega \tag{5}$$

$$a(x/\epsilon)\nabla u_\epsilon \cdot \nu = 0 \quad \text{on } \partial\Omega$$

and the solution to the corresponding homogenized problem

$$-\nabla \cdot A\nabla u_0 = f \quad \text{in } \Omega \tag{6}$$

$$A\nabla u_0 \cdot \nu = 0 \quad \text{on } \partial\Omega$$

where $f$ is a given function in $L^2(\Omega)$ with $\int_\Omega f\,dx = 0$. The solutions $u_\epsilon$ and $u$ are uniquely determined by the additional requirement that $\int_\Omega u_\epsilon\,dx = \int_\Omega u_0\,dx = 0$. It is well known that $u_\epsilon \to u_0$ in $L^2(\Omega)$, but we need much more precise estimates of the rate of convergence. Throughout this section we shall assume that the domain $\Omega$ is bounded and convex with a boundary that is a curvilinear polygon of class $C^\infty$ in the sense of Definition 1.4.1.5 of [5]. By a slight abuse of notation we shall refer to such a domain as a convex, curvilinear polygon of class $C^\infty$. Since $\Omega$ is not assumed to be globally smooth (later we shall restrict our attention to classical convex polygons) it is important that our estimates do not require the homogenized solution to be very regular. The procedure for obtaining these estimates is somewhat analogous to the procedure we used for the Dirichlet problem [11] and represents a nontrivial extension of the method developed in [3]. As in the Dirichlet case, we need a boundary corrector function to appropriately adjust the boundary data. In the Neumann case, however, it is quite a bit more difficult to prove that this boundary corrector exists and is of order $O(\epsilon)$ in $L^2(\Omega)$. The proof of these facts extensively uses the duality between this Neumann boundary corrector function and a corresponding Dirichlet boundary corrector function.

We rewrite (5) as a first order system and proceed with the same asymptotic expansion as in the proof of Proposition 1 of [11]. We define

$$u_1(x,y) = -\chi^j(y)\frac{\partial u_0}{\partial x_j}$$

$$\begin{aligned}(v_0(x,y))_i &= (a_{ij}(y) - a_{ik}(y)\frac{\partial \chi^j}{\partial y_k})\frac{\partial u_0}{\partial x_j} \\ &= A_{ij}(y)\frac{\partial u_0}{\partial x_j} \quad ,\end{aligned}$$

$$v_1^*(x,y) = r^j(y)\nabla_x^\perp \frac{\partial u_0}{\partial x_j}$$

where $r^j(y)$ is $Y$-periodic and solves

$$(\nabla_y^\perp r^j)_i = -A_{ij} + A_{ij}(y) \quad . \tag{7}$$

We note that the matrix $A_{ij}$ is symmetric, but that this is generally not the case with $A_{ij}(y)$. The symbol $\perp$ indicates a 90 degree rotation counter clockwise, *i.e.* $\nabla_x^\perp = (-\frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_1})$. The problem (7) has a periodic (smooth) solution due to the facts that $\frac{\partial}{\partial y_i}(-A_{ij} + A_{ij}(y)) = 0$ and $\int_Y (-A_{ij}+A_{ij}(y))\, dy = -A_{ij}+\overline{A_{ij}(y)} = 0$ .

Since $\Omega$ is convex it follows from a combination of Theorem 3.1.3.3 and Theorem 3.2.1.3 in [5] that $u_0$ is in $H^2(\Omega)$ with $\|u_0\|_{H^2(\Omega)} \le C\|f\|_{L^2(\Omega)}$ (the 0th order term in [5] is irrelevant as far as regularity is concerned). For the same reason we also get that $u_\epsilon \in H^2(\Omega)$. If we let

$$z_\epsilon(x) = u_\epsilon(x) - u_0(x) - \epsilon u_1(x, x/\epsilon) \quad \in H^1(\Omega)$$

$$\eta_\epsilon(x) = a(x/\epsilon)\nabla u_\epsilon(x) - v_0(x, x/\epsilon) - \epsilon v_1^*(x, x/\epsilon) \quad \in L^2(\Omega)$$

then it follows just as in the proof of Proposition 1 in [11] that

$$\nabla \cdot \eta_\epsilon = 0 \quad \text{in } \Omega \tag{8}$$

and

$$\|a(x/\epsilon)\nabla z_\epsilon - \eta_\epsilon\|_{L^2(\Omega)} \le C\epsilon\|u_0\|_{H^2(\Omega)} \ . \tag{9}$$

To adjust the Neumann data, we define the boundary corrector function $B_\epsilon \in H^1(\Omega)$ through the variational equation

$$\int_\Omega a(x/\epsilon)\nabla B_\epsilon \cdot \nabla\phi \ dx = \int_\Omega \frac{1}{\epsilon}\eta_\epsilon \cdot \nabla\phi \ dx \quad \text{ for all } \ \phi \in H^1(\Omega). \tag{10}$$

We note that the equation for $B_\epsilon$ is the natural weak formulation of

$$-\nabla \cdot a(x/\epsilon)\nabla B_\epsilon = 0 \quad \text{in } \Omega \tag{11}$$

$$a(x/\epsilon)\nabla B_\epsilon \cdot \nu = \frac{1}{\epsilon}\eta_\epsilon \cdot \nu \quad \text{ on } \partial\Omega.$$

This observation is consistent with the fact that for any field, $\zeta_\epsilon \in L^2(\Omega)$, with $\nabla \cdot \zeta_\epsilon = 0$ in $\Omega$ we may define $\zeta_\epsilon \cdot \nu \in H^{-1/2}(\Omega)$ in a weak sense by means of the formula

$$\int_{\partial\Omega} \zeta_\epsilon \cdot \nu\phi \ d\sigma = \int_\Omega \zeta_\epsilon \cdot \nabla\phi \ dx \quad \text{ for all } \quad \phi \in H^1(\Omega) \ .$$

Then we also have

$$\|\zeta_\epsilon \cdot \nu\|_{H^{-1/2}(\partial\Omega)} \le C\|\zeta_\epsilon\|_{L^2(\Omega)} \ .$$

In the particular case of the field $a(x/\epsilon)\nabla B_\epsilon$ this estimate becomes

$$\|a(x/\epsilon)\nabla B_\epsilon \cdot \nu\|_{H^{-1/2}(\partial\Omega)} \le C\|B_\epsilon\|_{H^1(\Omega)} \ . \tag{12}$$

To make $B_\epsilon$ unique we impose the additional requirement that $\int_\Omega \epsilon B_\epsilon \ dx = \int_\Omega z_\epsilon \ dx = -\epsilon\int_\Omega u_1 \ dx$ . Note that with this "convention"

$$\left|\int_\Omega \epsilon B_\epsilon \ dx\right| \le C\epsilon\|u_0\|_{H^1(\Omega)} \ . \tag{13}$$

**Proposition 1** *Let $u_0$ denote the solution to (6), let $u_1, \eta_\epsilon$ be defined as above, and let $B_\epsilon$ be the solution to (10). There exists a constant $C$, independent of $u_0$ and $\epsilon$ such that*

$$||u_\epsilon(\cdot) - u_0(\cdot) - \epsilon u_1(\cdot, \cdot/\epsilon) - \epsilon B_\epsilon(\cdot)||_{H^1(\Omega)} \leq C\epsilon ||u_0||_{H^2(\Omega)}$$

*Proof* Given $g \in L^2(\Omega)$ with $\int_\Omega g \; dx = 0$, we define $\phi_\epsilon \in H^1(\Omega)$ to be the solution to

$$\int_\Omega a(x/\epsilon)\nabla\phi_\epsilon \cdot \nabla\psi \; dx = \int_\Omega g\psi \; dx \quad \text{for all } \; \psi \in H^1(\Omega) \; . \tag{14}$$

Using this equation for $\phi_\epsilon$ and the symmetry of the matrix $a(x/\epsilon)$, we have

$$\begin{aligned} \int_\Omega (z_\epsilon - \epsilon B_\epsilon) g \; dx &= \int_\Omega a(x/\epsilon)\nabla(z_\epsilon - \epsilon B_\epsilon) \cdot \nabla\phi_\epsilon \; dx \\ &= \int_\Omega a(x/\epsilon)\nabla z_\epsilon \cdot \nabla\phi_\epsilon \; dx - \epsilon \int_\Omega a(x/\epsilon)\nabla B_\epsilon \cdot \nabla\phi_\epsilon \; dx \\ &= \int_\Omega a(x/\epsilon)\nabla z_\epsilon \cdot \nabla\phi_\epsilon \; dx - \int_\Omega \eta_\epsilon \cdot \nabla\phi_\epsilon \; dx \quad , \end{aligned}$$

where in the last step we also used (10). In combination with (9) these identities yield

$$\begin{aligned} \left| \int_\Omega (z_\epsilon - \epsilon B_\epsilon) g \; dx \right| &= \left| \int_\Omega (a(x/\epsilon)\nabla z_\epsilon - \eta_\epsilon)\nabla\phi_\epsilon \; dx \right| \\ &\leq ||a(x/\epsilon)\nabla z_\epsilon - \eta_\epsilon||_{L^2(\Omega)} ||\nabla\phi_\epsilon||_{L^2(\Omega)} \\ &\leq C\epsilon ||u_0||_{H^2(\Omega)} ||g||_{H^{-1}(\Omega)} \quad , \end{aligned} \tag{15}$$

where we use $H^{-1}$ to denote the dual of $H^1$ , and not of $H^1_0$ (as is customary). Since $\int_\Omega \epsilon B_\epsilon \; dx = \int_\Omega z_\epsilon \; dx$ this implies

$$||z_\epsilon - \epsilon B_\epsilon||_{H^1(\Omega)} \leq C\epsilon ||u_0||_{H^2(\Omega)} \quad ,$$

exactly as desired. □

Consider the vectorfield $a(x/\epsilon)\nabla B_\epsilon$. This has divergence zero in $\Omega$, and so there exists, for each $\epsilon$, a function $C_\epsilon \in H^1(\Omega)$ such that

$$\nabla^\perp C_\epsilon = a(x/\epsilon)\nabla B_\epsilon \quad \text{in } \Omega \quad . \tag{16}$$

Here we used that $\Omega$ is simply connected (since it is assumed to be convex). We note that at this point $C_\epsilon$ is only defined up to a constant. Let $Q$ denote the matrix

$$Q = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

representing a rotation by 90 degrees counter clockwise. The function $C_\epsilon$ then satisfies

$$\nabla \cdot Q^t a^{-1}(x/\epsilon) Q \nabla C_\epsilon = 0 \quad \text{in} \quad \Omega \ .$$

Define

$$\tilde{a}(x/\epsilon) = Q^t a^{-1}(x/\epsilon) Q \ . \tag{17}$$

Since we are assuming that $\Omega$ is a curvilinear polygon of class $C^\infty$, $\partial\Omega$ consists of $m$ "corners" and $m$ open smooth curves $\Gamma_1, \ldots \Gamma_m$ connecting these "corners". Let $p$ be an interior point of $\Gamma_i$ and suppose for a brief moment that we have $f \in H^1(\Omega)$. If $\omega$ denotes a sufficiently small neighborhood of $p$, then standard elliptic regularity theory ensures that $u_0$ is in $H^3(\Omega \cap \omega)$. It now follows that $\eta_\epsilon$ is in $H^1(\Omega \cap \omega)$, and so its normal component $\eta_\epsilon \cdot \nu$ is in $H^{1/2}(\Gamma_i \cap \omega)$. From standard elliptic regularity theory it now follows that $B_\epsilon$ is in $H^2(\Omega \cap \omega)$ in any slightly smaller neighborhood, for simplicity also denoted $\Omega \cap \omega$. The identity (16) now immediately givess that $C_\epsilon$ is in $H^2(\Omega \cap \omega)$. By taking the traces on $\Gamma_i \cap \omega$ it also follows that $C_\epsilon \in H^{3/2}(\Gamma_i \cap \omega)$, $\nabla B_\epsilon \in H^{1/2}(\Gamma_i \cap \omega)$ and that

$$\frac{\partial C_\epsilon}{\partial \tau} = -a(x/\epsilon) \nabla B_\epsilon \cdot \nu \quad \text{on} \quad \Gamma_i \cap \omega \ ,$$

where $\tau = Q\nu$ denotes the counter clockwise unit tangent to $\partial\Omega$. On $\Gamma_i \cap \omega$ we may calculate

$$\begin{aligned}
a(x/\epsilon) \nabla B_\epsilon \cdot \nu &= \frac{1}{\epsilon} \eta_\epsilon \cdot \nu \\
&= -\frac{1}{\epsilon} v_0 \cdot \nu - v_1^* \cdot \nu \\
&= \frac{1}{\epsilon} A \nabla u_0 \cdot \nu - \frac{1}{\epsilon} A(x/\epsilon) \nabla u_0 \cdot \nu - r^j \nabla_x^\perp \frac{\partial u_0}{\partial x_j} \cdot \nu \\
&= -\frac{1}{\epsilon} \nabla_y^\perp r^j \cdot \nu \frac{\partial u_0}{\partial x_j} - r^j \nabla_x^\perp \frac{\partial u_0}{\partial x_j} \cdot \nu \\
&= \frac{\partial}{\partial \tau} \left( r^j \frac{\partial u_0}{\partial x_j} \right) \ .
\end{aligned}$$

The extra regularity of $\nabla B_\epsilon$ and $\eta_\epsilon \cdot \nu$ (which is assured by the extra regularity of $f$) guarantees that all terms in the above calculation make sense as functions (in $H^{1/2}$) on $\Gamma_i \cap \omega$. By combining the past two identities (for all $p \in \Gamma_i$) we conclude that there exists some constant $d_i$ such that

$$C_\epsilon = -r^j(x/\epsilon) \frac{\partial u_0}{\partial x_j} + d_i \quad \text{on} \quad \Gamma_i \ .$$

Since we know that $C_\epsilon \in H^1(\Omega)$, its trace is in $H^{1/2}(\partial\Omega)$; similarly, since $r^j$ is smooth and $u_0$ is in $H^2(\Omega)$, the trace of the function $r^j(x/\epsilon)\frac{\partial u_0}{\partial x_j}$ is also in $H^{1/2}(\partial\Omega)$. As a consequence the trace of the function $C_\epsilon + r^j(x/\epsilon)\frac{\partial u_0}{\partial x_j}$ is in $H^{1/2}(\partial\Omega)$. This trace equals the constant $d_i$ on $\Gamma_i$, and since jump discontinuities are not permitted in $H^{1/2}(\partial\Omega)$ all the $d_i$'s must have the same value, $d$. As noted earlier, $C_\epsilon$ has only been determined up to an additive constant. We now make the choice of $C_\epsilon$ unique by requiring that $\int_{\partial\Omega} C_\epsilon \, ds = -\int_{\partial\Omega} r^j(x/\epsilon)\frac{\partial u_0}{\partial x_j} \, ds$, which means that $d = 0$. In summary we thus have

$$C_\epsilon = -r^j(x/\epsilon)\frac{\partial u_0}{\partial x_j} \quad \text{on} \quad \partial\Omega \tag{18}$$

provided $f$ is in $H^1(\Omega)$. The mapping

$$f \to u_0 \to C_\epsilon|_{\partial\Omega}$$

is continuous

$$L^2(\Omega) \cap \{\int_\Omega f \, dx = 0\} \to H^2(\Omega) \to H^{1/2}(\partial\Omega) \quad ,$$

and so is the mapping

$$f \to u_0 \to -r^j(x/\epsilon)\frac{\partial u_0}{\partial x_j} \quad .$$

Since $H^1(\Omega) \cap \{\int_\Omega f \, dx = 0\}$ is dense in $L^2(\Omega) \cap \{\int_\Omega f \, dx = 0\}$ it follows that (18) holds true even for $f$ that are only in $L^2$. We now return to assuming that $f$ is an $L^2$ function. Based on the previous discussion it follows that the function $C_\epsilon$ solves

$$\begin{aligned} \nabla \cdot \tilde{a}(x/\epsilon)\nabla C_\epsilon = 0 \quad & \text{in} \quad \Omega \\ C_\epsilon = -r^j(x/\epsilon)\frac{\partial u_0}{\partial x_j} \quad & \text{on} \quad \partial\Omega. \end{aligned} \tag{19}$$

Note that $C_\epsilon$ is a function similar to the boundary corrector for the Dirichlet problem. In fact, if we let $\tilde{\chi}^j(y)$ $j = 1, 2$ be the cell functions corresponding to the coefficient matrix $\tilde{a}(y)$, that is, we let $\tilde{\chi}^j$, $j = 1, 2$ be the periodic solutions to

$$\nabla_y \cdot \tilde{a}(y)\nabla_y \tilde{\chi}^j = \frac{\partial}{\partial y_i}\tilde{a}_{ij} \quad ,$$

then the functions $r^j$ are just linear combinations of these cell functions. To prove this we calculate

$$\begin{aligned} \nabla_y \cdot \tilde{a}(y)\nabla_y r^j &= \nabla_y \cdot Q^t a^{-1} Q \nabla_y r^j = \nabla_y \cdot Q^t a^{-1} \nabla_y^\perp r^j \\ &= \frac{\partial}{\partial y_k}\left[(Q^t a^{-1})_{ki}(a_{il}(\delta_{lj} - \frac{\partial \chi^j}{\partial y_l}) - A_{ij})\right] \\ &= \frac{\partial}{\partial y_k}\left[(Q^t a^{-1} a)_{kl}(\delta_{lj} - \frac{\partial \chi^j}{\partial y_l}) - (Q^t a^{-1} A)_{kj}\right] \quad . \end{aligned}$$

Since

$$(Q^t a^{-1} a)_{kl}(\delta_{lj} - \frac{\partial \chi^j}{\partial y_l}) = [-\nabla_y^\perp (y_j - \chi^j)]_k$$

it follows that

$$\begin{aligned}\nabla_y \cdot \tilde{a}(y)\nabla_y r^j &= -\frac{\partial}{\partial y_k}(Q^t a^{-1} A)_{kj} \\ &= \frac{\partial}{\partial y_k}(\tilde{a} Q A)_{kj} \ ,\end{aligned}$$

where in the last step we used the fact that $Q^{-1} = Q^t = -Q$. Hence we obtain

$$\nabla_y \cdot \tilde{a}(y)\nabla_y r^j = \frac{\partial}{\partial y_k}\tilde{a}_{ki}(QA)_{ij} \ ,$$

which means that, up to an additive constant,

$$r^j(y) = \tilde{\chi}^i(y)(QA)_{ij} \ . \tag{20}$$

**Lemma 1** *Let $r^j$ be defined by (7) and let $u_0 \in H^2(\Omega)$ be the solution to (6). Then*

$$\|r^j(x/\epsilon)\frac{\partial u_0}{\partial x_j}\|_{H^{1/2}(\partial\Omega)} \leq C\epsilon^{-1/2}\|u_0\|_{H^2(\Omega)}.$$

*Proof* Suppose the vectorfield $\mathbf{F} = \{F_j\}_{j=1}^2$ is in $H_0^1(\Gamma_i)$ for $i = 1,\ldots,m$ ( $\mathbf{F}$ is defined on all of $\partial\Omega$ and vanishes at the corners). We then have

$$\begin{aligned}\|r^j(x/\epsilon)F_j\|_{L^2(\Gamma_i)} &\leq C\|\mathbf{F}\|_{L^2(\Gamma_i)} \ , \\ \|r^j(x/\epsilon)F_j\|_{H_0^1(\Gamma_i)} &\leq \frac{C}{\epsilon}\|\mathbf{F}\|_{H_0^1(\Gamma_i)} \ .\end{aligned}$$

By interpolation it now follows that for any $\mathbf{F} \in H_{00}^{1/2}(\Gamma_i)$

$$\|r^j(x/\epsilon)F_j\|_{H_{00}^{1/2}(\Gamma_i)} \leq C\epsilon^{-1/2}\|\mathbf{F}\|_{H_{00}^{1/2}(\Gamma_i)} \ . \tag{21}$$

The space $H_{00}^{1/2}(\Gamma_i)$ consists roughly speaking of those elements of $H^{1/2}(\Gamma_i)$ that may be "extended to zero beyond $\Gamma_i$ and still be of class $H^{1/2}$ "; we refer to [9] Section 11.5 for the exact definition of this space and for the above interpolation result.

Since the vectorfield $A\nabla u_0$ is in $H^1(\Omega)$, since $A\nabla u_0 \cdot \nu = 0$ on $\partial\Omega$ it follows from Theorem 1.5.2.3 in [5] that the boundary trace of $A\nabla u_0$ is in $H_{00}^{1/2}(\Gamma_i)$, $i = 1,\ldots,m$, and that it satisfies the estimate

$$\sum_{i=1}^m \|A\nabla u_0\|_{H_{00}^{1/2}(\Gamma_i)} \leq C\|A\nabla u_0\|_{H^1(\Omega)} \ .$$

Due to the fact that $A$ is a constant, invertible matrix we also get that $\nabla u_0$ is in $H_{00}^{1/2}(\Gamma_i)$, $i = 1,\ldots,m$ and that the above estimate holds with $\nabla u_0$ in place of $A\nabla u_0$, *i.e.*,

$$\sum_{i=1}^m \|\nabla u_0\|_{H_{00}^{1/2}(\Gamma_i)} \leq C\|\nabla u_0\|_{H^1(\Omega)} \ . \tag{22}$$

From the proof of Theorem 1.5.2.3 in [5] it also follows that

$$||\phi||_{H^{1/2}(\partial\Omega)} \leq C \sum_{i=1}^{m} ||\phi||_{H_{00}^{1/2}(\Gamma_i)}$$

for any $\phi$ with $\phi \in H_{00}^{1/2}(\Gamma_i)$, $i = 1, \ldots, m$.

A combination of this last estimate with $\phi = r^j(x/\epsilon)\frac{\partial u_0}{\partial x_j}$, the estimate (21), with $\mathbf{F} = \nabla u_0$, and the estimate (22) now yields

$$\begin{aligned} ||r^j(x/\epsilon)\frac{\partial u_0}{\partial x_j}||_{H^{1/2}(\partial\Omega)} &\leq C\sum_{i=1}^{m} ||r^j(x/\epsilon)\frac{\partial u_0}{\partial x_j}||_{H_{00}^{1/2}(\Gamma_i)} \\ &\leq C\epsilon^{-1/2}\sum_{i=1}^{m} ||\nabla u_0||_{H_{00}^{1/2}(\Gamma_i)} \\ &\leq C\epsilon^{-1/2}||\nabla u_0||_{H^1(\Omega)} \quad . \end{aligned}$$

This concludes the proof of the lemma. □

The previous lemma in combination with (19) implies that

$$||C_\epsilon||_{H^1(\Omega)} \leq C\epsilon^{-1/2}||u_0||_{H^2(\Omega)}.$$

Due to (16) (the definition of $C_\epsilon$) and (13) this immediately yields

**Lemma 2** *Let $u_0 \in H^2(\Omega)$ be the solution to (6) and let $B_\epsilon$ be as defined previously. Then*

$$||\epsilon B_\epsilon||_{H^1(\Omega)} \leq C\epsilon^{1/2}||u_0||_{H^2(\Omega)} \quad .$$

We use this lemma to prove a corresponding $L^2$ estimate for $\epsilon B_\epsilon$.

**Lemma 3** *Let the hypotheses be as in Lemma 2. Then*

$$||\epsilon B_\epsilon||_{L^2(\Omega)} \leq C\epsilon||u_0||_{H^2(\Omega)} \quad .$$

*Proof* Let $g$ be given in $L^2(\Omega)$ with integral zero and define $\phi_\epsilon$ by (14). Then

$$\int_\Omega \epsilon B_\epsilon g \; dx = \int_\Omega \epsilon a(x/\epsilon)\nabla B_\epsilon \cdot \nabla \phi_\epsilon \; dx \quad .$$

Now define $\phi_0$,$\phi_1$, and $B_\epsilon^g$ to be analogous to $u_0$,$u_1$, and $B_\epsilon$ but corresponding to right hand side $g$ (instead of $f$). Since $\Omega$ is convex we know that $\phi_0$ is in $H^2(\Omega)$, with $||\phi_0||_{H^2(\Omega)} \leq C||g||_{L^2(\Omega)}$. From Proposition 1

$$||\phi_\epsilon - \phi_0 - \epsilon\phi_1 - \epsilon B_\epsilon^g||_{H^1(\Omega)} \leq C\epsilon||\phi_0||_{H^2(\Omega)}$$

and hence, by use of Lemma 2,

$$|\int_\Omega \epsilon B_\epsilon g \; dx| \leq |\int_\Omega \epsilon a(x/\epsilon)\nabla B_\epsilon \cdot \nabla(\phi_0 + \epsilon\phi_1 + \epsilon B_\epsilon^g) \; dx| \\ + C\epsilon^{3/2}\|u_0\|_{H^2(\Omega)}\|\phi_0\|_{H^2(\Omega)} \quad . \tag{23}$$

We bound the three terms in the above integral separately, starting with

$$\begin{aligned} \left|\int_\Omega \epsilon a(x/\epsilon)\nabla B_\epsilon \cdot \nabla\phi_0 \; dx\right| &= \left|\int_\Omega \epsilon\nabla^\perp C_\epsilon \cdot \nabla\phi_0 \; dx\right| \\ &= \left|\int_{\partial\Omega} \epsilon C_\epsilon \frac{\partial\phi_0}{\partial\tau} \; ds\right| \\ &= \left|\int_{\partial\Omega} \epsilon r^j \frac{\partial u_0}{\partial x_j}\frac{\partial\phi_0}{\partial\tau} \; ds\right| \\ &\leq C\epsilon\|u_0\|_{H^2(\Omega)}\|\phi_0\|_{H^2(\Omega)} \quad . \end{aligned} \tag{24}$$

The second term

$$\begin{aligned} \left|\int_\Omega \epsilon a(x/\epsilon)\nabla B_\epsilon \cdot \nabla(\epsilon\phi_1) \; dx\right| &\leq \left|\int_{\partial\Omega} \epsilon a(x/\epsilon)\nabla B_\epsilon \cdot \nu \; (\epsilon\phi_1) \; ds\right| \\ &\leq \|\epsilon a(x/\epsilon)\nabla B_\epsilon \cdot \nu\|_{H^{-1/2}(\partial\Omega)}\|\epsilon\phi_1\|_{H^{1/2}(\partial\Omega)} \end{aligned} \tag{25}$$

Since

$$\phi_1 = -\chi^j(x/\epsilon)\frac{\partial\phi_0}{\partial x_j} \quad ,$$

we get, by the analogue of Lemma 1,

$$\|\epsilon\phi_1\|_{H^{1/2}(\partial\Omega)} \leq C\epsilon^{1/2}\|\phi_0\|_{H^2(\Omega)} \quad .$$

Due to Lemma 2 and the inequality (12)

$$\|\epsilon a(x/\epsilon)\nabla B_\epsilon \cdot \nu\|_{H^{-1/2}(\partial\Omega)} \leq C\epsilon^{1/2}\|u_0\|_{H^2(\Omega)} \quad .$$

In combination with (25) the last two estimates yield the following upper bound for the second term of the integral in (23)

$$\left|\int_\Omega \epsilon a(x/\epsilon)\nabla B_\epsilon \cdot \nabla(\epsilon\phi_1) \; dx\right| \leq C\epsilon\|u_0\|_{H^2(\Omega)}\|\phi_0\|_{H^2(\Omega)}. \tag{26}$$

Finally, concerning the third term, an application of Lemma 2 and its analogue for $\epsilon B_\epsilon^g$ gives

$$\left|\int_\Omega \epsilon a(x/\epsilon)\nabla B_\epsilon \nabla(\epsilon B_\epsilon^g) \; dx\right| \leq C\epsilon\|u_0\|_{H^2(\Omega)}\|\phi_0\|_{H^2(\Omega)} \quad . \tag{27}$$

From (23),(24),(26), and (27) it follows that

$$\begin{aligned} \left|\int_\Omega \epsilon B_\epsilon g \; dx\right| &\leq C\epsilon\|u_0\|_{H^2(\Omega)}\|\phi_0\|_{H^2(\Omega)} \\ &\leq C\epsilon\|u_0\|_{H^2(\Omega)}\|g\|_{L^2(\Omega)} \end{aligned}$$

and thus

$$\|\epsilon B_\epsilon\|_{L^2(\Omega)} \leq C\epsilon \|u_0\|_{H^2(\Omega)} \quad .$$

Note that for this last estimate we also use the fact $|\int_\Omega \epsilon B_\epsilon \, dx| \leq C\epsilon\|u_0\|_{H^2(\Omega)}$ (from (13) it actually follows that this integral is bounded by $C\epsilon\|u_0\|_{H^1(\Omega)}$). $\square$

**Corollary 1** *Let the hypotheses be as in Proposition 1. Then there exists a constant $C$, independent of $u_0$ and $\epsilon$ such that*

$$\|u_\epsilon - u_0\|_{L^2(\Omega)} \leq C\epsilon\|u_0\|_{H^2(\Omega)}$$

*and*

$$\|u_\epsilon(\cdot) - u_0(\cdot) - \epsilon u_1(\cdot, \cdot/\epsilon)\|_{H^1(\Omega)} \leq C\epsilon^{1/2}\|u_0\|_{H^2(\Omega)} \quad .$$

*Proof* This follows directly from Proposition 1, Lemma 2, Lemma 3 and the fact that $u_1$ is bounded in $L^2(\Omega)$ independently of $\epsilon$. $\square$

We now assume that our solution $u_0$ of (6) is in $H^3(\Omega)$ and proceed with an asymptotic expansion analogous to that in the proof of Proposition 2 of [11]. We maintain the definitions of $u_1$ and $v_0$ given earlier in this paper and complement the expansion with

$$u_2 = \chi^{ij}(x/\epsilon)\frac{\partial^2 u_0}{\partial x_i \partial x_j}$$

where $\chi^{ij}$ is $Y$-periodic (smooth) and satisfies

$$\nabla_y \cdot a(y)\nabla_y \chi^{ij} = A_{ij} - A_{ij}(y) + \frac{\partial}{\partial y_k}(a_{ki}\chi^j) \quad .$$

In place of $v_1^*$, we define

$$(v_1(x,y))_k = c^{ijk}(y)\frac{\partial^2 u_0}{\partial x_i \partial x_j}$$

where

$$c^{ijk}(y) = -a_{ki}(y)\chi^j(y) + a_{kl}(y)\frac{\partial \chi^{ij}}{\partial y_l} \quad .$$

As in [11] we also define

$$v_2(x,y) = (\nabla_x p(x,y))^\perp$$

where $p(x,y)$ is a $Y$-periodic (smooth) solution to

$$(\nabla_y p)^\perp = v_1(x,y) - v_1^*(x,y) - \overline{v_1}(x) + \overline{v_1^*}(x) \quad .$$

If we let

$$\psi_\epsilon(x) = u_\epsilon(x) - u_0(x) - \epsilon u_1(x, x/\epsilon) - \epsilon^2 u_2(x, x/\epsilon) \quad \in H^1(\Omega)$$

$$\xi_\epsilon(x) = a(x/\epsilon)\nabla u_\epsilon(x) - v_0(x, x/\epsilon) - \epsilon v_1(x, x/\epsilon) - \epsilon^2 v_2(x, x/\epsilon) \quad \in L^2(\Omega) \quad ,$$

then it follows just as in [11] that

$$\nabla \cdot \xi_\epsilon = 0 \quad \text{in } \Omega \quad , \tag{28}$$

$$\|a(x/\epsilon)\nabla\psi_\epsilon - \xi_\epsilon\|_{L^2(\Omega)} \le C\epsilon^2 \|u_0\|_{H^3(\Omega)} \quad . \tag{29}$$

It is again necessary to adjust the boundary data. Let $\theta_\epsilon \in H^1(\Omega)$ satisfy

$$\int_\Omega a(x/\epsilon)\nabla\theta_\epsilon \cdot \nabla\phi \; dx = \frac{1}{\epsilon}\int_\Omega \xi_\epsilon \cdot \nabla\phi \; dx \quad \text{for all } \; \phi \in H^1(\Omega) \quad . \tag{30}$$

**Proposition 2** *Assume the solution $u_0$ to (6) is in $H^3(\Omega)$, and let $u_1$,$u_2$, and $\xi_\epsilon$ be as defined earlier. Let $\theta_\epsilon$ be the solution to (30). There exists a constant $C$, independent of $u_0$ and $\epsilon$, such that*

$$\|u_\epsilon(\cdot) - u_0(\cdot) - \epsilon u_1(\cdot, \cdot/\epsilon) - \epsilon^2 u_2(\cdot, \cdot/\epsilon) - \epsilon\theta_\epsilon(\cdot)\|_{H^1(\Omega)} \le C\epsilon^2 \|u_0\|_{H^3(\Omega)} \quad .$$

*Proof* The proof is exactly the same as the proof of Proposition 1, with $z_\epsilon$,$\eta_\epsilon$, $B_\epsilon$ replaced by $\psi_\epsilon$,$\xi_\epsilon$, $\theta_\epsilon$ respectively. □

Our goal now is to get an estimate, better than that which we could immediately derive from Proposition 1, for the expression

$$\|u_\epsilon(\cdot) - u_0(\cdot) - \epsilon u_1(\cdot, \cdot/\epsilon) - \epsilon B_\epsilon(\cdot)\|_{L^2(\Omega)}$$

when the homogenized solution is in $H^3(\Omega)$. Note that

$$\theta_\epsilon = B_\epsilon + R_\epsilon$$

where $R_\epsilon$ satisfies

$$\int_\Omega a(x/\epsilon)\nabla R_\epsilon \cdot \nabla\phi \; dx = \frac{1}{\epsilon}\int_\Omega (\xi_\epsilon - \eta_\epsilon) \cdot \nabla\phi \; dx \quad \text{for all } \; \phi \in H^1(\Omega) \quad . \tag{31}$$

We make $R_\epsilon$ (and therefore $\theta_\epsilon$) unique by requiring that $\int_\Omega R_\epsilon \; dx = 0$.

**Lemma 4** *Let $\xi_\epsilon$ and $\eta_\epsilon$ be as defined previously and let $R_\epsilon$ satisfy (31). There exists a constant $C$, independent of $u_0$ and $\epsilon$, such that*

$$\|R_\epsilon\|_{L^2(\Omega)} \le C\epsilon^{1/2}\|u_0\|_{H^3(\Omega)} \quad .$$

*Proof* Let $g$ be given in $L^2(\Omega)$ with integral zero, and let $\phi_\epsilon$ be the solution to (14). Also define $\phi_0$ and $\phi_1$ to be the homogenized solution and the first order (interior) corrector corresponding to right hand side $g$. From Corollary 1 we know that

$$\|\phi_\epsilon(\cdot)-\phi_0(\cdot)-\epsilon\phi_1(\cdot,\cdot/\epsilon)\|_{H^{1/2}(\partial\Omega)} \leq C\|\phi_\epsilon(\cdot)-\phi_0(\cdot)-\epsilon\phi_1(\cdot,\cdot/\epsilon)\|_{H^1(\Omega)} \leq C\epsilon^{1/2}\|\phi_0\|_{H^2(\Omega)} \ ,$$

and since

$$\|\epsilon\phi_1(\cdot,\cdot/\epsilon)\|_{H^{1/2}(\partial\Omega)} \leq C\epsilon^{1/2}\|\phi_0\|_{H^2(\Omega)},$$

(the analogue of Lemma 1) it follows that

$$\|\phi_\epsilon - \phi_0\|_{H^{1/2}(\partial\Omega)} \leq C\epsilon^{1/2}\|\phi_0\|_{H^2(\Omega)} \quad .$$

Therefore there exists some $\tilde{\phi}_\epsilon$ such that

$$\|\tilde{\phi}_\epsilon\|_{H^1(\Omega)} \leq C\epsilon^{1/2}\|\phi_0\|_{H^2(\Omega)}$$

and

$$\tilde{\phi}_\epsilon = \phi_\epsilon - \phi_0 \quad \text{on} \quad \partial\Omega \ .$$

We now calculate

$$\begin{aligned} \left|\frac{1}{\epsilon}\int_\Omega(\xi_\epsilon - \eta_\epsilon)\cdot\nabla(\phi_\epsilon - \phi_0)\ dx\right| &= \left|\frac{1}{\epsilon}\int_\Omega(\xi_\epsilon - \eta_\epsilon)\cdot\nabla\tilde{\phi}_\epsilon\ dx\right| \\ &= \left|\int_\Omega(v_1^* - v_1 - \epsilon v_2)(x,x/\epsilon)\cdot\nabla\tilde{\phi}_\epsilon\ dx\right| \\ &\leq C\|u_0\|_{H^3(\Omega)}\|\tilde{\phi}_\epsilon\|_{H^1(\Omega)} \\ &\leq C\epsilon^{1/2}\|u_0\|_{H^3(\Omega)}\|\phi_0\|_{H^2(\Omega)} \quad . \end{aligned} \tag{32}$$

In order to estimate the $L^2$ norm of $R_\epsilon$ we consider the integral $\int_\Omega R_\epsilon g\ dx$. Use of the equations for $\phi_\epsilon$ and $R_\epsilon$ gives

$$\begin{aligned} \int_\Omega R_\epsilon g\ dx &= \int_\Omega a(x/\epsilon)\nabla R_\epsilon\cdot\nabla\phi_\epsilon\ dx \\ &= \frac{1}{\epsilon}\int_\Omega(\xi_\epsilon - \eta_\epsilon)\cdot\nabla\phi_\epsilon\ dx \\ &= \frac{1}{\epsilon}\int_\Omega(\xi_\epsilon - \eta_\epsilon)\cdot\nabla\phi_0\ dx + \frac{1}{\epsilon}\int_\Omega(\xi_\epsilon - \eta_\epsilon)\cdot\nabla(\phi_\epsilon - \phi_0)\ dx \quad . \end{aligned}$$

The last term is, due to (32), bounded by $C\epsilon^{1/2}\|u_0\|_{H^3(\Omega)}\|\phi_0\|_{H^2(\Omega)}$. The first term equals

$$\begin{aligned} \frac{1}{\epsilon}\int_\Omega(\xi_\epsilon - \eta_\epsilon)\cdot\nabla\phi_0\ dx &= \int_\Omega(v_1^* - v_1 - \epsilon v_2)(x,x/\epsilon)\cdot\nabla\phi_0\ dx \\ &= \int_\Omega(v_1^* - v_1)(x,x/\epsilon)\cdot\nabla\phi_0\ dx - \epsilon\int_\Omega v_2(x,x/\epsilon)\cdot\nabla\phi_0\ dx \end{aligned}$$

where the last integral is bounded by $C\epsilon\|u_0\|_{H^3(\Omega)}\|\phi_0\|_{H^1(\Omega)}$. In summary, we have shown that

$$\int_\Omega R_\epsilon g\ dx \quad = \quad \int_\Omega (v_1^* - v_1)(x, x/\epsilon) \cdot \nabla \phi_0\ dx + O(\epsilon^{1/2}\|u_0\|_{H^3(\Omega)}\|\phi_0\|_{H^2(\Omega)}) \tag{33}$$

We proceed to estimate the integral on the right hand side of (33)

$$\begin{aligned}
&\int_\Omega (v_1^* - v_1)(x, x/\epsilon)\nabla \phi_0\ dx \\
&= \int_\Omega \left( -r^j(x/\epsilon)\frac{\partial^2 u_0}{\partial x_j \partial x_2}\frac{\partial \phi_0}{\partial x_1} + r^j(x/\epsilon)\frac{\partial^2 u_0}{\partial x_j \partial x_1}\frac{\partial \phi_0}{\partial x_2} - c^{ijk}(x/\epsilon)\frac{\partial^2 u_0}{\partial x_j \partial x_i}\frac{\partial \phi_0}{\partial x_k}\right)\ dx \\
&= \int_\Omega \left( -r^i(x/\epsilon)\frac{\partial^2 u_0}{\partial x_i \partial x_2}\frac{\partial \phi_0}{\partial x_1} + r^i(x/\epsilon)\frac{\partial^2 u_0}{\partial x_i \partial x_1}\frac{\partial \phi_0}{\partial x_2}\right)\ dx \\
&\quad - \int_\Omega \left( c^{i1k}(x/\epsilon)\frac{\partial^2 u_0}{\partial x_1 \partial x_i}\frac{\partial \phi_0}{\partial x_k} + c^{i2k}(x/\epsilon)\frac{\partial^2 u_0}{\partial x_2 \partial x_i}\frac{\partial \phi_0}{\partial x_k}\right)\ dx \\
&= \int_\Omega \left( (-r^i - c^{i21})(x/\epsilon)\frac{\partial^2 u_0}{\partial x_i \partial x_2}\frac{\partial \phi_0}{\partial x_1} + (r^i - c^{i12})(x/\epsilon)\frac{\partial^2 u_0}{\partial x_i \partial x_1}\frac{\partial \phi_0}{\partial x_2}\right)\ dx \\
&\quad - \int_\Omega \left( c^{i11}(x/\epsilon)\frac{\partial^2 u_0}{\partial x_i \partial x_1}\frac{\partial \phi_0}{\partial x_1} + c^{i22}(x/\epsilon)\frac{\partial^2 u_0}{\partial x_i \partial x_2}\frac{\partial \phi_0}{\partial x_2}\right)\ dx \quad .
\end{aligned} \tag{34}$$

The periodic functions $r^i$ were defined, by (7), only up to additive constants. We now choose these constants such that

$$\overline{r^i} = -\overline{c^{i21}}.$$

A simple computation (see pp. 9-10 of [11]) yields

$$\overline{c^{ijk}} = -\overline{c^{ikj}}$$

so that

$$\overline{r^i} = \overline{c^{i12}} \quad \text{and} \quad \overline{c^{ikk}} = 0 \quad .$$

In the last identity there is no summation. We now define a (smooth) $Y$-periodic function $H_i$ by

$$\Delta_y H_i = -r^i - c^{i21}$$

We substitute this into the first term in the above integral, (34), and integrate by parts once

$$\begin{aligned}
\int_\Omega (-r^i - c^{i21})(x/\epsilon)\frac{\partial^2 u_0}{\partial x_i \partial x_2}\frac{\partial \phi_0}{\partial x_1}\ dx \quad &= \quad \int_\Omega \epsilon^2 \Delta_x H_i(x/\epsilon)\frac{\partial^2 u_0}{\partial x_i \partial x_2}\frac{\partial \phi_0}{\partial x_1}\ dx \\
&= \quad -\epsilon \int_\Omega \epsilon \nabla_x H_i(x/\epsilon) \cdot \nabla \left(\frac{\partial^2 u_0}{\partial x_i \partial x_2}\frac{\partial \phi_0}{\partial x_1}\right)\ dx \\
&\quad\quad +\epsilon \int_{\partial\Omega} \epsilon \nabla_x H_i(x/\epsilon) \cdot \nu \frac{\partial^2 u_0}{\partial x_i \partial x_2}\frac{\partial \phi_0}{\partial x_1}\ ds \\
&= \quad O(\epsilon\|u_0\|_{H^3(\Omega)}\|\phi_0\|_{H^2(\Omega)}) \quad .
\end{aligned}$$

Since the periodic function in each term of (34) has cell average zero, we can bound these terms similarly to obtain

$$|\int_\Omega (v_1^* - v_1)\nabla\phi_0 \; dx| \leq C\epsilon \|u_0\|_{H^3(\Omega)}\|\phi_0\|_{H^2(\Omega)} \quad .$$

Combining this with (33) we get

$$\begin{aligned} |\int_\Omega R_\epsilon g \; dx| &\leq C\epsilon^{1/2}\|u_0\|_{H^3(\Omega)}\|\phi_0\|_{H^2(\Omega)} \\ &\leq C\epsilon^{1/2}\|u_0\|_{H^3(\Omega)}\|g\|_{L^2(\Omega)} \quad , \end{aligned}$$

which immediately gives the desired estimate . $\square$

For $u_0 \in H^3(\Omega)$ a combination of Proposition 2 and Lemma 4 gives

$$\begin{aligned} \|u_\epsilon - u_0 - \epsilon u_1 - \epsilon B_\epsilon\|_{L^2(\Omega)} &\leq C\epsilon^2\|u_0\|_{H^3(\Omega)} + \epsilon^2\|u_2\|_{L^2(\Omega)} + \epsilon\|R_\epsilon\|_{L^2(\Omega)} \\ &\leq C\epsilon^{3/2}\|u_0\|_{H^3(\Omega)} \quad . \end{aligned} \tag{35}$$

Consider the bounded linear operator

$$\Lambda_\epsilon(u_0) = u_\epsilon - u_0 - \epsilon u_1 - \epsilon B_\epsilon \quad ,$$

from $H^2(\Omega) \cap \{A\nabla u_0 \cdot \nu = 0 \;\text{ on } \partial\Omega\}$ into $L^2(\Omega)$ . Observe that even the terms $u_\epsilon$ and $B_\epsilon$ depend linearly (and continuously) on the homogenized function $u_0$. From Proposition 1 and the previous estimate

$$\|\Lambda_\epsilon(u_0)\|_{L^2(\Omega)} \leq C\epsilon\|u_0\|_{H^2(\Omega)}$$

and

$$\|\Lambda_\epsilon(u_0)\|_{L^2(\Omega)} \leq C\epsilon^{3/2}\|u_0\|_{H^3(\Omega)} \quad .$$

By interpolating between the spaces $H^2(\Omega)\cap\{A\nabla u_0\cdot\nu = 0 \;\text{ on } \partial\Omega\}$ and $H^3(\Omega)\cap \{A\nabla u_0 \cdot \nu = 0 \;\text{ on } \partial\Omega\}$ using the subspace interpolation result in Section 14.3 of [9] we arrive at the estimate

$$\|\Lambda_\epsilon(u_0)\|_{L^2(\Omega)} \leq C\epsilon^{1+\frac{\omega}{2}}\|u_0\|_{H^{2+\omega}(\Omega)}$$

for $u_0 \in H^{2+\omega}(\Omega) \cap \{A\nabla u_0 \cdot \nu = 0 \;\text{ on } \partial\Omega\}$, $0 < \omega < 1$. We have thus verified

**Theorem 1** *Let $\Omega \subset I\!R^2$ be a bounded, convex curvilinear polygon of class $C^\infty$ and let $u_0$ denote the solution to (6). Suppose $u_0 \in H^{2+\omega}(\Omega)$ for some $0 \leq \omega \leq 1$. Set $u_1(x,y) = -\chi^j(y)\frac{\partial u_0}{\partial x_j}(x)$ and let $B_\epsilon \in H^1(\Omega)$ denote the solution to (10). There exists a constant $C_\omega$, independent of $u_0$ and $\epsilon$, such that*

$$\|u_\epsilon(\cdot) - u_0(\cdot) - \epsilon u_1(\cdot, \cdot/\epsilon) - \epsilon B_\epsilon(\cdot)\|_{L^2(\Omega)} \leq C_\omega\epsilon^{1+\frac{\omega}{2}}\|u_0\|_{H^{2+\omega}(\Omega)} \quad .$$

**Remark**

We note that if $\Omega$ is a bounded, convex classical polygon (with sides that are line segments) and if $f \in H^1(\Omega)$ then it follows from Theorem 5.1.3.5 in [5] that $u_0 \in H^{2+\omega}(\Omega)$ for some $\omega > 0$. Here we use that all the interior angles satisfy $0 < \alpha_i < \pi$. Theorem 1 therefore automatically guarantees that

$$||u_\epsilon(\cdot) - u_0(\cdot) - \epsilon u_1(\cdot, \cdot/\epsilon) - \epsilon B_\epsilon(\cdot)||_{L^2(\Omega)} = o(\epsilon) \quad .$$

We expect that an analysis very similar to that in [5] would give $u_0 \in H^{2+\omega}(\Omega)$, for some $\omega > 0$, even when $\Omega$ is a bounded, convex curvilinear polygon. As a consequence we expect that the above estimate would remain valid (for $f \in H^1(\Omega)$ without any explicit assumptions on $u_0$). $\square$

# 3 Application of Osborn's Formula

We will first state an eigenvalue approximation result due to Osborn [14], and then we proceed to use this result to derive a representation formula for the first order eigenvalue correction. The procedure here is virtually identical to that for the Dirichlet problem, and we refer to [11] for more details.

Suppose $X$ is a Banach space and $T_n : X \to X$ is a sequence of compact linear operators such that $T_n \to T$ pointwise, where $T$ is also a compact linear operator from $X$ to itself. Suppose furthermore that the sequence $\{T_n\}$ is collectively compact in the sense that the set $\{T_n f : ||f|| \leq 1 \; n = 1, 2, ...\}$ is precompact. Finally, suppose that the adjoints $T_n^* \to T^*$ pointwise and that the sequence $\{T_n^*\}$ is also collectively compact.

Let $\mu$ be a nonzero eigenvalue of $T$ of algebraic multiplicity $m$. Under the above assumptions it is well known that, for $n$ large enough, there exist $m$ eigenvalues of $T_n$, $\mu_1^n, ...\mu_m^n$ (counted according to algebraic multiplicity) such that $\mu_j^n \to \mu$ as $n \to \infty$, for each $1 \leq j \leq m$ .

Let $E$ be the spectral projection onto the generalized eigenspace of $T$ corresponding to the eigenvalue $\mu$. The space $X$ can be decomposed in terms of the range and the null space of $E$: $X = R(E) \bigoplus N(E)$, *i.e.* any $f \in X$ can uniquely be written as $f = g + h$, $g \in R(E)$, $h \in N(E)$. For $\phi^* \in R(E)^*$ (the dual space of $R(E)$) we extend $\phi^*$ to all of $X$ by $\phi^* f = \phi^* E f$.

**Theorem 2** *(Osborn) Let $\phi_1, \phi_2, ...\phi_m$ be a basis for $R(E)$ and let $\phi_1^*, \phi_2^*, ...\phi_m^*$ denote the corresponding dual basis (each $\phi_j^*$ extended as above). There exists a constant $C$ such that*

$$\begin{aligned} |\mu - \frac{1}{m}\Sigma_{j=1}^m \mu_j^n - \frac{1}{m}\Sigma_{j=1}^m < (T - T_n)\phi_j, \phi_j^* > | \\ \leq C||(T - T_n)|_{R(E)}|| \cdot ||(T^* - T_n^*)|_{R(E^*)}|| \end{aligned} \qquad (36)$$

We now use this theorem to derive a formula for the first order eigenvalue corrections. Let $X$ be the real Hilbert space $\tilde{L}^2(\Omega) = \{f \in L^2(\Omega) \ : \ \int_\Omega f \ dx = 0\}$. For a given $f \in X$, we define $T_\epsilon f = u_\epsilon \in H^1(\Omega)$ to be the variational solution to

$$-\nabla \cdot a(x/\epsilon)\nabla u_\epsilon = f \ \text{ in } \Omega \ , \quad a(x/\epsilon)\nabla u_\epsilon \cdot \nu = 0 \ \text{ on } \partial\Omega \ , \tag{37}$$

normalized by $\int_\Omega u_\epsilon \ dx = 0$. Similarly, we define $Tf = u_0$ to be the variational solution to the homogenized problem

$$-\nabla \cdot A\nabla u_0 = f \ \text{ in } \Omega \ , \quad A\nabla u_0 \cdot \nu = 0 \ \text{ on } \partial\Omega \ , \tag{38}$$

normalized by $\int_\Omega u_0 \ dx = 0$. The operators $T_\epsilon$ and their limit $T$ are compact, self adjoint operators and they satisfy all of the hypotheses of the theorem for any sequence $\epsilon_n \to 0$ (according to Corollary 1, and the fact that $\|u_0\|_{H^2(\Omega)} \leq C\|f\|_{L^2(\Omega)}$, we actually have that $T_\epsilon$ converge to $T$ in the operator norm). It is easy to see that $\lambda$ is a nonzero eigenvalue for (2) if and only if $1/\lambda$ is an eigenvalue of $T$. Similarly, $\lambda_\epsilon$ is a nonzero eigenvalue for (1) if and only if $1/\lambda_\epsilon$ is an eigenvalue of $T_\epsilon$.

For the remainder of this section we assume that $\lambda$ is a simple, nonzero eigenvalue for (2) and we assume that the corresponding eigenfunction $v$ is in $H^{2+\omega}(\Omega)$ for some $0 < \omega \leq 1$. Let $v$ be normalized by the requirement that $\|v\|_{L^2(\Omega)} = 1$. We use the same notation as above, only with $f$ replaced by $v$, so that $u_0 = Tv$ and $u_\epsilon = T_\epsilon v$; since $Tv = \frac{1}{\lambda}v$ we note that $u_0 = \frac{1}{\lambda}v$. Osborn's formula yields

$$\left|\frac{1}{\lambda} - \frac{1}{\lambda_\epsilon} - < u_0 - u_\epsilon, v > \right| \leq C\|u_0 - u_\epsilon\|^2_{L^2(\Omega)} \ , \tag{39}$$

where $\lambda_\epsilon$ is the family of eigenvalues of $T_\epsilon$ that converge towards $\lambda$ (it makes no difference that $T_\epsilon$ is parametrized by a continuous parameter). From Corollary 1 it immediately follows that

$$\left|\frac{1}{\lambda} - \frac{1}{\lambda_\epsilon} - < u_0 - u_\epsilon, v > \right| \leq C\epsilon^2 \ . \tag{40}$$

Since we assume that $u_0$ is in $H^{2+\omega}(\Omega)$ for some $0 < \omega \leq 1$, we may use Theorem 1 to estimate the term $< u_0 - u_\epsilon, v >$

$$\begin{aligned} < u_0 - u_\epsilon, v > &= \int_\Omega (u_0 - u_\epsilon + \epsilon u_1 + \epsilon B_\epsilon)v \ dx - \int_\Omega (\epsilon u_1 + \epsilon B_\epsilon)v \ dx \\ &= -\int_\Omega (\epsilon u_1 + \epsilon B_\epsilon)v \ dx \quad + O(\epsilon^{1+\frac{\omega}{2}}) \ . \end{aligned}$$

The cell functions $\chi^j(y)$ have only been determined up to an additive constant; for simplicity let us now select them to have integral zero over a period cell. In this case one may find (smooth) $Y$-periodic functions $b^j(y)$ that satisfy $\Delta_y b^j(y) = \chi^j(y)$, and therefore

$$\int_\Omega u_1 v \ dx \quad = \quad -\int_\Omega \chi^j(x/\epsilon)\frac{\partial u_0}{\partial x_j}v \ dx$$

$$
\begin{aligned}
&= -\int_\Omega \epsilon^2 \Delta_x b^j(x/\epsilon) \frac{\partial u_0}{\partial x_j} v \; dx \\
&= \epsilon \int_\Omega \epsilon \nabla_x b^j(x/\epsilon) \cdot \nabla \left( \frac{\partial u_0}{\partial x_j} v \right) \; dx - \epsilon \int_{\partial\Omega} \epsilon \nabla_x b^j(x/\epsilon) \cdot \nu \frac{\partial u_0}{\partial x_j} v \; ds \quad .
\end{aligned}
$$

It follows that

$$
< u_0 - u_\epsilon, v >= - \int_\Omega \epsilon B_\epsilon v \; dx + O(\epsilon^{1+\frac{\omega}{2}}) \quad .
$$

In combination with (40) this gives

$$
\frac{1}{\lambda} - \frac{1}{\lambda_\epsilon} + \epsilon \int_\Omega B_\epsilon v \; dx = O(\epsilon^{1+\frac{\omega}{2}})
$$

that is

$$
\lambda_\epsilon - \lambda = -\epsilon \lambda_\epsilon \lambda \int_\Omega B_\epsilon v \; dx + O(\epsilon^{1+\frac{\omega}{2}}) \quad .
$$

From Lemma 3 it now follows that

$$
|\lambda_\epsilon - \lambda| \leq C\epsilon \quad .
$$

Inserting this into the previous identity and using Lemma 3 once more we get

$$
\frac{\lambda_\epsilon - \lambda}{\epsilon} = -\lambda^2 \int_\Omega B_\epsilon v \; dx + O(\epsilon^{\frac{\omega}{2}}) \quad . \tag{41}
$$

The first order corrections, that we seek to determine, are therefore simply all the possible limit points of the expression $-\lambda^2 \int_\Omega B_\epsilon v \; dx$. Remember that $u_0 = \frac{1}{\lambda} v$ and that $u_0$ was used to define $B_\epsilon$. Notice also that the above formula is similar to that derived for the case of Dirichlet eigenvalues (*cf.* [11]).

# 4 Eigenvalue Corrections in One Dimension

In [15] we started our study of the first order corrections to the Dirichlet eigenvalues with a detailed analysis of the one dimensional case (with $\Omega = (0,1)$). This case served as a simple illustration of what occured in two dimensions for a convex polygon (see [15] and [11]). Given a sequence $\epsilon_k \to 0$ with

$$
\frac{1}{\epsilon_k} = M_k + \delta_k \; , \quad M_k \; \text{ integer} \; , \quad 0 \leq \delta_k < 1
$$

and

$$
\delta_k \to \delta_0 \quad .
$$

we showed that the first order correction to any Dirichlet eigenvalue $\lambda$ has the form

$$
\lambda_1 = \lim_{\epsilon_k \to 0} \frac{\lambda_{\epsilon_k} - \lambda}{\epsilon_k} = 2\lambda[\chi(\delta_0) - \chi(0)] \quad ,
$$

where $\chi$ is the (single) periodic cell function defined by (3). Generically this correction depends on the sequence $\epsilon_k$ (more specifically it depends on the value of $\delta_0$). There is not a single correction, but a multitude of limit points.

We shall now consider the first order corrections to the nontrivial, homogenized Neumann eigenvalues in the one dimensional case. The problem (1) reduces to

$$-(a(x/\epsilon)v_\epsilon')' = \lambda_\epsilon v_\epsilon \quad \text{in } \Omega = (0,1)$$
$$v_\epsilon'(0) = v_\epsilon'(1) = 0 \quad ,$$

with the corresponding homogenized problem

$$-\overline{a^{-1}}^{-1} v'' = \lambda v \quad \text{in } \Omega = (0,1)$$
$$v'(0) = v'(1) = 0 \quad .$$

Any nontrivial, homogenized eigenvalue has the form $\lambda = \overline{a^{-1}}^{-1} n^2\pi^2$ for some integer $n \geq 1$, and as a corresponding eigenfunction we have

$$v = \sqrt{2}\cos n\pi x \quad .$$

In line with our earlier notation we let $u_\epsilon$ denote the solution to

$$-(a(x/\epsilon)u_\epsilon')' = v \quad \text{in } \Omega = (0,1)$$
$$u_\epsilon'(0) = u_\epsilon'(1) = 0 \quad ,$$

with $\int_0^1 u_\epsilon \, dx = 0$. We are able to compute $u_\epsilon$ explicitly

$$\begin{aligned} (a(x/\epsilon)u_\epsilon')' &= -\sqrt{2}\cos n\pi x \\ a(x/\epsilon)u_\epsilon' &= -\frac{\sqrt{2}}{n\pi}\sin n\pi x + c_\epsilon \quad . \end{aligned}$$

The boundary conditions $u_\epsilon'(0) = u_\epsilon'(1) = 0$ imply $c_\epsilon = 0$, so

$$u_\epsilon(z) = -\frac{\sqrt{2}}{n\pi}\int_0^z \frac{\sin n\pi x}{a(x/\epsilon)}\, dx + d_\epsilon \quad .$$

For $u_0$ (the limit of $u_\epsilon$) we correspondingly have

$$u_0(z) = -\frac{\sqrt{2}}{n\pi}\,\overline{a^{-1}}\int_0^z \sin n\pi x \, dx + d_0 \quad .$$

Using Osborn's eigenvalue estimate (Theorem 2) we get

$$|\frac{1}{\lambda} - \frac{1}{\lambda_\epsilon}| \leq |< u_0 - u_\epsilon, v >| + C\|u_0 - u_\epsilon\|^2_{L^2(0,1)} \quad . \tag{42}$$

According to our previous calculations

$$\begin{aligned} u_0(z) - u_\epsilon(z) &= -\frac{\sqrt{2}}{n\pi}\,\overline{a^{-1}} \int_0^z \sin n\pi x \; dx + \frac{\sqrt{2}}{n\pi} \int_0^z \frac{\sin n\pi x}{a(x/\epsilon)} \; dx + d_0 - d_\epsilon \\ &= \frac{\sqrt{2}}{n\pi} \int_0^z \left[ a^{-1}(x/\epsilon) - \overline{a^{-1}} \right] \sin n\pi x \; dx + d_0 - d_\epsilon \quad . \end{aligned}$$

Define

$$\frac{d^2 b}{dy^2}(y) = a^{-1}(y) - \overline{a^{-1}} \quad ,$$

so that

$$\epsilon^2 \frac{d^2}{dx^2} b(x/\epsilon) = a^{-1}(x/\epsilon) - \overline{a^{-1}} \quad ,$$

and therefore

$$\begin{aligned} < u_0 - u_\epsilon, v > &= \frac{2}{n\pi} \int_0^1 \int_0^z \epsilon^2 \frac{d^2}{dx^2} b(x/\epsilon) \sin n\pi x \; dx \; \cos n\pi z \; dz \\ &= \frac{2}{n\pi} \int_0^1 \epsilon^2 \frac{d^2}{dx^2} b(x/\epsilon) \sin n\pi x \int_x^1 \cos n\pi z \; dz \; dx \\ &= -\frac{2}{n^2\pi^2} \int_0^1 \epsilon^2 \frac{d^2}{dx^2} b(x/\epsilon) \sin^2 n\pi x \; dx \quad . \end{aligned}$$

After integration by parts twice

$$< u_0 - u_\epsilon, v > = -\frac{2}{n\pi} \int_0^1 \epsilon^2 b(x/\epsilon) \frac{d}{dx} \left[ 2 \sin n\pi x \cos n\pi x \right] \; dx \quad , \tag{43}$$

since all the boundary terms vanish. We thus have

$$< u_0 - u_\epsilon, v > = O(\epsilon^2) \quad .$$

Noting that Corollary 1 also holds in one dimension, we conclude by a combination of (42) and (43) that

$$\left|\frac{1}{\lambda} - \frac{1}{\lambda_\epsilon}\right| = O(\epsilon^2) \quad ,$$

and hence

$$|\lambda_\epsilon - \lambda| = O(\epsilon^2) \quad .$$

This implies that the first order corrections to all Neumann eigenvalues are zero, independent of the sequence $\epsilon_k \to 0$. The phenomenon that occured for the Dirichlet eigenvalues does not occur for the one dimensional Neumann eigenvalues. Numerical computations demonstrate this difference. We used a shooting method to compute the eigenvalues for the two problems for a specific choice of $a(\cdot)$ (Figures 1 and 2). Our coefficient, $a(\cdot)$, is defined as the periodic extension of the step function

$$a(y) = \begin{cases} 1 & 0 \le y < 1/2 \\ 3 & 1/2 \le y < 1 \end{cases}$$

The small dots in the figures represent a "generic" sequence $\epsilon_k$, that is, one for which $\delta_k$ does not have a limit. The other symbols each represent a sequence with fixed $\delta_k = \delta_0$ (for different values of $\delta_0$). In Figure 1 one clearly sees the different slopes of the lines (the different values of the first order correction) for different values of $\delta_0$.

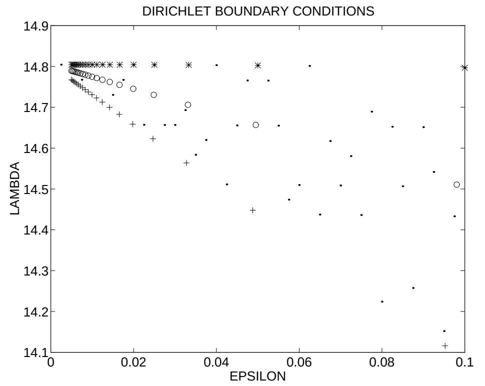


Figure 1

By contrast, we see the quadratic convergence of the Neumann eigenvalues in Figure 2.

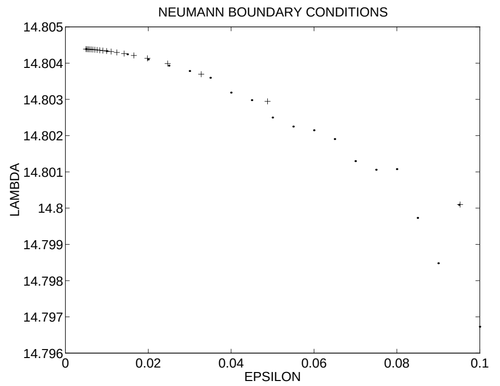


Figure 2

One might expect that the first order Neumann eigenvalue corrections are somehow always zero, also in higher dimensions, *but this is not the case.* In the following section we show that, for a two dimensional convex polygon with sides of rational (or infinite) slope, the Neumann corrections display a behaviour much like that of the Dirichlet eigenvalue corrections. That is, the values of the corrections depend crucially on the interaction of the microstructure with the boundary.

## 5 The Case of Convex Polygons

In this section we obtain an explicit formula for the first order eigenvalue correction when the domain $\Omega \subset I\!R^2$ is a classical, convex polygon with sides that have rational (or infinite) slopes. We exploit the duality between the Neumann- and the Dirichlet boundary corrector functions, and then we we use a theorem from [11] (and [15]) about the $L^2$ limit points of the Dirichlet boundary corrector functions.

Since we are restricting attention to convex polygons in two dimensions, it is well known that the eigenfunctions of (2) are more regular than $H^2$. It follows from Theorem 5.1.3.5 of [5] with homogeneous Neumann boundary data and right hand side $\lambda v$ that any Neumann eigenfunction of (2) is in $H^{2+\omega}(\Omega)$ for some $0 < \omega \leq 1$. It is absolutely essential here that the polygon be convex; $\omega$ depends on the interior angles of the domain, and in general it approaches zero if any angle approaches $\pi$. Suppose now the homogenized eigenvalue $\lambda$ is simple. The above type of regularity is exactly what is required for the formula (41) to hold, therefore we have

$$\frac{\lambda_\epsilon - \lambda}{\epsilon} = -\lambda^2 \int_\Omega B_\epsilon v \; dx + O(\epsilon^{\frac{\omega}{2}}) \quad ,$$

where $B_\epsilon$ is the solution to the Neumann boundary value problem (11) with $u_0 = \frac{1}{\lambda} v$ $(f = v)$.

We wish to obtain a quite explicit formula for the limit points of these ratios. At this point it becomes essential for our analysis that we assume that the sides of the polygon have rational (or infinite) slopes. Consider the functions $C_\epsilon$ defined by the Dirichlet boundary value problems (19) with $u_0 = \frac{1}{\lambda} v$. These functions are quite similar to the the boundary corrector functions for the Dirichlet eigenvalue problem. Indeed, it follows directly from the proof of Proposition 3.1 in [15] that, given any arbitrary sequence, one may always extract a subsequence $\epsilon_k$ so that the funtions $C_{\epsilon_k}$ converge in $L^2(\Omega)$ to a function $C_*$, which (is in $H^1(\Omega)$ and) solves a boundary value problem of the form

$$\nabla \cdot \tilde{A} \nabla C_* = 0 \qquad \text{in} \qquad \Omega \tag{44}$$

$$C_* = -r_*^j \frac{\partial u_0}{\partial x_j} = -\frac{1}{\lambda} r_*^j \frac{\partial v}{\partial x_j} \qquad \text{on} \qquad \partial\Omega \quad . \tag{45}$$

We also refer the reader to Theorem 5 in [11]. Going back to (19) and (20) we can rewrite the boundary condition for $C_\epsilon$

$$\begin{aligned} C_\epsilon = -r^j(x/\epsilon) \frac{\partial u_0}{\partial x_j} &= -\tilde{\chi}^i(x/\epsilon) Q_{il} A_{lj} \frac{\partial u_0}{\partial x_j} \\ &= -\tilde{\chi}^i(x/\epsilon) Q_{il} \; \tau_l (A \nabla u_0) \cdot \tau \\ &= \tilde{\chi}^i(x/\epsilon) \nu_i (A \nabla u_0) \cdot \tau \qquad \text{on } \partial\Omega \quad , \end{aligned} \tag{46}$$

where $\tilde{\chi}^i$ are the standard cell functions corresponding to the coefficient $\tilde{a}(y) = Q^t a^{-1}(y) Q$. In the second equality we have used that $A\nabla u_0 \cdot \nu = 0$ on $\partial\Omega$. The boundary condition for the limit $C_* = \lim_{\epsilon_k \to 0} C_{\epsilon_k}$ may now also be written

$$C_* = \tilde{\chi}_*(A\nabla u_0) \cdot \tau \quad \text{on } \partial\Omega \quad , \tag{47}$$

a form which is very similar to that of Theorem 5 in [11]. We should note that in the results in [15] and [11] the function $u_0 = \frac{1}{\lambda} v$ is a Dirichlet eigenfunction, whereas here it is a Neumann eigenfunction, but that is of absolute no relevance to the proofs. The constant matrix $\tilde{A}$ is the homogenized matrix corresponding to coefficient $\tilde{a}(y)$. The $r_*^j$'s and $\tilde{\chi}_*$ are constant on each side of the boundary of $\Omega$. In order to describe the exact values of the constants associated with the $r_*^j$'s and $\tilde{\chi}_*$ we need some more notation, which we now briefly introduce. This notation is completely identical to that of section 4 in [11], the reader may consult that paper for more background material.

Let $\Gamma$ be a particular side of $\partial\Omega$ with unit outward normal $\nu$, and let $N = (p, q)$ denote the minimal outward normal with integer entries. Such a normal exists since $\Gamma$ has rational (or infinite) slope. Functions that are periodic in $x$ with period $\epsilon$ are periodic with period $\epsilon(p^2 + q^2)^{1/2} = \epsilon T$ in the rotated coordinate system with basis vector $\nu$ and $\nu^\perp$, in particular they are periodic with period $\epsilon T$ on $\Gamma$. Suppose the line on which $\Gamma$ lies is given by $\nu \cdot x = s$, and let the fractional part of a number $t$ be defined by $\delta(t) = t - [t]$, where we briefly have used $[\cdot]$ to denote the integer part. The remainders $\delta_\epsilon = T\delta(\frac{s}{\epsilon T})$ will play a special role in determining $r_*^j$. Namely, suppose that the subsequence $\epsilon_k$ is such that, for each side of $\partial\Omega$, there exists a $0 \le \delta_0 \le T$ such that

$$\delta_{\epsilon_k} \to \delta_0 \quad \text{as} \quad \epsilon_k \to 0 \quad . \tag{48}$$

The value of $\delta_0$ changes from one side of $\partial\Omega$ to another. We note that, given any arbitrary sequence, this kind of limiting behaviour may be obtained by extraction of a subsequence. Let $P$ denote the halfplane

$$P = \{ y \ : \ -\nu \cdot y > 0 \ \} \quad ,$$

with boundary $\partial P = \{ y \ : \ -\nu \cdot y = 0 \ \}$ , and let $G$ denote the semi infinite strip

$$G = \{ y \ : \ -\nu \cdot y > 0 \ , \ 0 < -\nu^\perp \cdot y < T \ \} \quad .$$

Corresponding to each side $\Gamma$ we define functions $w_j$, $j = 1, 2$, by the requirements that $w_j$ be periodic, with period $T$, in the $\nu^\perp$ direction and satisfy

$$\nabla_y \cdot \tilde{a}(y + \delta_0 \nu) \nabla_y w_j = 0 \quad \text{in} \quad P \quad , \tag{49}$$

$$w_j = r^j(y + \delta_0 \nu) \quad \text{on} \quad \partial P \quad , \tag{50}$$

and

$$e^{\gamma(-\nu \cdot y)} \frac{\partial w_j}{\partial y_i} \in L^2(G) \quad i = 1, 2 \quad , \tag{51}$$

for some $\gamma > 0$. The existence and uniqueness of such solutions $w_j$ is well known [8]; it is also known that the $w_j$ has a constant limit as $-\nu \cdot y$ approaches $+\infty$ [11]. The value of $r_*^j$ on the side $\Gamma$ is exactly that constant

$$r_*^j|_\Gamma = \lim_{-\nu \cdot y \to +\infty} w_j(y) \quad .$$

The value of $\tilde{\chi}_*$ on the side $\Gamma$ is given by

$$\tilde{\chi}_*|_\Gamma = \lim_{-\nu \cdot y \to +\infty} w(y) \quad ,$$

where $w$ is the solution to (49)–(51) when $r^j(\cdot)$ is replaced by $\tilde{\chi}^i(\cdot)\nu_i$. Since the functions $r^j(\cdot)$ and $\tilde{\chi}^i(\cdot)$ are related by $r^j(\cdot) = \tilde{\chi}^i(\cdot)(QA)_{ij}$ it is not difficult to see that

$$\tilde{\chi}_*|_\Gamma = r_*^j|_\Gamma \xi_j \quad ,$$

where $\xi_j$ are the coordinates of the vector $\xi = -A^{-1}\tau$. The $r_*^j|_\Gamma$'s and $\tilde{\chi}_*|_\Gamma$ may be expressed as integrals over $G$ and its vertical boundary, *cf.* the appendix in [11]. For $\tilde{\chi}_*|_\Gamma$ one obtains for example

$$\tilde{\chi}_*|_\Gamma = \frac{1}{< \tilde{A}\nu, \nu >} \left( \frac{1}{T} \int_0^T \left[ \tilde{\chi}^i \tilde{a}_{jm} (\delta_{ml} - \frac{\partial \tilde{\chi}^l}{\partial y_m}) \nu_i \nu_j \nu_l |_{y_1' = -\delta_0} \right] \, dy_2' \right.$$
$$\left. + \frac{1}{T} \int_G \tilde{a}(y + \delta_0 \nu) \nabla w \cdot \nabla w \; dy \right) \quad , \quad (52)$$

where $w$ again is the $y_2'$ periodic solution to (49)–(51) when $r^j(\cdot)$ is replaced by $\tilde{\chi}^i(\cdot)\nu_i$. Here we have used the notation $y_1' = -\nu \cdot y$, $y_2' = -\nu^\perp \cdot y$ for a new set of rotated coordinates. When $\tilde{\chi}_*|_\Gamma$ is decomposed as (52) the first term may also be written as

$$\text{weak} \lim_{\epsilon_k \to 0} \left[ \tilde{\chi}^i \tilde{a}_{jm} (\delta_{ml} - \frac{\partial \tilde{\chi}^l}{\partial y_m}) \nu_i \nu_j \nu_l \right] (x/\epsilon_k)|_\Gamma \quad .$$

The boundary value problem (44), (45) (or (44), (47)) with all possible $r_*^j$ (or $\chi_*$) inserted, characterizes the limit points of the functions $C_\epsilon$ (taking different subsequences $\epsilon_k \to 0$). We shall devote the rest of this section to finding a simple expression for the first order eigenvalue correction

$$\lambda_1 = \lim_{\epsilon_k \to 0} \frac{\lambda_{\epsilon_k} - \lambda}{\epsilon_k} = -\lambda^2 \lim_{\epsilon_k \to 0} \int_\Omega B_{\epsilon_k} v \; dx \qquad (53)$$

in terms of $C_*$ and the eigenfunction (or rather in terms of their Dirichlet boundary data). In doing so we shall always assume that (48) is satisfied for all sides of $\Omega$. As a first step in this direction we prove the following lemma

**Lemma 5** *Suppose $\Omega$ is a convex polygon with sides of rational (or infinite) slopes. Let $C_{\epsilon_k}$ be a convergent sequence of solutions to the boundary value problem (19), and let $C_*$ be the solution to the corresponding boundary value problem (44), (45). Then*

$$\tilde{a}(x/\epsilon)\nabla C_\epsilon \to \tilde{A}\nabla C_*$$

*in $\left[H^{-1}(\Omega)\right]^2$.*

*Proof* Let $\mathbf{F}$ be given in $\left[H_0^1(\Omega)\right]^2$ and let $\tilde{\chi}^j$ be the cell functions for the coefficient $\tilde{a}(\cdot)$, then

$$\int_\Omega \tilde{a}(x/\epsilon)\nabla C_\epsilon \cdot \mathbf{F}\ dx = \int_\Omega (\tilde{a}_{ij} - \tilde{a}_{jk}\frac{\partial \tilde{\chi}^i}{\partial y_k})(x/\epsilon)\frac{\partial C_\epsilon}{\partial x_j}F_i\ dx + \int_\Omega \tilde{a}_{jk}\frac{\partial \tilde{\chi}^i}{\partial y_k}(x/\epsilon)\frac{\partial C_\epsilon}{\partial x_j}F_i\ dx \quad . \tag{54}$$

Since

$$\frac{\partial}{\partial x_j}\left((\tilde{a}_{ij} - \tilde{a}_{jk}\frac{\partial \tilde{\chi}^i}{\partial y_k})(x/\epsilon)\right) = 0$$

for each $i$, integration by parts yields

$$\int_\Omega (\tilde{a}_{ij} - \tilde{a}_{jk}\frac{\partial \tilde{\chi}^i}{\partial y_k})(x/\epsilon)\frac{\partial C_\epsilon}{\partial x_j}F_i\ dx = -\int_\Omega (\tilde{a}_{ij} - \tilde{a}_{jk}\frac{\partial \tilde{\chi}^i}{\partial y_k})(x/\epsilon)C_\epsilon\frac{\partial F_i}{\partial x_j}\ dx \quad . \tag{55}$$

The boundary terms disappear since $\mathbf{F}$ is zero on $\partial\Omega$. We know that

$$(\tilde{a}_{ij} - \tilde{a}_{jk}\frac{\partial \tilde{\chi}^i}{\partial y_k})(x/\epsilon) \rightharpoonup \tilde{A}_{ij}$$

weak* in $L^\infty(\Omega)$ as $\epsilon \to 0$, and we also know that $C_{\epsilon_k} \to C_*$ (strongly) in $L^2(\Omega)$. The integral (55), which is exactly the first term on the right hand side of (54), therefore converges to

$$-\int_\Omega \tilde{A}_{ij}C_*\frac{\partial F_i}{\partial x_j}\ dx = \int_\Omega \tilde{A}\nabla C_* \cdot \mathbf{F}\ dx \quad ,$$

along the sequence $\epsilon_k$. In order to complete the proof of this lemma it only remains to prove that the second term on the right hand side of (54) goes to zero as $\epsilon_k \to 0$. Towards this end we have

$$\begin{aligned}
\int_\Omega \tilde{a}_{jk}(x/\epsilon)\frac{\partial \tilde{\chi}^i}{\partial y_k}(x/\epsilon)\frac{\partial C_\epsilon}{\partial x_j}F_i\ dx &= \epsilon\int_\Omega \tilde{a}_{jk}(x/\epsilon)\frac{\partial}{\partial x_k}\tilde{\chi}^i(x/\epsilon)\frac{\partial C_\epsilon}{\partial x_j}F_i\ dx \\
&= \epsilon\int_\Omega \tilde{a}_{jk}(x/\epsilon)\frac{\partial C_\epsilon}{\partial x_j}\frac{\partial}{\partial x_k}\left(\tilde{\chi}^i(x/\epsilon)F_i\right)\ dx \\
&\quad -\epsilon\int_\Omega \tilde{a}_{jk}(x/\epsilon)\frac{\partial C_\epsilon}{\partial x_j}\tilde{\chi}^i(x/\epsilon)\frac{\partial F_i}{\partial x_k}\ dx \\
&= -\epsilon\int_\Omega \tilde{a}_{jk}(x/\epsilon)\frac{\partial C_\epsilon}{\partial x_j}\tilde{\chi}^i(x/\epsilon)\frac{\partial F_i}{\partial x_k}\ dx \ ,
\end{aligned}$$

where we have used the equation for $C_\epsilon$ and the fact that $F_i = 0$ on $\partial\Omega$ to derive the last identity. Recall that

$$||C_\epsilon||_{H^1(\Omega)} \leq C\epsilon^{-1/2},$$

and so the second term on the right hand side of (54) is clearly bounded by $C\epsilon^{1/2}$, in particular it goes to zero with $\epsilon$. $\Box$

As stated previously the first order corrections to the Neumann eigenvalues are the limit points of the expression

$$-\lambda^2 \int_\Omega B_\epsilon v \ dx \quad ,$$

as $\epsilon \to 0$. As before $u_0 = Tv (= \frac{1}{\lambda} v)$ is the solution to (6) with right hand side $f = v$, and thus

$$\int_\Omega B_\epsilon v \ dx = \int_\Omega \nabla B_\epsilon \cdot (A\nabla u_0) \ dx \quad . \tag{56}$$

We now use a result from [1] to give us the existence of a vector field $G \in \left[H^1(\Omega)\right]^2$ satisfying

$$\begin{aligned} \nabla \cdot G &= 0 \quad \text{in} \quad \Omega \\ G &= A\nabla u_0 \quad \text{on} \quad \partial\Omega \quad . \end{aligned}$$

The existence of $G$ follows immediately from Theorem 6.2 in [1] (with $s = 2$ and $p = 2$) since $A\nabla u_0 \in \left[H^{1/2}(\partial\Omega)\right]^2$ and $A\nabla u_0 \cdot \nu = 0$ on $\partial\Omega$. Note that since $G$ has divergence zero and $G \cdot \nu$ is zero on the boundary, $G$ is orthogonal to any gradient (in $L^2$). This, in combination with (16) and (56), now gives

$$\begin{aligned} \int_\Omega B_\epsilon v \ dx &= \int_\Omega \nabla B_\epsilon \cdot (A\nabla u_0 - G) \ dx \\ &= \int_\Omega a^{-1}(x/\epsilon) \nabla^\perp C_\epsilon \cdot (A\nabla u_0 - G) \ dx \quad . \end{aligned}$$

From the definition of $\tilde{a}$

$$a^{-1}(x/\epsilon)\nabla^\perp C_\epsilon = (\tilde{a}(x/\epsilon)\nabla C_\epsilon)^\perp$$

and since $(A\nabla u_0 - G)$ is in $\left[H_0^1(\Omega)\right]^2$, we can now use the previous lemma to get

$$\begin{aligned} \lim_{\epsilon_k \to 0} \int_\Omega B_{\epsilon_k} v \ dx &= \int_\Omega (\tilde{A}\nabla C_*)^\perp \cdot (A\nabla u_0 - G) \ dx \\ &= \int_\Omega (\tilde{A}\nabla C_*)^\perp \cdot (A\nabla u_0) \ dx - \int_\Omega (\tilde{A}\nabla C_*)^\perp \cdot G \ dx \quad . \end{aligned} \tag{57}$$

Since

$$\nabla \cdot \tilde{A} \nabla C_* = 0 \quad \text{in } \Omega \quad ,$$

there exists some function $D \in H^1(\Omega)$ with

$$(\tilde{A} \nabla C_*)^\perp = \nabla D \quad .$$

This implies that the second integral in (57) is zero, and hence

$$\lim_{\epsilon_k \to 0} \frac{\lambda_{\epsilon_k} - \lambda}{\epsilon_k} = -\lambda^2 \lim_{\epsilon_k \to 0} \int_\Omega B_{\epsilon_k} v \; dx = -\lambda^2 \int_\Omega (\tilde{A} \nabla C_*)^\perp \cdot A \nabla u_0 \; dx \quad . \tag{58}$$

**Lemma 6** *Let $Q$ be the rotation matrix $Q = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and let $A$ and $\tilde{A}$ be the homogenized matrices for $a(x/\epsilon)$ and $\tilde{a}(x/\epsilon)$ respectively. Then*

$$AQ\tilde{A} = Q.$$

*Proof* From the definition (7) of $r^j(y)$ and (20) we know that

$$\left[\nabla_y^\perp (\tilde{\chi}^i(y)(QA)_{ij})\right]_k = A_{kj}(y) - A_{kj}$$

or

$$Q_{kp} \frac{\partial \tilde{\chi}^i}{\partial y_p} (QA)_{ij} = A_{kj}(y) - A_{kj}.$$

Multiplying both sides by $Q^t a^{-1}(y)$ we get

$$\tilde{a}_{mp}(y) \frac{\partial \tilde{\chi}^i}{\partial y_p} (QA)_{ij} = Q_{lm}(a^{-1})_{lk}(y)(A_{kj}(y) - A_{kj}) \quad ,$$

which gives us

$$\begin{aligned} -\tilde{a}_{mi}(y)(QA)_{ij} \quad &+ \quad \tilde{a}_{mp}(y) \frac{\partial \tilde{\chi}^i}{\partial y_p}(QA)_{ij} \\ &= \quad -\tilde{a}_{mi}(y)(QA)_{ij} + Q_{lm}(a^{-1})_{lk}(y)(A_{kj}(y) - A_{kj}) \quad , \end{aligned}$$

or

$$\begin{aligned} -\tilde{A}_{mi}(y)(QA)_{ij} \quad &= \quad -(Q^t a^{-1}(y) Q)_{mi}(QA)_{ij} - (Q^t a^{-1}(y) A)_{mj} \\ &\quad\; + Q_{lm}(a^{-1})_{lk}(y) A_{kj}(y) \quad . \end{aligned}$$

The first two terms on the right hand side cancel, so

$$\begin{aligned} -\tilde{A}_{mi}(y)(QA)_{ij} \quad &= \quad Q_{lm}(a^{-1})_{lk}(y) \left[ a_{kp}(y)\delta_{pj} - a_{kp}(y) \frac{\partial \chi^j}{\partial y_p} \right] \\ &= \quad Q_{lm}\delta_{lp}\delta_{pj} - Q_{lm}\delta_{lp} \frac{\partial \chi^j}{\partial y_p} \quad , \end{aligned}$$

and consequently

$$-\tilde{A}_{mi}(y)(QA)_{ij} = Q_{jm} - Q_{pm}\frac{\partial \chi^j}{\partial y_p} \quad .$$

By taking $Y$-averages of both sides we get

$$-\tilde{A}QA = Q^t \quad .$$

Transposition of this identity finally leads to

$$AQ\tilde{A} = -A^tQ^t\tilde{A}^t = (-\tilde{A}QA)^t = Q \quad ,$$

as desired. □

From the above lemma,

$$A(\tilde{A}\nabla C_*)^\perp = AQ\tilde{A}\nabla C_* = \nabla^\perp C_* \quad ,$$

which in combination with (58) and (45) gives

$$\begin{aligned}
\lambda_1 = \lim_{\epsilon_k \to 0} \frac{\lambda_{\epsilon_k} - \lambda}{\epsilon_k} &= -\lambda^2 \int_\Omega \nabla^\perp C_* \cdot \nabla u_0 \; dx \\
&= -\lambda^2 \int_{\partial\Omega} C_* \frac{\partial u_0}{\partial \tau} \; ds \\
&= \lambda^2 \int_{\partial\Omega} r_*^j \frac{\partial u_0}{\partial x_j} \frac{\partial u_0}{\partial \tau} \; ds \\
&= \int_{\partial\Omega} r_*^j \frac{\partial v}{\partial x_j} \frac{\partial v}{\partial \tau} \; ds \quad .
\end{aligned} \tag{59}$$

If we take the alternate form, (47), of the boundary condition for $C_*$ and insert this into (58) we arrive at

$$\begin{aligned}
\lambda_1 &= -\lambda^2 \int_{\partial\Omega} C_* \frac{\partial u_0}{\partial \tau} \; ds \\
&= -\lambda^2 \int_{\partial\Omega} \tilde{\chi}_* A\nabla u_0 \cdot \tau \frac{\partial u_0}{\partial \tau} \; ds \\
&= -\int_{\partial\Omega} \tilde{\chi}_* A\nabla v \cdot \tau \frac{\partial v}{\partial \tau} \; ds \quad .
\end{aligned} \tag{60}$$

Simple algebraic manipulations give that

$$A\nabla v \cdot \tau = \alpha(\tau, \nu)\frac{\partial v}{\partial \tau} \quad ,$$

with

$$\alpha(\tau, \nu) = \frac{< A\tau, \tau >< A\nu, \nu > - < A\nu, \tau >^2}{< A\nu, \nu >} \quad .$$

In combination with (60) this yields

$$\lambda_1 = \lim_{\epsilon_k \to 0} \frac{\lambda_{\epsilon_k} - \lambda}{\epsilon_k} = -\int_{\partial\Omega} \tilde{\chi}_* \ \alpha(\tau,\nu) \left[\frac{\partial v}{\partial \tau}\right]^2 \ ds \ . \tag{61}$$

This clearly demonstrates the similarity between the first order Dirichlet and first order Neumann eigenvalue corrections. For the first order corrections, $\lambda_1^D$, to a simple Dirichlet eigenvalue we proved in [11] that

$$\lambda_1^D = \int_{\partial\Omega} \chi_* < A\nu, \nu > \left[\frac{\partial v^D}{\partial \nu}\right]^2 \ ds \ ,$$

where $v^D$ is the (normalized) homogenized Dirichlet eigenfunction, and $\chi_*$ is defined exactly as $\tilde{\chi}_*$ only with $\tilde{a}(y)$ replaced by $a(y)$ and $\tilde{\chi}^i(y)$ replaced by $\chi^i(y)$.

## 6 A Simple Example

In this section we present a simple example which demonstrates that the first order Neumann eigenvalue correction is not always zero. In fact, we observe the same phenomenon as in the Dirichlet problem, namely that (for a polygon with sides of rational or infinite slope) the correction depends on the interaction of the periodic microstructure with the boundary. We choose as our domain

$$\Omega = (0,1) \times (0,1)$$

and take our periodic coefficient matrix to be isotropic and to depend only on one variable, *i.e.*, $a$ is a scalar function of the form

$$a = a(y_1) \ .$$

Clearly, $\chi^2 = 0$, $\chi^1 = \chi^1(y_1)$, and

$$A(y) = \begin{pmatrix} \overline{a^{-1}}^{-1} & 0 \\ 0 & a(y_1) \end{pmatrix}$$

so that the homogenized eigenvalue problem becomes

$$\overline{a^{-1}}^{-1} \frac{\partial^2 v}{\partial x_1^2} + \overline{a} \frac{\partial^2 v}{\partial x_2^2} = -\lambda v \ ,$$

with the boundary condition

$$\overline{a^{-1}}^{-1} \frac{\partial v}{\partial x_1} \nu_1 + \overline{a} \frac{\partial v}{\partial x_2} \nu_2 = 0 \ .$$

In view of the special form of $\Omega$ these boundary conditions translate into

$$\frac{\partial v}{\partial x_1}(0,x_2) = \frac{\partial v}{\partial x_1}(1,x_2) = 0 \quad \text{and} \quad \frac{\partial v}{\partial x_2}(x_1,0) = \frac{\partial v}{\partial x_2}(x_1,1) = 0 \quad .$$

Consider

$$v = 2\cos \pi x_1 \cos \pi x_2.$$

One easily checks that $v$ satisfies the equation and the boundary conditions, and hence is a Neumann eigenfunction, corresponding to $\lambda = \overline{a^{-1}}^{-1}\pi^2 + \overline{a}\pi^2$. Indeed $\lambda$ is the smallest eigenvalue with a corresponding eigenfunction that is truly a function of two variables. It is not difficult to see that that if $\overline{a} \neq (k^2-1)\overline{a^{-1}}^{-1}$, for all integer $k$, then $\lambda$ is simple. For the remainder of this section we suppose that this assumption holds true. Note that we have chosen $v$ so that $\|v\|_{L^2(\Omega)} = 1$. We now use the formula (59) to compute the possible corrections $\lambda_1 = \lim_{\epsilon_k \to 0} \frac{\lambda_{\epsilon_k} - \lambda}{\epsilon_k}$. Denote the right edge of $\Omega$ by $S_1$, and label the other edges $S_2, S_3, S_4$ in the counter clockwise direction. Each $r_*^j$ is constant on any edge of $\partial\Omega$; we denote the constant on the edge $S_l$ by $r_{*l}^j$.

$$\lambda_1 = r_{*l}^j \int_{S_l} \frac{\partial v}{\partial x_j}\frac{\partial v}{\partial \tau}\, ds \quad .$$

We know that $\frac{\partial v}{\partial x_1} = 0$ on $S_1$ ,$S_3$ and $\frac{\partial v}{\partial x_2} = 0$ on $S_2$,$S_4$. We also know that $\tau_1 = 0$ on $S_1$ ,$S_3$ and $\tau_2 = 0$ on $S_2$,$S_4$. The expression for $\lambda_1$ simplifies to

$$\begin{aligned}\lambda_1 = r_{*1}^2 \int_{S_1} (\frac{\partial v}{\partial x_2})^2\, ds \quad &- \quad r_{*3}^2 \int_{S_3} (\frac{\partial v}{\partial x_2})^2\, ds \\ &- \quad r_{*2}^1 \int_{S_2} (\frac{\partial v}{\partial x_1})^2\, ds + r_{*4}^1 \int_{S_4} (\frac{\partial v}{\partial x_1})^2\, ds \quad . \end{aligned} \tag{62}$$

We may compute

$$\begin{aligned}\int_{S_1} (\frac{\partial v}{\partial x_2})^2\, ds \quad &= \quad 4\pi^2 \int_{S_1} \cos^2 \pi x_1 \sin^2 \pi x_2\, dx_2 \\ &= \quad 4\pi^2 \int_0^1 \sin^2 \pi x_2\, dx_2 = 2\pi^2 \quad .\end{aligned}$$

In fact, each edge integral in (62) has this same value, and thus

$$\lambda_1 = (r_{*1}^2 - r_{*3}^2 - r_{*2}^1 + r_{*4}^1)2\pi^2 \quad . \tag{63}$$

From the definition of $r^j$, (7), it is in this case easy to see that $r^1$ =const, and that $r^2 = r(y_1)$ is a function of $y_1$ alone, satisfying

$$\frac{d}{dy_1} r = a(y_1) - \overline{a} \quad .$$

Suppose now that $\frac{1}{\epsilon_k} = M_k + \delta_k$, $M_k$ integer, $0 \leq \delta_k < 1$ with $\delta_k \to \delta_0$ as $\epsilon_k \to 0$. Since $r^1$ =const it follows immediately that $r_{*2}^1 = r_{*4}^1$. Let us consider

$r_{*1}^2$. The function $w_2$ defined by (49)-(51) is now simply $w_2 \equiv r(\delta_0)$ and therefore $r_{*1}^2 = r(\delta_0)$. For the case of $r_{*3}^2$ the corresponding function, $w_2$, is also constant, namely $w_2 \equiv r(0)$, and so $r_{*3}^2 = r(0)$. Inserting these values of $r_{*l}^j$ into (63) we finally arrive at

$$\lambda_1 = \lim_{\epsilon_k \to 0} \frac{\lambda_{\epsilon_k} - \lambda}{\epsilon_k} = (r(\delta_0) - r(0))2\pi^2 \quad .$$

This correction clearly depends on $\delta_0$, that is, it depends on the interaction between the microstructure and the macroscopic boundary (more specifically, the piece $\{x_1 = 1,\ 0 \le x_2 \le 1\}$ ).

# 7 Discussion

We have derived formulas for the first order corrections to the (simple) homogenized Neumann eigenvalues in two dimensions ((53), (59) and (61)). Whereas the first order Neumann eigenvalue corrections are always zero in one dimension, our formulas clearly show that this is not (always) the case in two dimensions: if the domain is a convex polygon with sides of rational (or infinite) slope then one encounters the same phenomenon as for the Dirichlet eigenvalue corrections. That is, one discovers that the exact value of the first order correction to any simple eigenvalue depends on the interaction between the microstructure and the macroscopic boundary. We refer to [10] for a heuristic explanation of the difference between the behavior of the Neumann corrections in one dimension and in two dimensions. Such a difference is not seen for the Dirichlet eigenvalue corrections. We also refer to [15] and [10] for numerical computations of the boundary corrector functions associated with the eigenvalue corrections. That part of our work which leads to the most explicit formulas for the first order eigenvalue corrections, (59) and (61), is currently limited to polygonal domains whose sides have rational or infinite slopes (integer normals). It would be very intersting to investigate to what extent this work may carry over to polygons with sides of irrational slope and to more general (smooth) domains. In particular it would be very interesting to see if it is still possible to have many different first order corrections when the domain is smooth and without any straight boundary parts of rational slope. There are some heuristic arguments that might suggest this is not the case, and that the first order correction is unique (*i.e.*, independent of the sequence $\epsilon_k$) for such domains. However, it should be emphasized that nothing rigorous is known to this effect.

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