

ASYMPTOTIC BEHAVIOR OF SOLUTIONS TO THE σ_k -YAMABE EQUATION NEAR ISOLATED SINGULARITIES

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ABSTRACT. σ_k -Yamabe equations are conformally invariant equations generalizing the classical Yamabe equation. In [38] YanYan Li proved that an admissible solution with an isolated singularity at $0 \in \mathbb{R}^n$ to the σ_k -Yamabe equation is asymptotically radially symmetric. In this work we prove that such a solution is asymptotic to a radial solution to the same equation on $\mathbb{R}^n \setminus \{0\}$. These results generalize earlier pioneering work in this direction on the classical Yamabe equation by Caffarelli, Gidas, and Spruck. In extending the work of Caffarelli et al, we formulate and prove a general asymptotic approximation result for solutions to certain ODEs which include the case for scalar curvature and σ_k curvature cases. An alternative proof is also provided using analysis of the linearized operators at the radial solutions, along the lines of approach in a work by Korevaar, Mazzeo, Pacard, and Schoen.

1. DESCRIPTION OF THE RESULTS

In a classic paper [4] Caffarelli, Gidas, and Spruck proved the asymptotic radial symmetry of positive singular solutions u to the conformal scalar curvature equation

$$(1) \quad \Delta u(x) + \frac{n(n-2)}{4} u^{\frac{n+2}{n-2}}(x) = 0$$

on a punctured ball, and further proved that such solutions are asymptotic to radial singular solutions to (1) on $\mathbb{R}^n \setminus \{0\}$. To describe the results of [4] more precisely, we first describe the radial solutions to (1) on $B_R(0) \setminus \{0\}$ for $0 < R \leq \infty$. A positive solution u to (1) corresponds to a conformal metric

$$g = u^{\frac{4}{n-2}}(x) |dx|^2$$

with scalar curvature $n(n-1)$. Using the polar coordinates $x = r\theta$, with $r = |x|$, and $\theta \in \mathbb{S}^{n-1}$, we can introduce cylindrical variable $t = -\ln r$, so that

$$(2) \quad g = u^{\frac{4}{n-2}}(x) |dx|^2 = \left[r^{\frac{n-2}{2}} u(r\theta) \right]^{\frac{4}{n-2}} (r^{-2} dr^2 + d\theta^2) = U^{\frac{4}{n-2}}(t, \theta) (dt^2 + d\theta^2),$$

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where $U(t, \theta) = r^{\frac{n-2}{2}} u(r\theta)$. Computing the scalar curvature of g in terms of U and the background cylindrical metric $dt^2 + d\theta^2$, we can transform (1) into

$$(3) \quad U_{tt}(t, \theta) + \Delta_{\mathbb{S}^{n-1}} U(t, \theta) - \frac{(n-2)^2}{4} U(t, \theta) + \frac{n(n-2)}{4} U^{\frac{n+2}{n-2}}(t, \theta) = 0.$$

If $u(x) = u(|x|)$ is a radial positive solution to (1) in some $B_R(0) \setminus \{0\}$, then $\psi(t) := U(t, \theta) = r^{\frac{n-2}{2}} u(r\theta)$ is a positive solution to the ODE

$$(4) \quad \psi_{tt}(t) - \frac{(n-2)^2}{4} \psi(t) + \frac{n(n-2)}{4} \psi^{\frac{n+2}{n-2}}(t) = 0,$$

for $t > -\ln R$. Solutions to (4) has a first integral:

$$H := \psi_t^2(t) + \frac{(n-2)^2}{4} \left[\psi^{\frac{2n}{n-2}}(t) - \psi^2(t) \right] \equiv \text{const.}$$

along any positive solution $\psi(t)$. In fact, positive solutions U to (3) globally defined on the entire cylinder $\mathbb{R} \times \mathbb{S}^{n-1}$ (thus positive solutions u to (1) on $\mathbb{R}^n \setminus \{0\}$) are classified in [4].

Theorem A. ([4]) *Let $U(t, \theta)$ be any positive solution to (3) defined on the entire cylinder $\mathbb{R} \times \mathbb{S}^{n-1}$. If 0 is a non-removable singularity of u in the sense that*

$$\liminf_{t \rightarrow \infty} \min_{\theta \in \mathbb{S}^{n-1}} U(t, \theta) > 0,$$

then U is independent of θ . Moreover $U(t)$ is a periodic solution of (4) with $0 < U(t) \leq 1$ for all $t \in \mathbb{R}$ and the first integral $H < 0$. We refer to these solutions as global singular positive solutions to (3). If 0 is a removable singularity of u in the sense that

$$\liminf_{t \rightarrow \infty} \min_{\theta \in \mathbb{S}^{n-1}} U(t, \theta) = 0,$$

then the corresponding $u(x) = e^{\frac{n-2}{2}t} U(t, \theta) = |x|^{-\frac{n-2}{2}} U(-\ln|x|, \frac{x}{|x|})$ is identically equal to $\left(\frac{2a}{1+a^2|x-\bar{x}|^2} \right)^{\frac{n-2}{2}}$

in \mathbb{R}^n for some $\bar{x} \in \mathbb{R}^n$ and $a > 0$.

Thus global singular positive solutions $U(t, \theta)$ to (3) defined on the entire cylinder $\mathbb{R} \times \mathbb{S}^{n-1}$ can be parametrized by two parameters: its minimum value $\varepsilon > 0$ and a moment T when it attains this minimum value. It turns out that the minimum value $\varepsilon > 0$ of global singular positive solutions to (3) has the restriction $0 < \varepsilon \leq \varepsilon_0$, where

$\varepsilon_0 = \left(\frac{n-2}{n}\right)^{\frac{n-2}{4}}$. For any such ε , let $\psi_\varepsilon(t)$ denote the solution to (4) such that $\psi_\varepsilon(0) = \varepsilon$ and $\psi'_\varepsilon(0) = 0$. Then any global singular positive solution to (3) can be represented as $\psi_\varepsilon(t + \tau)$ for some $0 < \varepsilon \leq \varepsilon_0$ and τ .

The main results in [4] on the asymptotic behavior of a positive solution to (1) in $B_R \setminus \{0\}$ can be stated as

Theorem B. ([4]) *Suppose that $u(x)$ is a positive solution to (1) in $B_R \setminus \{0\}$ and does not extend to a smooth solution to (1) over 0, then*

$$(5) \quad u(x) = \bar{u}(|x|) (1 + O(|x|)) \quad \text{as } x \rightarrow 0,$$

with

$$\bar{u}(|x|) = \int_{\mathbb{S}^{n-1}} u(|x|\theta) d\theta$$

being the spherical average of u over the sphere $\partial B_{|x|}(0)$; furthermore, there exists a radial singular solution

$u^*(|x|)$ to (1) on $\mathbb{R}^n \setminus \{0\}$ and some $\alpha > 0$, $0 < \varepsilon \leq \varepsilon_0$ and τ such that

$$(6) \quad u(x) = u^*(|x|) (1 + O(|x|^\alpha)) \quad \text{as } |x| \rightarrow 0.$$

and

$$u^*(|x|) = |x|^{-\frac{n-2}{2}} \psi_\varepsilon(-\ln|x| + \tau) \quad \text{as } |x| \rightarrow 0.$$

A key ingredient in the proof in [4] of Theorem B uses a ‘‘measure theoretic’’ variation of the moving plane technique, which had been developed by Alexandrov [1], Serrin [49], and Gidas-Ni-Nirenberg [17] to prove symmetries of solutions to certain elliptic PDEs. Subsequent to [4] there have been many papers related to the theme of Theorem B, including [13], [37], [32], and [50], among others. In particular [32] gives a proof of (6) and provides an expansion of u after the order $u^*(|x|)$ using rescaling analysis, classification of global singular solutions as given by Theorem A, and analysis of linearized operators at these global singular solutions.

Our objective in this paper is to study similar problems for singular solutions in a punctured ball to a family of conformally invariant equations which include (1). More specifically, we consider singular solutions to the equation

$$(7) \quad \sigma_k(g^{-1} \circ A_g) = \text{constant},$$

on a punctured ball $\{x \in \mathbb{R}^n : 0 < |x| < R\}$, where

$$A_g = \frac{1}{n-2} \left\{ Ric_g - \frac{R_g}{2(n-1)} g \right\},$$

is the Weyl-Schouten tensor of the conformal metric

$$g = u^{\frac{4}{n-2}}(x) |dx|^2,$$

Ric_g and R_g denote respectively the Ricci and scalar curvature of g , and $\sigma_k(g^{-1} \circ A_g)$ denotes the k -th elementary symmetric function of the eigenvalues of A_g with respect to g , and $0 < R \leq \infty$. Due to the transformation law, see e.g. [52],

$$A_g = A_{g_0} + \left[\nabla^2 w + dw \otimes dw - \frac{1}{2} |\nabla w|^2 g_0 \right],$$

when $g = e^{-2w} g_0$, and

$$\sigma_k(g^{-1} \circ A_g) = e^{2kw} \sigma_k(g_0^{-1} \circ A_g),$$

(7) is equivalent to

$$(8) \quad \sigma_k(g_0^{-1} \circ \{A_{g_0} + \nabla^2 w + dw \otimes dw - \frac{1}{2} |\nabla w|^2 g_0\}) = ce^{-2kw},$$

for some constant c , where we will often take g_0 to be the round cylindrical metric $dt^2 + d\theta^2$ on $\mathbb{R} \times \mathbb{S}^{n-1}$, which is pointwise conformal to the flat metric $|dx|^2$ on \mathbb{R}^n as seen through (2). For ease of reference, we also record an equivalent formulation of (8) in terms of $u(x)$ through the transformation (2):

$$(8') \quad \sigma_k \left(-(n-2)u(x)\nabla^2 u(x) + n\nabla u(x) \otimes \nabla u(x) - |\nabla u(x)|^2 \text{Id} \right) = \frac{2^k c}{(n-2)^{2k}} u^{\frac{2kn}{n-2}}(x).$$

The study of singular solutions of equations of the above type is related to the characterization of the size of the limit set of the image domain in \mathbb{S}^n of the developing map of a locally conformally flat n -manifold. More specifically, one is led to looking for necessary/sufficient conditions on a domain $\Omega \subset \mathbb{S}^n$ so that it admits a metric g which is pointwise conformal to the standard metric on \mathbb{S}^n , complete, and with its Weyl-Schouten tensor A_g in

the Γ_k^\pm class, *i.e.*, the eigenvalues, $\lambda_1 \leq \dots \leq \lambda_n$, of A_g at each $x \in \Omega$ satisfy $\sigma_j(\lambda_1, \dots, \lambda_n) > 0$ for all $j, 1 \leq j \leq k$,

in the case of Γ_k^+ ; and $(-1)^j \sigma_j(\lambda_1, \dots, \lambda_n) > 0$ for all $j, 1 \leq j \leq k$, in the case of Γ_k^- . For $k \geq 2$, it is often restricted

to metrics whose Weyl-Schouten tensor is in the Γ_k^\pm class, because, for a metric in such a class, (8) becomes a fully nonlinear PDE in w that is *elliptic*. In the case of $k = 1$, $\sigma_1(A_g)$ is simply a positive constant multiple of

the scalar curvature of g ; so A_g in the Γ_k^\pm class is a generalization of the notion that the scalar curvature R_g of g having a fixed \pm sign. For the positive scalar curvature case, Schoen and Yau proved in [48] that if a complete conformal metric g exists on a domain $\Omega \subset \mathbb{S}^n$ with $\sigma_1(A_g)$ having a positive lower bound, then the Hausdorff dimension of $\partial\Omega$ has to be $\leq (n-2)/2$. In [47] Schoen constructed complete conformal metrics on $\mathbb{S}^n \setminus \Lambda$ when Λ is either a finite discrete set on \mathbb{S}^n containing at least two points or a set arising as the limit set of a Kleinian group action. Later Mazzeo and Pacard [41] proved that if $\Omega \subset \mathbb{S}^n$ is a domain such that $\mathbb{S}^n \setminus \Omega$ consists a finite number of disjoint smooth submanifolds of dimension $1 \leq k \leq (n-2)/2$, then one can find a complete conformal metric g on Ω with its scalar curvature identical to $+1$. For the negative scalar curvature case, the results of Loewner-Nirenberg [39], Aviles [2], and Veron [51] imply that if $\Omega \subset \mathbb{S}^n$ admits a complete, conformal metric

with negative constant scalar curvature, then the Hausdorff dimension of $\partial\Omega > (n-2)/2$. Loewner-Nirenberg [39] also proved that if $\Omega \subset \mathbb{S}^n$ is a domain with smooth boundary $\partial\Omega$ of dimension $> (n-2)/2$, then there exists a complete conformal metric g on Ω with $\sigma_1(A_g) = -1$. This result was later generalized by D. Finn [16] to the case of $\partial\Omega$ consisting of smooth submanifolds of dimension $> (n-2)/2$ and with boundary. For more recent development related to the negative scalar curvature case, see [33], [34], [40] and the references therein. The consideration of singular solutions of equations of type (8) can be considered as a natural generalization of these known results. In fact, in [11], Chang, Hang, and Yang proved that if $\Omega \subset \mathbb{S}^n$ ($n \geq 5$) admits a complete, conformal metric g with

$$(9) \quad \sigma_1(A_g) \geq c_1 > 0, \quad \sigma_2(A_g) \geq 0, \quad \text{and} \\ |R_g| + |\nabla_g R|_g \leq c_0,$$

then $\dim(\mathbb{S}^n \setminus \Omega) < (n-4)/2$. This has been generalized by M. Gonzalez [19] and Guan, Lin and Wang [21] to the case of $2 < k < n/2$: if $\Omega \subset \mathbb{S}^n$ admits a complete, conformal metric g with

$$(9) \quad \sigma_1(A_g) \geq c_1 > 0, \quad \sigma_2(A_g), \dots, \sigma_k(A_g) \geq 0, \quad \text{and} \\ \dim(\mathbb{S}^n \setminus \Omega) < (n-2k)/2.$$

We restrict our attention in this paper to singular solutions with isolated singularity. We say a solution to (8) or (8') is in the Γ_k^+ class in some region if its associated Weyl-Schouten tensor is in the Γ_k^+ there; for a positive solution u to (8'), this means that the matrix $(-(n-2)u(x)\nabla^2 u(x) + n\nabla u(x) \otimes \nabla u(x) - |\nabla u(x)|^2 \text{Id})$ belongs to Γ_k^+ . Our main result is

Theorem 1. *Let $w(t, \theta)$ be a smooth solution to (8) on $\{t > t_0\} \times \mathbb{S}^{n-1}$ in the Γ_k^+ class, where $n \geq 3$, $2 \leq k \leq n$,*

and the constant $c > 0$. Then there exist a radial solution $w^(t)$ to (8) on $\mathbb{R} \times \mathbb{S}^{n-1}$ in the Γ_k^+ class, and constants $\alpha > 0$, $C > 0$ such that*

$$(10) \quad |w(t, \theta) - w^*(t)| \leq Ce^{-\alpha t} \quad \text{for } t > t_0 + 1.$$

We can formulate Theorem 1 in terms of the variable $u(x)$ defined on $B_R \setminus \{0\}$ through (2):

Theorem 1'. *Let $u(x)$ be a positive smooth solution to (8') on $B_R \setminus \{0\}$ in the Γ_k^+ class, where $n \geq 3$, $2 \leq k \leq n$,*

$R > 0$, and $c > 0$, then there exist a positive radial smooth solution $u^(|x|)$ to (8') on $\mathbb{R}^n \setminus \{0\}$ in the Γ_k^+ class, and constants $\alpha > 0$, $C > 0$ such that*

$$(11) \quad |u(x) - u^*(|x|)| \leq C|x|^\alpha u^*(|x|) \quad \text{for } |x| < R/2.$$

Remark. *As a consequence of Theorem 2 below, the α in Theorem 1 and 1' can be any number in $(0, 1)$, while the constant C depends also on α .*

Remark. *The $k = 1$ case of Theorems 1 and 1' was proved in [4], as remarked earlier. In the situation of Theorem 1', the asymptotic symmetry of $u(x)$, a positive solution to (8') on $B_R \setminus \{0\}$ in the Γ_k^+ class, was proved in [38], namely, for some constant $C > 0$,*

$$|u(x) - \bar{u}(|x|)| \leq C|x|\bar{u}(|x|),$$

where $\bar{u}(|x|)$ is the spherical average of $u(x)$. This is a generalization of the result (5) in [4] for solutions to (1), and will be a starting point for our proof of Theorem 1.

We can describe the asymptotic behavior of solutions in Theorems 1 and 1' in more explicit terms after describing the classification results from [8] on the radial solutions to (8) for $k > 1$ in the Γ_k^+ class globally defined

on $\mathbb{R} \times \mathbb{S}^{n-1}$. Let us first work out (8) more explicitly in the case of radial solutions.

To fix our notations, we introduce new variables $v(t, \theta)$ and $w(t, \theta)$ such that

$$(2') \quad g = u^{\frac{4}{n-2}}(x)|dx|^2 = v^{-2}(x)|dx|^2 = U^{\frac{4}{n-2}}(t, \theta)(dt^2 + d\theta^2) = e^{-2w(t, \theta)}(dt^2 + d\theta^2),$$

where $t = -\ln|x|$ and $\theta = x/|x|$. Thus

$$(12) \quad |x|^{\frac{n-2}{2}}u(x) = \left(\frac{|x|}{v(x)}\right)^{\frac{n-2}{2}} = U(t, \theta) = e^{-\frac{n-2}{2}w(t, \theta)}.$$

Following the notation in [8], the Schouten tensor of g can be computed as, when $v = v(|x|)$,

$$A_{ij} = \frac{v_{ij}}{v} - \frac{|\nabla v|^2}{2v^2}\delta_{ij} = \lambda\delta_{ij} + \mu\frac{x_i x_j}{|x|^2},$$

with $\lambda = \frac{v_r}{v}(1 - \frac{rv_r}{2v})$ and $\mu = \frac{v_{rr}}{v} - \frac{v_r}{v}$. The eigenvalues of A with respect to $|dx|^2$ are λ with multiplicity $(n-1)$,

and $\lambda + \mu$ with multiplicity 1. The formula for $\sigma_k(g^{-1} \circ A_g)$ can be found easily by the binomial expansion of

$$(x - \lambda)^{n-1}(x - \lambda - \mu):$$

$$(13) \quad \sigma_k(g^{-1} \circ A_g) = c_{n,k} v^{2k} \lambda^{k-1} (n\lambda + k\mu),$$

where $c_{n,k} = \frac{(n-1)!}{k!(n-k)!} = \frac{1}{n} \binom{n}{k}$.

Remark. The convention $t = -\ln r$ here is off by a sign with the convention in [8], and the transformation from v to w is adjusted from [8] accordingly.

$$v_r(x) = e^{w(t, \theta)}(-w_t(t, \theta) + 1) = (-w_t(t, \theta) + 1)v(x)/r,$$

and

$$v_{rr}(x) = e^{w(t, \theta)+t}[w_{tt}(t, \theta) - w_t(t, \theta)(-w_t(t, \theta) + 1)] = [w_{tt}(t, \theta) - w_t(t, \theta)(-w_t(t, \theta) + 1)]v(x)e^{2t}.$$

Thus when $v = v(|x|)$, we find $w(t, \theta) = t + \ln v(e^{-t}) =: \xi(t)$ is a function of t , and

$$\lambda = \frac{1}{2}e^{2t}(1 - \xi_t^2), \quad \text{and} \quad \mu = e^{2t}(\xi_{tt} + \xi_t^2 - 1).$$

Using (13), (8) in the radial case then becomes

$$(14) \quad \begin{aligned} c = \sigma_k(A_g) &= c_{n,k} e^{2k(\xi+t)} \frac{(1-\xi_t^2)^{k-1}}{2^{k-1} e^{2(k-1)t}} \left[n \frac{1-\xi_t^2}{2e^{2t}} + k \frac{\xi_{tt} + \xi_t^2 - 1}{e^{2t}} \right] \\ &= c'_{n,k} (1-\xi_t^2)^{k-1} \left[\frac{k}{n} \xi_{tt} + \left(\frac{1}{2} - \frac{k}{n} \right) (1-\xi_t^2) \right] e^{2k\xi}, \end{aligned}$$

where $c'_{n,k} = nc_{n,k} 2^{1-k} = 2^{1-k} \binom{n}{k}$. Thus (14) is the radial case of (7), written in cylindrical coordinate t and the variable $w(t, \theta) = \xi(t)$. In general, we will allow ourselves the flexibility of treating (7) either as an equation for $u(x)$ on $B_R \setminus \{0\}$ or as an equation for $U(t, \theta)$ or $w(t, \theta)$ on a cylinder $\{(t, \theta) : t > -\ln R, \theta \in \mathbb{S}^{n-1}\}$ with respect to the background metric $dt^2 + d\theta^2$.

Now we record the relevant part of the results in [8] regarding radial solutions of (14) in the Γ_k^+ class on the entire $\mathbb{R} \times \mathbb{S}^{n-1}$, when $k > 1$ and σ_k is a *positive* constant, normalized to be $2^{-k} \binom{n}{k}$.

Theorem C. ([8]) *Any radial solution $\xi(t) := w(t, \theta)$ of (8) in the Γ_k^+ class on the entire $\mathbb{R} \times \mathbb{S}^{n-1}$, when $k > 1$ and c is a positive constant, normalized to be $2^{-k} \binom{n}{k}$, has the property that $1 - \xi_t^2 > 0$ for all t . Furthermore, $h := e^{(2k-n)\xi(t)} (1 - \xi_t^2(t))^k - e^{-n\xi(t)}$ is a nonnegative constant. Moreover*

(1) *If $h = 0$, then $u^{\frac{4}{n-2}}(|x|) = \left(\frac{2\rho}{|x|^2 + \rho^2} \right)^2$ for some positive parameter ρ . So these solutions give rise to the*

round spherical metric on $\mathbb{R}^n \cup \{\infty\} = \mathbb{S}^n$.

(2) *If $h > 0$, then the behavior of u is classified according to the relation between $2k$ and n :*

(a) *If $2k < n$, then h has the further restriction $h \leq h^* := \frac{2k}{n-2k} \left(\frac{n-2k}{n} \right)^{\frac{n}{2k}}$ and $\xi(t)$ is a periodic function*

of t , giving rise to a metric $g = \frac{e^{-2\xi(\ln|x|)}}{|x|^2} |dx|^2$ on $\mathbb{R}^n \setminus \{0\}$ which is complete. Note that the case

$h = h^$ gives rise to the cylindrical metric $\frac{|dx|^2}{|x|^2}$ on $\mathbb{R}^n \setminus \{0\}$.*

- (b) If $2k = n$, then h satisfies the further restriction $h < 1$ and as $|x| \rightarrow 0$, $g = u^{\frac{4}{n-2}}(|x|)|dx|^2$ has the asymptotic

$$g \sim |x|^{-2(1-\sqrt{1-\sqrt[k]{h}})}|dx|^2 = e^{-(2\sqrt{1-\sqrt[k]{h}})t}(dt^2 + d\theta^2),$$

and as $|x| \rightarrow \infty$, $g = u^{\frac{4}{n-2}}(|x|)|dx|^2$ has the asymptotic

$$g \sim |x|^{-2(1+\sqrt{1-\sqrt[k]{h}})}|dx|^2 = e^{2\sqrt{1-\sqrt[k]{h}}t}(dt^2 + d\theta^2).$$

Thus g gives rise to a metric on $\mathbb{R}^n \setminus \{0\}$ singular at 0 and at ∞ which behaves like the cone metric, is incomplete with finite volume.

- (c) If $2k > n$, then $u^{\frac{4}{n-2}}(|x|)$ has an asymptotic expansion of the form

$$u^{\frac{4}{n-2}}(|x|) = \rho^{-2} \left\{ 1 - \sqrt[k]{h} \frac{k}{2k-n} \left(\frac{|x|}{\rho} \right)^{2-\frac{n}{k}} + \dots \right\}$$

as $|x| \rightarrow 0$, where $\rho > 0$ is a positive parameter; thus $u(|x|)$ has a positive, finite limit, but $u_{rr}(|x|)$ blows up at $|x| \rightarrow 0$. The behavior of u as $|x| \rightarrow \infty$ can be described similarly. Putting together, we

conclude that $u^{\frac{4}{n-2}}(|x|)|dx|^2$ extends to a $C^{2-\frac{n}{k}}$ metric on \mathbb{S}^n .

We can parametrize the global singular radial solutions to (8) in a similar way as before: for each $0 < h$, subject to any further constraints depending on $2k < \text{or} = n$, as given in Theorem C, let $\xi_h(t)$ denote the solution to (8) with its first integral equal to h and such that $\xi_h(0)$ equals $\min_{\mathbb{R}} \xi_h(t)$. We can now reformulate Theorem 1 with more explicit information as

Theorem 1''. Let $w(t, \theta)$ be a smooth solution to (8) on $\{t > t_0\} \times \mathbb{S}^{n-1}$ in the Γ_k^+ class, where $n \geq 3$, $2 \leq k \leq n$, and $c > 0$. Then there exist $\alpha > 0$, $h \geq 0$, τ and $C > 0$ such that

$$(15) \quad |w(t, \theta) - \xi_h(t + \tau)| \leq Ce^{-\alpha t} \quad \text{for } t > t_0 + 1.$$

As mentioned earlier, the α in the above theorem can be taken as any number in $(0, 1)$, while the constant C then depends on α as well.

Using the transformation (2') and our knowledge of $\xi_h(t + \tau)$ as given in Theorem C, we can also formulate the result in Theorem 1'' in terms of the variable $u(x)$ defined on $B_R \setminus \{0\}$.

Corollary. Let $u(x)$ be a positive smooth solution to (8') on $B_R \setminus \{0\}$ in the Γ_k^+ class, where the constant c is

normalized to be $2^{-k} \binom{n}{k}$. If

$$(16) \quad \liminf_{x \rightarrow 0} |x|^{\frac{n-2}{2}} u(x) > 0,$$

then $2k < n$, furthermore, there exist $\alpha > 0$, $h^* \geq h \geq 0$, τ and $C > 0$ such that

$$(17) \quad u(x) = (1 + o(|x|^\alpha)) |x|^{-\frac{n-2}{2}} e^{-\frac{n-2}{2}\xi_h(-\ln|x|+\tau)}$$

as $x \rightarrow 0$;

If $2k > n$, or $2k < n$ and

$$(18) \quad \liminf_{x \rightarrow 0} |x|^{\frac{n-2}{2}} u(x) = 0,$$

then $\lim_{x \rightarrow 0} u(x)$ exists and equals some $a > 0$, and there exist some $\alpha > 0$ and $C > 0$ such that

$$(19) \quad |u(x) - a| \leq C|x|^\alpha;$$

If $2k = n$, then there exist some $0 \leq h < 1$ and $\alpha > 0$ such that

$$|x|^{\frac{n-2}{2}(1-\sqrt{1-\frac{k}{h}})} u(x)$$

extends to a C^α positive function over B_R .

Remark. In the case $2k > n$ Gursky and Viaclovsky [29], YanYan Li [38] had obtained (19) earlier, with $\alpha = 2 - \frac{n}{k}$.

In the case $2k < n$, M. Gonzalez [20] proved that if u is a solution to (13) in $B_R \setminus \{0\}$ in the Γ_k^+ class such that $u^{4/(n-2)}|dx|^2$ has finite volume over $B_R \setminus \{0\}$, then u is bounded in $B_R \setminus \{0\}$.

As in [32], we also obtain higher order expansions for solutions to (8) in the case $2k \leq n$.

Theorem 2. Let $w(t, \theta)$ be a solution to (8) on $\{t > t_0\} \times \mathbb{S}^{n-1}$ in the Γ_k^+ class, where $n \geq 3$, $2 \leq k \leq n/2$, and

the constant c is normalized to be $2^{-k} \binom{n}{k}$, and let $w^*(t) = \xi_h(t + \tau)$ be the radial solution to (8) on $\mathbb{R} \times \mathbb{S}^{n-1}$ in

the Γ_k^+ class for which (10) holds. Let $\{Y_j(\theta) : j = 0, 1, \dots\}$ denote the set of normalized spherical harmonics, and ρ be the infimum of the positive characteristic exponents defined through Floquet theory to the linearized equation of (8) at $w^*(t)$ corresponding to higher order spherical harmonics $Y_j(\theta)$, $j > n$ — see the paragraph before Lemma 1 in Section 4 for more detail. Then $\rho > 1$, and when $h > 0$, there is a

$$w_1(t, \theta) = \sum_{j=1}^n a_j e^{-t-\tau} \left(1 + \xi_h'(t + \tau)\right) Y_j(\theta),$$

which is a solution to the linearized equation of (8) at $w^*(t)$, such that

$$(20) \quad |w(t, \theta) - w^*(t) - w_1(t, \theta)| \leq C e^{-\min\{2, \rho\}t} \quad \text{for } t > t_0 + 1,$$

provided $\rho \neq 2$; when $\rho = 2$, (20) continues to hold if the right hand side is modified into Cte^{-2t} .

Theorem 2 requires some knowledge on the spectrum of the linearized operator to (8). We are able to provide the needed analysis, and will state them as Propositions 2 and 3 in section 4. Such analysis will also be needed in constructing solutions to (8) on $\mathbb{S}^n \setminus \Lambda$, and in analysing the moduli space of solutions to (8) on $\mathbb{S}^n \setminus \Lambda$, when Λ is a finite set. Our knowledge of the spectrum of the linearized operator to (8) immediately yields Fredholm mapping properties of these operators on appropriately defined weighted spaces, as those in [44], [43], and [32]. We will pursue these problems in a different paper.

It turns out that either of the approaches in [4] and [32] can be adapted to prove the main part of Theorem 1. We will provide proofs along both lines.

The approach in [4] first proves that the radial average of the solution is a good approximation to the solution, and satisfies an ODE which is an approximation to the ODE (4) satisfied by a radial solution to (3); from this approximate ODE one proves that the radial average is approximated by a (translated) radial solution to (3). More specifically, [4] first proves (5) for a positive solution to (1) in the punctured ball $B_2(0) \setminus \{0\}$.

In terms of $U(t, \theta) = r^{\frac{n-2}{2}} u(r\theta)$, $t = -\ln r = -\ln|x|$, and

$$\beta(t) := |\mathbb{S}^{n-1}|^{-1} \int_{\mathbb{S}^{n-1}} U(t, \theta) d\theta,$$

(5) is reformulated as

$$(21) \quad |U(t, \theta) - \beta(t)| \leq C\beta(t)e^{-t}.$$

Using gradient estimates and (5), [4] further deduces that for some constant $C > 0$

$$|\nabla(u(x) - \bar{u}(|x|))| \leq C\bar{u}(|x|),$$

which, in terms of $U(t, \theta)$, $t = -\ln r = -\ln|x|$, and $\beta(t)$, is reformulated as

$$(22) \quad |\nabla_{t,\theta}(U(t, \theta) - \beta(t))| \leq C\beta(t)e^{-t}.$$

It follows from (3), (21) and a version of (22) for derivatives up to order 2 that

$$(23) \quad \beta''(t) - \frac{(n-2)^2}{4}\beta(t) + \frac{n(n-2)}{4}\beta^{\frac{n+2}{n-2}}(t) = O(\beta(t)e^{-t}).$$

It is then routine to deduce from (23) the following approximate first integral, which, up to a constant, is referred to as (7.14) in [4]: for some constant D_∞ ,

$$(24) \quad \beta'^2(t) = \frac{(n-2)^2}{4} \left[\beta^2(t) - \beta^{\frac{2n}{n-2}}(t) \right] + D_\infty + \left(\beta^2(t) + \beta'^2(t) \right) O(e^{-t}),$$

Since $\beta(t)$ remains positive for all large t , (24) demands that $0 \geq D_\infty \geq D^*$, where

$$D^* = -\frac{(n-2)^2}{4} \sup_{\beta \geq 0} \left[\beta^2 - \beta^{\frac{2n}{n-2}} \right] = -\frac{n-2}{2} \left(\frac{n-2}{n} \right)^{\frac{n}{2}}$$

is determined so that for $0 \geq D_\infty \geq D^*$,

$$\sup_{\beta \geq 0} \left(\frac{(n-2)^2}{4} \left[\beta^2 - \beta^{\frac{2n}{n-2}} \right] + D_\infty \right) \geq 0.$$

Then [4] indicates that when $0 > D_\infty \geq D^*$, $\beta(t)$ is asymptotic to a translated solution $\psi(t)$ to (4) whose first integral is the same as D_∞ . When $D_\infty = 0$, [4] gives a detailed argument that 0 is a removable singularity.

We will formulate and prove a general asymptotic approximation result for solutions to certain ODEs which include the case for scalar curvature and σ_k curvature cases.

Consider a solution $\beta(t)$ to

$$(25) \quad \beta''(t) = f(\beta'(t), \beta(t)) + e_1(t), \quad t \geq 0,$$

where f is locally Lipschitz, and $e_1(t)$ is considered as a perturbation term with $e_1(t) \rightarrow 0$ as $t \rightarrow \infty$ at a sufficiently fast rate to be specified later. Suppose that $|\beta(t)| + |\beta'(t)|$ is bounded over $t \in [0, \infty)$. Then by a compactness argument there exist a sequence of $t_i \rightarrow \infty$ and a solution $\psi(t)$ to

$$(26) \quad \psi''(t) = f(\psi'(t), \psi(t))$$

which exists for all $t \in \mathbb{R}$ such that

$$(27) \quad \beta(t_i + \cdot) \rightarrow \psi \quad \text{in } C_{loc}^1(-\infty, \infty) \text{ as } i \rightarrow \infty.$$

Theorem 3. *Suppose that, for the $\beta(t)$, $\psi(t)$ and $\{t_i\}$ above, $\psi(t)$ is a periodic solution to (26) with (minimal) period $T \geq 0$. Thus for some finite $m \leq M$,*

$$(28) \quad \psi(\mathbb{R}) = [m, M].$$

We may do a time translation for $\psi(t)$ so that $\psi(0) = m$, $\psi'(0) = 0$, then the approximation property (27) can be reformulated as, for some s ,

$$(29) \quad \beta(t_i + \cdot) - \psi(-s + \cdot) \rightarrow 0, \text{ in } C_{loc}^1(-\infty, \infty), \text{ as } i \rightarrow \infty.$$

Suppose that $\psi(t)$ has a first integral in the form of

$$(30) \quad H(\psi'(t), \psi(t)) = 0, \quad \text{for some continuous function } H(x, y),$$

where H satisfies the following non-degeneracy condition, depending on

case (i). ($\psi(t) \equiv m$ is a constant): *there exist some $\varepsilon_1 > 0$, $A > 0$, $l > 0$,*

$$(31) \quad H(x, y) \geq A \left(|x|^l + |y - m|^l \right), \text{ for any } (x, y) \text{ with } |x| + |y - m| \leq \varepsilon_1;$$

case (ii). ($\psi(t)$ is non-constant): *there exist some $\varepsilon_1 > 0$, $A > 0$ and $l > 0$,*

$$(32) \quad |H(0, y)| = |H(0, y) - H(0, m)| \geq A|y - m|^l, \text{ for any } y \text{ with } |y - m| \leq \varepsilon_1.$$

Suppose also that $\beta(t)$ has H as an approximate first integral

$$(33) \quad |H(\beta'(t), \beta(t))| \leq e_2(t), \quad \text{for } t \geq 0,$$

where $e_2(t) \rightarrow 0$ as $t \rightarrow \infty$. Without loss of generality, we may suppose that $e_2(t)$ is monotone non-increasing in t . Finally suppose that

$$(34) \quad \int_0^\infty \left((e_2(t))^{1/l} + \sup_{\tau \geq t} |e_1(\tau)| \right) dt < \infty.$$

Then, there exist some s_∞ and $C > 0$ such that,

$$(35) \quad \begin{aligned} & |\beta(t) - \psi(t - s_\infty)| + |\beta'(t) - \psi'(t - s_\infty)| \\ & \leq C \int_{t-1}^\infty \left((e_2(t'))^{1/l} + \sup_{\tau \geq t'} |e_1(\tau)| \right) dt' \rightarrow 0, \quad \text{as } t \rightarrow \infty. \end{aligned}$$

In the case of (1), we can take

$$H(x, y) = x^2 + \frac{(n-2)^2}{4} \left[y^{\frac{2n}{n-2}} - y^2 \right] - D_\infty,$$

according to (24). Then $\beta(t)$ and H satisfy the conditions in Theorem 3. We will indicate how Theorem 3 can be applied to prove Theorem 1, after we provide more background information on solutions to (8) and (8').

Some comments on Theorem 3 are appropriate here.

Remark. (a). (32) would be satisfied with $l = 1$ if $H_y(0, m) \neq 0$, for instance. Assumptions (30), (32), and (33) are used only near $(0, m)$, and need not be posed near the minimum m of $\psi(t)$: the argument would go through if they are posed near any critical value of $\psi(t)$. (b). Our proof gives an exponential decay rate for

$|\beta(t) - \psi(t - s_\infty)| + |\beta' - \psi'(t - s_\infty)|$ when $|e_1(t)|$ and $e_2(t)$ have exponential decay rates.

Here is a brief description of our plan for the remaining part of the paper. We will first summarize some needed preliminary properties for solutions to (8) and (8') in section 2, then provide a proof for Theorem 1 in section 3, using Theorem 3 and several other ingredients, the proof for which we supply in this section. In section 4, we provide the analysis for the linearized operator for (8) at entire radial solutions, and use them to provide an alternative proof for Theorem 1 along the approach of [32]. We will also provide a proof for Theorem 2 here. Finally in section 5 we provide a proof for Theorem 3.

2. SEVERAL PRELIMINARY PROPERTIES OF SOLUTIONS TO (7)

To adapt either of the approaches in [4] or [32] to our situation, we will need several key properties of solutions derived in [38]. The following is a special case of Theorem 1.2 in [38].

Theorem D. ([38]) *Let $U(t, \theta)$ be any positive solution to (7) defined on the entire cylinder $\mathbb{R} \times \mathbb{S}^{n-1}$. Suppose*

that $U^{\frac{4}{n-2}}(t, \theta)(dt^2 + d\theta^2)$ is in the Γ_k^+ class, and

$$u(x) = |x|^{-\frac{n-2}{2}} U(-\ln|x|, \frac{x}{|x|})$$

can not be extended as a C^2 positive function near 0, then U is independent of θ .

Remark. *Theorem D is an analogue of Theorem A. In the setting of Theorem D, it follows from Theorem 1.3 in [35] that if $u(x)$ can be extended as a C^2 positive function near 0, then $u(x)$ is a constant multiple of*

$$\left(\frac{a}{1+a^2|x-\bar{x}|^2}\right)^{\frac{n-2}{2}} \text{ in } \mathbb{R}^n \text{ for some } \bar{x} \in \mathbb{R}^n \text{ and } a > 0.$$

When a solution u in Theorem D can not be extended as a C^2 positive function near 0, we refer to the corresponding $U(t, \theta) = U(t)$ as a global singular positive solution to (7). Using Theorem C above, when $2k < n$, $U(t)$ is a periodic solution of (7) with $0 < U(t) \leq 1$ for all $t \in \mathbb{R}$ and the first integral $h > 0$.

Another needed estimate, generalizing estimate (5) from solutions to (1) to solutions to (7), is drawn from Theorems 1.1' and 1.3 of [38]:

Theorem E. ([38]) *Suppose that $u \in C^2(B_2 \setminus \{0\})$ is a positive solution to (8'). Then*

$$(36) \quad \limsup_{x \rightarrow 0} |x|^{\frac{n-2}{2}} u(x) < \infty;$$

and there exists some constant $C > 0$ such that

$$(37) \quad |u(x) - \bar{u}(|x|)| \leq C|x|\bar{u}(|x|),$$

for $0 < |x| \leq 1$, where

$$\bar{u}(|x|) = \frac{1}{|\partial B_{|x|}(0)|} \int_{\partial B_{|x|}(0)} u(y) d\sigma(y)$$

is the spherical average of $u(x)$ over $\partial B_{|x|}(0)$.

As in the previous section, in terms of $t = -\ln r = -\ln|x|$,

$$U(t, \theta) = r^{\frac{n-2}{2}} u(r\theta) = e^{-\frac{n-2}{2}w(t, \theta)},$$

$$\beta(t) := |\mathbb{S}^{n-1}|^{-1} \int_{\mathbb{S}^{n-1}} U(t, \theta) d\theta,$$

and the spherical average

$$(38) \quad \gamma(t) := |\mathbb{S}^{n-1}|^{-1} \int_{\mathbb{S}^{n-1}} w(t, \theta) d\theta,$$

of $w(t, \theta)$, (36) is reformulated as

$$(39) \quad U(t, \theta) \leq C \quad \text{and} \quad e^{-2w(t, \theta)} \leq C.$$

We also derive from (37) that

$$(40) \quad |U(t, \theta) - \beta(t)| \leq C\beta(t)e^{-t},$$

and

$$(41) \quad |\widehat{w}(t, \theta)| := |w(t, \theta) - \gamma(t)| \leq \tilde{C}e^{-t}.$$

(40) is simply a reformulation of (37) in terms of $U(t, \theta)$ and $\beta(t)$. In terms of $w(t, \theta)$, (40) becomes

$$|e^{-\frac{n-2}{2}w(t, \theta) - \ln \beta(t)} - 1| \leq Ce^{-t},$$

from which it follows that, for some $\tilde{C} > 0$,

$$(42) \quad |w(t, \theta) + \frac{2}{n-2} \ln \beta(t)| \leq \tilde{C}e^{-t}.$$

Integrating over $\theta \in \mathbb{S}^{n-1}$, we obtain

$$(43) \quad |\gamma(t) + \frac{2}{n-2} \ln \beta(t)| \leq \tilde{C}e^{-t}.$$

(42) and (43) imply (41).

We also have a counterpart to (22) for positive singular solutions $u(x)$ in the Γ_k^+ class to (8') on $B_R(0) \setminus \{0\}$.

Proposition 1. *Let $u(x)$ be a positive singular solution to (8') on $B_2(0) \setminus \{0\}$ in the Γ_k^+ class, $U(t, \theta)$, $\beta(t)$, $w(t, \theta)$, and $\gamma(t)$ be defined above. Then for any $0 < \delta$ small, there exists a constant $C > 0$ depending on δ such that*

$$(44) \quad |\nabla_{t, \theta}^l (U(t, \theta) - \beta(t))| \leq C\beta(t)e^{-(1-\delta)t}, \quad \text{for all } t \geq 0 \text{ and } 1 \leq l \leq 2,$$

and

$$(45) \quad |\nabla_{t, \theta}^l (w(t, \theta) - \gamma(t))| \leq Ce^{-(1-\delta)t} \quad \text{for all } t \geq 0 \text{ and } 1 \leq l \leq 2.$$

We now provide an argument for (45). First, (39) and the gradient estimates for solutions to (8), see [23], give a bound $B > 0$ depending on $l > 0$ and C in (39), such that

$$(46) \quad |\nabla_{t,\theta}^l w(t, \theta)| \leq B.$$

This obviously leads to

$$|\nabla_{t,\theta}^l \gamma(t)| \leq B,$$

which, together with (46), implies that

$$|\nabla_{t,\theta}^l (w(t, \theta) - \gamma(t))| \leq 2B.$$

This estimate, together with (41) and interpolation, proves (45).

3. FIRST PROOF OF THEOREM 1: EXPLOITING THE ODE SATISFIED BY THE RADIAL AVERAGE

Let $u(x)$ be a positive solution to (8') on $B_R \setminus \{0\}$ in the Γ_k^+ class, where the constant c is normalized to be $2^{-k} \binom{n}{k}$, and $\gamma(t)$ is defined as in (38). We first make

Claim 1.

$$(47) \quad \left\{ 2(1 - \gamma_t^2)^{k-1} \left[\frac{k}{n} \gamma_{tt} + \frac{n-2k}{2n} (1 - \gamma_t^2) \right] + \eta_1(t) \right\} e^{2k\gamma} = 1 + \eta_2(t),$$

and

$$(48) \quad e^{(2k-n)\gamma} \left\{ (1 - \gamma_t^2)^k + \eta_3(t) \right\} - e^{-n\gamma} \{ 1 + \eta_4(t) \} = h,$$

for some constant h , where $\eta_i(t)$, for $i = 1, \dots, 4$, have the decay rate $\eta_i(t) = O(e^{-2(1-\delta)t})$ as $t \rightarrow \infty$, and $\delta > 0$ can be made as small as one needs, as in (45).

We will postpone a proof for (47) and (48) to the end of this section. We can think of h as a numerical characteristic to each potential isolated singularity. We make another claim relating the asymptotic behavior of u with that of $\gamma(t)$, h , and k .

Claim 2. *Let u be a positive solution to (8') in the Γ_k^+ class in a punctured ball $B_R \setminus \{0\}$.*

(i) *If (16) holds, namely*

$$\liminf_{x \rightarrow 0} |x|^{\frac{n-2}{2}} u(x) > 0,$$

then $2k < n$ and $h > 0$. Furthermore, for some $\varepsilon > 0$,

$$(49) \quad 1 - \gamma_t^2(t) \geq \varepsilon$$

for all sufficiently large t .

(ii) In the case $2k < n$, condition (18) holds, namely

$$\liminf_{x \rightarrow 0} |x|^{\frac{n-2}{2}} u(x) = 0$$

iff $h = 0$; in such cases, we furthermore have,

$$(50) \quad \lim_{x \rightarrow 0} |x|^{\frac{n-2}{2}} u(x) = 0, \quad \gamma_t(t) > 0 \quad \text{for } t \text{ large, and} \quad \lim_{t \rightarrow \infty} \gamma(t) = \infty.$$

Combining (i) and (ii), we see that in the case $2k < n$, we always have $h \geq 0$, with $h = 0$ iff (18) holds.

Proof. (49) is proved by noting that (16) and (39) imply that

$$(51) \quad -C \leq \gamma(t) \leq C$$

for some C , which, together with (47), implies that, for large t , $1 - \gamma_t^2$ never changes sign, which, in turn with

(51), (46) and (47), implies that, for some $\varepsilon > 0$, $1 - \gamma_t^2 > \varepsilon$ for all sufficiently large t .

The part $h > 0$ in (i) can be proved in one of two ways. The first proof uses rescaling and compactness arguments on the translations to $\gamma(t)$, with the help of (51), (46) and (48) to produce a limiting $\widehat{\gamma}(t)$ which exists and is bounded for all $t \in \mathbb{R}$ and satisfies (14) with

$$e^{(2k-n)\widehat{\gamma}(t)} (1 - \widehat{\gamma}_t^2(t))^k - e^{-n\widehat{\gamma}(t)} = h$$

for the same h . But the classification result, Theorem C, says that no bounded solution of (14) exists for all $t \in \mathbb{R}$ with $h \leq 0$. Since no bounded solution exists to (14) for all $t \in \mathbb{R}$ when $2k \geq n$ according to Theorem C, this argument also shows that (16) implies that $2k < n$; equivalently, (18) must hold in the case $2k \geq n$.

The second proof regards (47) as a perturbation of (14), and makes a continuous dependence argument, with the help of (48), (49), and (51) to prove that, when $h < 0$, either $1 - \gamma_t^2(t) \rightarrow 0$ as $t \rightarrow \infty$, which contradicts (49),

or $1 - \gamma_t^2(t) \rightarrow -\infty$ as $t \rightarrow \infty$ in the case $1 - \gamma_t^2(t) < 0$ and k is odd, which contradicts (46). The case $h = 0$ can also be ruled out along similar lines by a more careful argument.

(18) is equivalent to

$$(52) \quad \limsup_{t \rightarrow \infty} \gamma(t) = \infty.$$

So when (18) holds and $2k < n$, it follows from (52), (48) and (46) directly that $h = 0$. For the converse in (ii), when $h = 0$, it follows from (i) that (18) must hold, thus proving (ii). In addition, it follows from (48) that, for

sufficiently large t , $\gamma_t(t) = 0$ can occur only near $\gamma(t) = 0$. Together with (52), we see that $\gamma_t(t) > 0$ for sufficiently large t and $\lim_{t \rightarrow \infty} \gamma(t) = \infty$. \square

We now proceed to prove (15). Our proof will handle four cases slightly differently: *Case (a)*. $h = 0$; *Case (b)*. $h > 0$ and $2k < n$; *Case (c)*. $2k > n$ and $h \neq 0$; and *Case (d)*. $2k = n$ and $h \neq 0$. Cases (a), (c) and (d) are proved by finding the asymptotics of $\gamma(t)$ directly using (48), while Case (b) will need the help of Theorem 3.

Case (a). $h = 0$. Using $\lim_{t \rightarrow \infty} \gamma(t) = \infty$ back into (48), which now takes the form

$$e^{2k\gamma} \left\{ (1 - \gamma_t^2)^k + \eta_3(t) \right\} - \{1 + \eta_4(t)\} = 0,$$

we see that $1 - \gamma_t^2(t) =: \eta(t) \rightarrow 0$ as $t \rightarrow \infty$. Since $\gamma_t(t) > 0$ for sufficiently large t , we conclude that $1 - \gamma_t(t) \rightarrow 0$ as $t \rightarrow \infty$. As a consequence, $\gamma(t) \geq (1 - \varepsilon)t + \gamma_0$ for large t and some $\varepsilon > 0$ small and γ_0 . This would imply through (48) that

$$|\eta(t)| \leq C e^{-\frac{2(1-\delta)}{k}t}$$

for some constant $C > 0$ and for large t . Finally, we have

$$|\gamma_t(t) - 1| = |\sqrt{1 - \eta(t)} - 1| \leq C e^{-\frac{2(1-\delta)}{k}t},$$

from which we conclude that

$$\gamma(t) - t = \tau + O(e^{-\frac{2(1-\delta)}{k}t}),$$

for some τ as $t \rightarrow \infty$.

Note that $\xi_0(t)$, the solution to (14) with $h = 0$, to which (47) is a perturbation, satisfies $\xi_0(t) = t - \ln 2 + O(e^{-2t})$. Therefore, using also (41),

$$w(t, \theta) = \gamma(t) + \widehat{w}(t, \theta) = \xi_0(t + \tau + \ln 2) + O(e^{-\frac{2(1-\delta)}{k}t}),$$

as $t \rightarrow \infty$, which is (15). Furthermore

$$\begin{aligned} u(x) &= e^{-\frac{n-2}{2}(w(t, \theta) - t)} = e^{-\frac{n-2}{2} \left(\xi_0(t + \tau + \ln 2) - t + O(e^{-\frac{2(1-\delta)}{k}t}) \right)} \\ &= u^*(|x|) e^{O(e^{-\frac{2(1-\delta)}{k}t})} = u^*(|x|) \left(1 + O(e^{-\frac{2(1-\delta)}{k}t}) \right) \\ &= u^*(|x|) \left(1 + O(|x|^{\frac{2(1-\delta)}{k}}) \right) \end{aligned}$$

where

$$u^*(|x|) = e^{-\frac{n-2}{2}(\xi_0(t+\tau+\ln 2)-t)}$$

is a positive radial solution to (8') on $\mathbb{R}^n \setminus \{0\}$. We also find in this case that

$$\lim_{x \rightarrow 0} u(x) = e^{-\frac{n-2}{2}\tau} =: u(0) > 0$$

exists, with

$$\begin{aligned} |u(x) - u(0)| &\leq |u(0)| \left| e^{-\frac{n-2}{2}(\widehat{w}(t,\theta) + O(e^{-\frac{2(1-\delta)}{k}t}))} - 1 \right| \leq C e^{-\frac{2(1-\delta)}{k}t} \\ &\leq C|x|^{\frac{2(1-\delta)}{k}}. \end{aligned}$$

□

Case (b). $2k < n$ and $h > 0$. Here h is subject to the further bound

$$h \leq h^* = \frac{2k}{n-2k} \left(\frac{n-2k}{n} \right)^{\frac{n}{2k}},$$

with h^* determined by

$$h^* := \sup \{ h : \min_{\gamma} (e^{-2k\gamma} + h e^{(n-2k)\gamma}) \leq 1 \}.$$

Set

$$H(x, y) = h + e^{-ny} - e^{(2k-n)y} (1-x^2)^k.$$

For $0 < h < h^*$, $H(0, \xi) = 0$ has two simple roots $\xi_- < \xi_+$ and H satisfies the conditions in case (ii) of Theorem 3 with $m = \xi_-$ and $l = 1$; for $h = h^*$, $H(0, \xi) = 0$ has a double root $m = \xi_- = \xi_+$, $H(0, m) = 0$ and $H(x, y) \geq 0$ satisfies the conditions in case (i) of Theorem 3 as well with $l = 2$. Thus, thanks to (47) and (48), we can apply Theorem 3 to conclude Theorem 1 in this case. □

Case (c). $2k > n$ and $h \neq 0$. As remarked earlier, (18) holds, which implies (52). (48) implies that, for large t , $\gamma'(t) = 0$ can occur only when $\gamma(t)$ is near certain finite value. Together with (52), this implies (50), which,

together with (48), implies that $(1 - \gamma_t^2(t))^k \rightarrow 0$ as $t \rightarrow \infty$. Then the conclusions (15) and (19) are proved in almost identical way as was done above for the $h = 0$ case. □

Case (d). $2k = n$ and $h \neq 0$. We first make

Claim 3. If $e^{-2w(t,\theta)}(dt^2 + d\theta^2) \in \Gamma_2^+$ for all $\theta \in \mathbb{S}^{n-1}$ at some t , then

$$(53) \quad 1 - \gamma_t^2(t) + \int_{\mathbb{S}^{n-1}} |\nabla \widehat{w}(t, \theta)|^2 d\theta \geq 0,$$

where $\gamma(t)$ and $\widehat{w}(t, \theta)$ are defined as before.

Assuming (53) now, then (53), (45) and (48) imply that $0 \leq h \leq 1$. The case $h = 1$ can be ruled out after further analysis of (48). We can again argue as above that (50) holds. Then (48) implies that $(1 - \gamma_t^2(t))^k \rightarrow h$ as $t \rightarrow \infty$;

and $e^{-n\gamma(t)} = O(e^{-\alpha t})$ as $t \rightarrow \infty$ for some $\alpha > 0$ depending on $0 < h < 1$. Now with $\eta(t) := 1 - \gamma_t^2(t)$, we find

$$\eta^k(t) = h + e^{-n\gamma(t)}(1 + \eta_4(t)) - \eta_3(t) = h + O(e^{-\alpha t})$$

as $t \rightarrow \infty$, and

$$\gamma(t) = \sqrt{1 - \eta(t)} = \sqrt{1 - \sqrt[k]{h}} + O(e^{-\alpha t}),$$

which implies that $\gamma(t) = \sqrt{1 - \sqrt[k]{h}} + \gamma_0 + O(e^{-\alpha t})$, for some γ_0 . Similarly, $\xi_h(t)$ satisfies $\xi_h(t) = \sqrt{1 - \sqrt[k]{h}} + \xi_0 + O(e^{-\alpha t})$, for some ξ_0 . Thus for some τ , we have

$$\gamma(t) = \xi_h(t + \tau) + O(e^{-\alpha t}),$$

and

$$u(x) = e^{-\frac{n-2}{2}(w(t,\theta)-t)} = e^{-\frac{n-2}{2}(\gamma(t)-t+\widehat{w}(t,\theta))},$$

from which we find that

$$|x|^{\frac{n-2}{2}}(1 - \sqrt{1 - \sqrt[k]{h}})u(x) = e^{-\frac{n-2}{2}(\gamma(t) - \sqrt{1 - \sqrt[k]{h}} + \widehat{w}(t,\theta))},$$

extends to a $C^\alpha(B_R)$ positive function for some $\alpha > 0$. \square

We now provide proofs for (47), (48) and (53) in Claims 1 and 3.

Proof of (47). (47) is derived from (14), (41) and (45) as follows. First, with $\widehat{w}(t, \theta) := w(t, \theta) - \gamma(t)$, it follows from (41) and (45) that

$$(54) \quad \sigma_k(A_{w(t,\theta)}) = \sigma_k(A_{\gamma(t)}) + L_{\gamma(t)}[\widehat{w}(t, \theta)] + \widehat{\eta}_1(t, \theta),$$

where $L_{\gamma(t)}$ denotes the linearized operator for $\sigma_k(A_{\gamma(t)})$ at $\gamma(t)$, and $\widehat{\eta}_1(t, \theta)$ satisfies $|\widehat{\eta}_1(t, \theta)| = O(e^{-2(1-\delta)t})$. Next,

$$(55) \quad e^{2kw(t,\theta)} = e^{2k\gamma(t)} \cdot e^{2k\widehat{w}(t,\theta)},$$

and

$$(56) \quad e^{-2k\widehat{w}(t,\theta)} = 1 - 2k\widehat{w}(t, \theta) + \widehat{\eta}_2(t, \theta),$$

where

$$|\widehat{\eta}_2(t, \theta)| = O(e^{-2t}).$$

Putting (54), (55), and (56) into (8), integrating over $\theta \in \mathbb{S}^{n-1}$, and noting that

$$(57) \quad \int_{\mathbb{S}^{n-1}} \widehat{w}(t, \theta) d\theta = 0,$$

and

$$(58) \quad \int_{\mathbb{S}^{n-1}} L_{\gamma(t)}[\widehat{w}(t, \theta)] d\theta = 0,$$

we arrive at (47). □

Proof of (48). A crude variant of (48) in the case $2k \leq n$ can be derived from (47) by elementary means as follows. Multiplying both sides of (47) by $ne^{-n\gamma(t)}\gamma(t)$, one has

$$[e^{(2k-n)\gamma(t)}(1 - \gamma_r^2(t))^k - e^{-n\gamma(t)}]_t = ne^{-n\gamma(t)}\gamma(t) [e^{2k\gamma(t)}\eta_1(t) - \eta_2(t)],$$

the right hand of which is of the order $O(e^{-2(1-\delta)t})$ as $t \rightarrow \infty$ in the case of $2k \leq n$, from (39), the gradient estimates, and the decay rates of $\eta_1(t)$, $\eta_2(t)$. It then follows that for some constant h , we have

$$(59) \quad e^{(2k-n)\gamma(t)}(1 - \gamma_r^2(t))^k - e^{-n\gamma(t)} = h + O(e^{-2(1-\delta)t}).$$

The more precise version, (48), is needed only to handle the case $2k > n$, or the $h = 0$ case when $2k < n$. It is derived from a Pohozaev type identity for solutions $w(t, \theta)$ to (8) when σ_k is a constant, which takes the form

$$(60) \quad \int_{\mathbb{S}^{n-1}} \left[\frac{n}{2k\sigma_k} e^{(2k-n)w(t, \theta)} \sum_{a=1}^n T_{a1}[w(t, \theta)] w_{at}(t, \theta) - e^{-nw(t, \theta)} \right] d\theta = \widehat{h}$$

for some constant \widehat{h} independent of t , where $T_{a1}[w(t, \theta)]$ are the components of the Newton tensor associated with $\sigma_k(A_{w(t, \theta)})$. Identities of the form (60) for solutions to (8) were first derived by Viaclovsky in [53]. (60) is a version from [31]. We assume (60) now and postpone a proof to the end of this section. Using (41) and (45), we

find that

$$\begin{aligned} & \sum_{a=1}^n T_{a1}[w(t, \theta)]w_{at}(t, \theta) \\ &= T_{11}[\gamma(t)]\gamma_t + \widehat{L}_{\gamma(t)}[\widehat{w}(t, \theta)] + O(e^{-2(1-\delta)t}) \\ &= \frac{2k\sigma_k}{n}(1 - \gamma_t^2(t))^{k-1}\gamma_t + \widehat{L}_{\gamma(t)}[\widehat{w}(t, \theta)] + O(e^{-2(1-\delta)t}), \end{aligned}$$

where $\widehat{L}_{\gamma(t)}$ stands for the linearization at $w(t, \theta) = \gamma(t)$ to $\sum_{a=1}^n T_{a1}[w(t, \theta)]w_{at}(t, \theta)$, and we have used that

$T_{a1}[\gamma(t)] = 2k\sigma_k\delta_{a1}(1 - \gamma_t^2(t))^{k-1}/n$. Using (47) to solve for $(1 - \gamma_t^2(t))^{k-1}\gamma_t$, we find that

$$(1 - \gamma_t^2(t))^{k-1}\gamma_t = -\frac{n-2k}{2k}(1 - \gamma_t^2(t))^k + \frac{n}{2k}e^{-2k\gamma(t)}(1 + \eta_2(t)) - \eta_1(t).$$

Using these in (60), and with estimates like (56), (57) and (58), we arrive at (48), in the case $2k \neq n$, with

$$\widehat{h} = \frac{2k-n}{2k}|\mathbb{S}^{n-1}|h.$$

When $2k = n$, (48) is covered by (59). □

Proof of (53). (53) is proved by noting that, if $A_g(t, \theta) \in \Gamma_2^+$ for all $\theta \in \mathbb{S}^{n-1}$, then

$$\int_{\mathbb{S}^{n-1}} A_g(t, \theta) d\theta \in \Gamma_2^+,$$

due to the convexity of Γ_2^+ . In our case the matrix for the Schouten tensor of the metric $g = e^{-2w(t, \theta)}(dt^2 + d\theta^2)$ is

$$(61) \quad A_g = \begin{bmatrix} w_{tt} + w_t^2 - \frac{1}{2}|\nabla w|^2 - \frac{1}{2} & w_{t\theta_j} + w_t w_{\theta_j} \\ w_{\theta_i t} + w_{\theta_i} w_t & w_{\theta_i \theta_j} + w_{\theta_i} w_{\theta_j} + \frac{1}{2}(1 - |\nabla w|^2)\delta_{ij} \end{bmatrix}.$$

Using $w(t, \theta) = \gamma(t) + \widehat{w}(t, \theta)$ and $\int_{\mathbb{S}^{n-1}} \widehat{w}(t, \theta) d\theta = 0$, we find

$$\int_{\mathbb{S}^{n-1}} A_g(t, \theta) d\theta = \text{diag}\left[\gamma_{tt}(t) - \frac{1 - \gamma_t^2(t)}{2}, \frac{1 - \gamma_t^2(t)}{2}, \dots, \frac{1 - \gamma_t^2(t)}{2}\right] - \frac{a(t)}{2}I_{n \times n} + \begin{bmatrix} b_{11}(t) & b_{1j}(t) \\ b_{i1}(t) & b_{ij}(t) \end{bmatrix},$$

where

$$a(t) = \int_{\mathbb{S}^{n-1}} |\nabla \widehat{w}(t, \theta)|^2 d\theta,$$

$$b_{11}(t) = \int_{\mathbb{S}^{n-1}} |\widehat{w}_1(t, \theta)|^2 d\theta,$$

$$b_{1i}(t) = b_{i1}(t) = \int_{\mathbb{S}^{n-1}} \widehat{w}_i(t, \theta) \widehat{w}_{\theta_i}(t, \theta) d\theta, \quad \text{for } i > 1,$$

$$b_{ij}(t) = \int_{\mathbb{S}^{n-1}} \widehat{w}_{\theta_i}(t, \theta) \widehat{w}_{\theta_j}(t, \theta) d\theta, \quad \text{for } i, j > 1.$$

But

$$a(t)I_{n \times n} - \begin{bmatrix} b_{11}(t) & b_{1j}(t) \\ b_{i1}(t) & b_{ij}(t) \end{bmatrix} \geq 0$$

as a matrix, so

$$\text{diag}[\gamma_t - \frac{1 - \gamma_t^2(t)}{2}, \frac{1 - \gamma_t^2(t)}{2}, \dots, \frac{1 - \gamma_t^2(t)}{2}] + \frac{a(t)}{2} I_{n \times n} \in \Gamma_2^+.$$

Computing the σ_1 and σ_2 of this tensor as in (13) it follows that

$$\gamma_t(t) + \frac{n-2}{2} (1 - \gamma_t^2(t) - a(t)) > 0,$$

and

$$(1 - \gamma_t^2(t) + a(t)) \left[1 - \gamma_t^2(t) + a(t) + \frac{4}{n} (\gamma_t(t) + \gamma_t^2(t) - 1) \right] > 0.$$

Simple algebra from these two inequalities implies (53). □

Finally we sketch a proof for (60). It follows from equation (3) in [31] that $\nabla_a Y^a = 0$, where

$$Y^a = T_b^a \nabla^b (\text{div}_g X) + 2k \sigma_k X^a,$$

for any conformal Killing vector field X^a on (M, g) with $\sigma_k(A_g) \equiv \text{constant}$ on M . We will take $M = \mathbb{R} \times \mathbb{S}^{n-1}$,

$g = e^{-2w(t, \theta)}(dt^2 + d\theta^2)$, and $X = \partial_t$. Thus

$$(62) \quad \int_{\mathbb{S}^{n-1}} Y^1(t, \theta) \sqrt{g(t, \theta)} d\theta = \text{constant},$$

independent of t . In addition, $\text{div}_g X = -nw_t(t, \theta)$, and (62) would take the form

$$\int_{\mathbb{S}^{n-1}} \left(-n \sum_{b=1}^n T_{1b} w_{tb}(t, \theta) e^{2kw(t, \theta)} + 2k\sigma_k \right) e^{-nw(t, \theta)} d\theta = \text{constant},$$

which gives (60).

4. PROOF OF THEOREM 2 AND SECOND PROOF OF THEOREM 1

We can now present our second proof for the main part of Theorem 1: the case $2k < n$ and $h > 0$. Let $w(t, \theta)$

be a solution to $\sigma_k(g^{-1} \circ A_g) = 2^{-k} \binom{n}{k}$, with

$$g = u^{\frac{4}{n-2}}(x) |dx|^2 = e^{-2w(t, \theta)}(dt^2 + d\theta^2)$$

for x to be over the punctured unit ball $x \in B^n \setminus \{0\}$. It is assumed that g is in the Γ_k^+ class over $B^n \setminus \{0\}$. Then

$w(t, \theta)$ is defined for $(t, \theta) \in \mathbb{R}^+ \times \mathbb{S}^{n-1}$, as $t = -\ln|x|$. It follows from Theorem E that (39) holds, i.e., for some constant $C_2 > 0$,

$$e^{-2w(t, \theta)} \leq C_2$$

for all $(t, \theta) \in \mathbb{R}^+ \times \mathbb{S}^{n-1}$. It follows from our discussion in the beginning of the previous section that $h > 0$ implies (16), i.e., for some $C_1 > 0$

$$e^{-2w(t, \theta)} \geq C_1$$

for $(t, \theta) \in \mathbb{R}^+ \times \mathbb{S}^{n-1}$, namely, $w(t, \theta)$ is bounded over $(t, \theta) \in \mathbb{R}^+ \times \mathbb{S}^{n-1}$. As in [32], We make the following assertions about the behavior of $w(t, \theta)$ as $t \rightarrow \infty$.

(a) Let $t_j \rightarrow \infty$ be any sequence tending to ∞ , then $\{w_j(t, \theta) := w(t + t_j, \theta)\}$ has a subsequence converging

to a bounded limiting solution $\xi(t)$ of (14) defined for $(t, \theta) \in \mathbb{R} \times \mathbb{S}^{n-1}$. The convergence is uniform on

any compact subset of $\mathbb{R}^+ \times \mathbb{S}^{n-1}$.

(b) Any angular derivative $\partial_\theta w(t, \theta)$ of w converges to 0 as $t \rightarrow \infty$.

(c) There exists $S > 0$ such that for any infinitesimal rotation ∂_θ of \mathbb{S}^{n-1} , and for any $t_j \rightarrow \infty$, if we set

$$A_j = \sup_{t \geq 0} |\partial_\theta w_j(t, \theta)|, \text{ and if } |\partial_\theta w_j(s_j, \theta_j)| = A_j \text{ for some } (s_j, \theta_j) \in \mathbb{R}^+ \times \mathbb{S}^{n-1}, \text{ then } s_j \leq S.$$

(d) $\partial_\theta w(t, \theta)$ converges to 0 at an exponential rate as $t \rightarrow \infty$, and

$$|w(t, \theta) - |\mathbb{S}^{n-1}|^{-1} \int_{\mathbb{S}^{n-1}} w(t, \omega) d\omega|$$

converges to 0 at an exponential rate as $t \rightarrow \infty$.

(e) There exists a bounded (periodic) solution $\xi(t)$ of (14) and $\tau \geq 0$ such that $w(t, \theta)$ converges to $\xi(t + \tau)$ at an exponential rate as $t \rightarrow \infty$.

(a)–(e) are proved along almost identical lines as in the proof for Proposition 5 in [32], provided some analytical preparations are established. In our case here, proofs for (a) and (b) can be provided using the local derivative estimates of [23] for solutions of (7) and the classification result in [38], reformulated as Theorem D above; proofs for (c), (d) and (e) will need an analysis of the linearized operator of (8) at a radial solution $\xi(t)$, as characterized by Proposition 2 below. We remark that an alternative proof for (d) without using the analysis of the linearized operator is in fact already contained in (41) and (45).

To compute the linearized operator of (8) at a radial solution $\xi(t)$, we use (61). When $w(t, \theta) = \xi(t)$, A_g becomes a block diagonal matrix

$$\begin{bmatrix} \xi_{tt} + \frac{1}{2}(\xi_t^2 - 1) & 0 \\ 0 & \frac{1}{2}(1 - |\xi_t|^2)\delta_{ij} \end{bmatrix}.$$

When we linearize $\sigma_k(A_g)$ at such a block diagonal matrix, the coefficient matrix consisting of the coefficients of the Newton tensor

$$T_{ij} = \frac{1}{(k-1)!} \delta_{j_1 \dots j_{k-1} j}^{i_1 \dots i_{k-1} i} A_{i_1 j_1} \dots A_{i_{k-1} j_{k-1}}$$

is also diagonal:

$$T_{11} = \binom{n-1}{k-1} \frac{1}{2^{k-1}} (1 - |\xi_t|^2)^{k-1},$$

while for $i \geq 2$,

$$\begin{aligned} T_{ii} &= \binom{n-1}{k-1} \frac{(n-k)(1 - |\xi_t|^2)^{k-1}}{2^{k-1}(n-1)} + \binom{n-2}{k-2} \frac{(1 - |\xi_t|^2)^{k-2}}{2^{k-2}} \left[\xi_{tt} + \frac{1}{2}(\xi_t^2 - 1) \right] \\ &= \binom{n-1}{k-1} \frac{(1 - |\xi_t|^2)^{k-2}}{2^{k-2}(n-1)} \left[(k-1)\xi_{tt} + \frac{n-2k+1}{2} (1 - |\xi_t|^2) \right]. \end{aligned}$$

So the linearization of $\sigma_k(A_g)$ at $g = e^{-2\xi(t)}(dt^2 + d\theta^2)$ is

$$\begin{aligned} L_\xi(\phi) &= T_{11}(t) [\phi_{tt}(t, \theta) + \xi_t(t) \phi_t(t, \theta)] + \sum_{i \geq 2} T_{ii}(t) [\phi_{\theta_i \theta_i}(t, \theta) - \xi_t(t) \phi_t(t, \theta)] \\ &= T_{11}(t) \phi_{tt}(t, \theta) + [T_{11}(t) - (n-1)T_{22}(t)] \xi_t(t) \phi_t(t, \theta) + T_{22}(t) \Delta_\theta \phi(t, \theta), \\ &= \frac{(1 - |\xi_t(t)|^2)^{k-2}}{2^{k-2}} \binom{n-1}{k-1} [A(t) \phi_{tt}(t, \theta) + B(t) \phi_t(t, \theta) + C(t) \Delta_\theta \phi(t, \theta)], \end{aligned}$$

where

$$(63) \quad A(t) = \frac{(1 - |\xi_t(t)|^2)}{2},$$

$$(64) \quad B(t) = -\xi_t(t) \left[(k-1) \xi_{tt}(t) + \frac{n-2k}{2} (1 - |\xi_t(t)|^2) \right],$$

$$(65) \quad C(t) = \frac{k-1}{n-1} \xi_{tt}(t) + \frac{n-2k+1}{n-1} \cdot \frac{1 - |\xi_t(t)|^2}{2}.$$

When $\xi(t)$ is a solution to $\sigma_k(g^{-1} \circ A_g) = \text{const.}$, normalized to be $2^{-k} \binom{n}{k}$, the linearization of the nonlinear PDE (8) at $\xi(t)$ is then

$$L_\xi(\phi) + 2^{1-k} k \binom{n}{k} e^{-2k\xi(t)} \phi(t, \theta) = 0.$$

If we take the projections of $\phi(t, \cdot)$ into spherical harmonics:

$$\phi(t, \theta) = \sum_j \phi_j(t) Y_j(\theta), \quad \text{where } Y_j(\theta) \text{ are the normalized eigenfunctions of } \Delta_\theta \text{ on } L^2(\mathbb{S}^{n-1}).$$

then $\phi_j(t)$ satisfies the ODE

$$(66) \quad L_j[\phi_j] := \phi_j''(t) + \frac{B(t)}{A(t)} \phi_j'(t) + \left\{ -\lambda_j \frac{C(t)}{A(t)} + \frac{ne^{-2k\xi(t)}}{2A(t)(1 - \xi_t^2(t))^{k-2}} \right\} \phi_j(t) = 0,$$

where λ_j are the eigenvalues of Δ_θ on $L^2(\mathbb{S}^{n-1})$ associated with $Y_j(\theta)$, thus

$$\lambda_0 = 0, \quad \lambda_1 = \dots = \lambda_n = n - 1, \quad \lambda_j \geq 2n, \quad \text{for } j > n.$$

Similar to properties of the linearized operator to the scalar curvature operator used in [32], we have the following properties for the L_j 's.

Proposition 2. *For all solutions $\xi_h(t)$ to (14) with $h > 0$, $k < n$, and $j \geq 1$, the following holds:*

- (i) $L_j[\phi] = 0$ has a pair of linearly independent solution basis on \mathbb{R} , one of which grows unbounded and the other one decays exponentially as $t \rightarrow \infty$;
- (ii) Any solution of $L_j[\phi] = 0$ which is bounded for \mathbb{R}^+ must decay exponentially;
- (iii) Any solution of $L_j[\phi] = 0$ which is bounded for all of \mathbb{R} must be identically 0;
- (iv) Any solution of $L_j[\phi] = 0$ which is bounded for all of \mathbb{R}^+ must be unbounded on \mathbb{R}^- .

These conclusions remain true for solutions $\xi_h(t)$ to (14) with $h = 0$ and $\lambda_j \geq 2n$.

While Proposition 2 is sufficient for providing a proof for Theorem 1, Theorem 2 requires some more detailed knowledge about the linearized operator to (8). More specifically, the decay rates of bounded solutions to $L_j[\phi] = 0$ on \mathbb{R}^+ need to be faster than e^{-t} when $\lambda_j \geq 2n$. L_j is an ordinary differential operator with period coefficient, so, by Floquet theory, has a set of well defined characteristic roots which give the exponential decay/grow rates to solutions ϕ of $L_j[\phi] = 0$ on \mathbb{R} , see, for instance, Theorem 5.1 in Chapter 3 of [15]. In fact, Theorem 5.1 in Chapter 3 of [15] and (5.11) on p. 81 of [15] applied to L_j implies that $L_j[\phi] = 0$ has a set of fundamental

solutions in the form of $e^{\rho_j t} p_1(t)$ and $e^{-\rho_j t} p_2(t)$ for some periodic functions $p_1(t)$ and $p_2(t)$, when $\rho_j \neq 0$. We have the following

Lemma 1. *When $2k \leq n$ and $h > 0$, there is a $\beta_* > \sqrt{2}$ such that for all $\lambda_j \geq 2n$, the associated ρ_j satisfies $\rho_j \geq \beta_*$.*

We can also formulate and prove a version that does not need L_j to have the structure to apply the Floquet theory.

Lemma 2. *Define*

$$V(t) = e^{(1-\frac{n}{2k})\xi(t)} \left(e^{-n\xi(t)} + h \right)^{\frac{k-1}{2k}}.$$

Then

$$(67) \quad V(t)L_j[V^{-1}(t)\psi(t)] = \psi_t(t) + E(t)\psi(t),$$

where we can estimate $E(t) \leq -C_n < -2$ when $\lambda_j \geq 2n$ and $2k \leq n$. As a consequence, when $\beta^2 < C_n$, $e^{-\beta t}V^{-1}(t)$ satisfies

$$L_j[e^{-\beta t}V^{-1}(t)] = [\beta^2 + E(t)]e^{-\beta t}V^{-1}(t)$$

is a supersolution to $L_j[\phi] = 0$ on \mathbb{R}^+ , therefore when $2k \leq n$ and $h > 0$, for any $\beta < \beta_* := \sqrt{C_n}$, and for all

$\lambda_j \geq 2n$, any bounded solution ϕ of $L_j[\phi] = 0$ on \mathbb{R}^+ satisfies $|\phi(t)| \lesssim e^{-\beta t}$.

Remark. Lemma 1 is an immediate consequence of (67) in Lemma 2. It is not immediately clear that the characteristic roots ρ_j of L_j are monotone increasing as the λ_j increases. But in the case $2k \leq n$ and $h > 0$, (67) allows a variational construction of a bounded (in fact, decaying) fundamental solution to $L_j[\phi] = 0$ on \mathbb{R}^+ which is positive. Thus in such cases comparison theorems show that the ρ_j indeed is monotone increasing as λ_j increases.

With such knowledge, we can now establish

Proposition 3. Suppose that $\phi(t, \theta)$ satisfies

$$(68) \quad L_\xi(\phi) + 2^{1-k}k \binom{n}{k} e^{-2k\xi(t)} \phi(t, \theta) = r(t, \theta), \quad \text{for } t \geq t_0 \text{ and } \theta \in \mathbb{S}^{n-1}.$$

Suppose that for some $0 < \beta < \beta_*$ and $\beta \neq 1$, $|r(t, \theta)| \lesssim e^{-\beta t}$. Then there exist constants a_j for $j = 1, \dots, n$, such that

$$(69) \quad |\phi(t, \theta) - \sum_{j=1}^n a_j e^{-t} (1 + \xi_t(t)) Y_j(\theta)| \lesssim e^{-\beta t}.$$

In fact, when $\beta_* \leq \beta < \rho_{n+1}$, (69) continues to hold, and when $\beta > \rho_{n+1}$, we will have

$$(70) \quad |\phi(t, \theta) - \sum_{j=1}^n a_j e^{-t} (1 + \xi_t(t)) Y_j(\theta)| \lesssim e^{-\rho_{n+1} t}.$$

When $\beta = 1$, (69) continues to hold if the right hand side is modified into te^{-t} ; and when $\beta = \rho_{n+1}$, (70) continues to hold if the right hand side is modified into $te^{-\rho_{n+1} t}$.

We now first prove Proposition 2, then provide a proof for Proposition 3 and Theorem 2, while deferring the proof for Lemma 2 to the appendix. For $j \leq n$, (i)–(iv) of Proposition 2 follow from an explicit solution basis to (66); for $j \geq n+1$, the arguments in [32] relies on the sign of the coefficient of the zeroth order term of L_j to be negative. Our computations below verify the same properties for the σ_k curvature problem when $k < n$, although we will only use these properties for the case $2k < n$ here.

Since $\xi(t)$ satisfies (14), with $\sigma_k = 2^{-k} \binom{n}{k}$, (14) becomes

$$(71) \quad 2(1 - \xi_t^2)^{k-1} \left[\frac{k}{n} \xi_{tt} + \left(\frac{1}{2} - \frac{k}{n} \right) (1 - \xi_t^2) \right] e^{2k\xi} = 1.$$

Due to the translation invariance of (71) in t , $\phi_0^+ = \partial_t \xi(t)$ is a solution to (66) for $\lambda_0 = 0$; since $\xi_h(t)$ is another family of solution to (71), we find $\phi_0^- = \partial_h \xi_h(t)$ to be another solution to (66) for $\lambda_0 = 0$. Differentiating the

first integral $e^{(2k-n)\xi}(1-\xi^2)^k = e^{-n\xi} + h$ with respect to t and h , respectively, one finds that $\{\phi_0^+(t), \phi_0^-(t)\}$ is linearly independent, thus forms a solution basis to (66) for $\lambda_0 = 0$.

Proof of Proposition 2. For $h > 0$, and $\lambda_j = n - 1$, which corresponds to $Y_j(\theta) = \theta_j$, we claim that

$$[1 - \partial_t \xi(t)] e^t \quad \text{and} \quad [1 + \partial_t \xi(t)] e^{-t}$$

form a basis to (66). This is due the translation invariance of (7): if $u(x)$ is a solution to (7), so is $u(x+a)$ for any $a \in \mathbb{R}^n$. In terms of $w(t, \theta)$, this means that

$$w_a(t, \theta) := -\ln|x| - \frac{2}{n-2} \ln u(x+a)$$

is a solution to (8). Thus $\partial_{a_j}|_{a=0} w_a(t, \theta)$ is a solution to the linearized equation of (8). But when $w(t, \theta) = \xi(t)$, we have

$$\begin{aligned} \partial_{a_j}|_{a=0} w_a(t, \theta) &= -\frac{2}{n-2} \partial_{x_j} \ln u(x) = \partial_{x_j} [\ln|x| + w(t, \theta)] \\ &= \frac{x_j}{|x|^2} + \partial_t \xi(t) \frac{\partial t}{\partial x_j} = [1 - \partial_t \xi(t)] e^t \theta_j. \end{aligned}$$

Thus $[1 - \partial_t \xi(t)] e^t$ is a solution of (66) with $\lambda_j = n - 1$. Since we have normalized $\xi(t)$ such that it is even in t , we find that $[1 + \partial_t \xi(t)] e^{-t}$ is another solution of (66) with $\lambda_j = n - 1$. $\{[1 - \partial_t \xi(t)] e^t, [1 + \partial_t \xi(t)] e^{-t}\}$ becomes linearly dependent only when they are identical (as they are both equal to 1 at $t = 0$), which is the case only when $\partial_t \xi(t) = \tanh(t)$ and $h = 0$. When $h > 0$ and $k < n$, one can use the asymptotic expansion in [8] to see that $[1 - \partial_t \xi(t)] e^t$ is exponentially growing. When $h > 0$ and $2k \leq n$, this can be seen more directly: the solution

$\xi(t)$ has the bound $C_h^{-1} \leq 1 \pm \partial_t \xi(t) \leq C_h$ for some $C_h > 0$, so $\{[1 + \partial_t \xi(t)] e^{-t}, [1 - \partial_t \xi(t)] e^t\}$ forms a solution basis for (66) with $\lambda_j = n - 1$, with one exponentially decaying and the other one exponentially growing, and the conclusion of the Proposition in the case $\lambda_j = n - 1$ follows from the explicit basis.

For $\lambda_j \geq 2n$, we will verify that

(72) the coefficient of ϕ_j in (66) has a negative upper bound.

Assuming (72) for now, we sketch a proof for properties (i)-(iv) of L_j for the case $\lambda_j \geq 2n$. The key is to check

that for $0 < \lambda$ small, $e^{\pm \lambda t}$ are supersolutions of $L_j[e^{\pm \lambda t}] \leq 0$. This is because

$$L_j[e^{\pm \lambda t}] = \lambda^2 \pm \lambda \left[1 - (n-1) \frac{C(t)}{A(t)} \right] \xi_t(t) - \lambda_j \frac{C(t)}{A(t)} + n e^{-2k\xi(t)} (1 - \xi_t^2)^{1-k},$$

and it follows from (72) that for $\lambda_j \geq 2n$,

$$-\lambda_j \frac{C(t)}{A(t)} + ne^{-2k\xi(t)}(1 - \xi_t^2)^{1-k}$$

has a negative upper bound. Furthermore, using (71), we have

$$\begin{aligned} C(t) &= \frac{k-1}{n-1} \xi_{tt}(t) + \frac{n-2k+1}{n-1} \cdot \frac{1 - |\xi_t(t)|^2}{2} \\ &= \frac{n(k-1)}{k(n-1)} \left[\frac{e^{-2k\xi}}{2(1 - \xi_t^2)^{k-1}} - \frac{n-2k}{2n} (1 - \xi_t^2) \right] + \frac{n-2k+1}{2(n-1)} (1 - \xi_t^2) \\ &= \frac{n(k-1)}{2k(n-1)} \frac{e^{-2k\xi}}{(1 - \xi_t^2)^{k-1}} + \frac{(n-k)(1 - \xi_t^2)}{2k(n-1)}. \end{aligned}$$

Thus

$$\begin{aligned} \frac{C(t)}{A(t)} &= \frac{n(k-1)}{k(n-1)} \frac{e^{-2k\xi}}{(1 - \xi_t^2)^k} + \frac{(n-k)}{k(n-1)} \\ &= \frac{n(k-1)}{k(n-1)} \frac{e^{-n\xi}}{e^{-n\xi} + h} + \frac{(n-k)}{k(n-1)} \end{aligned}$$

is bounded from above. Here we used the first integral $e^{(2k-n)\xi}(1 - \xi_t^2)^k = e^{-n\xi} + h$ and as a consequence $\xi \geq 0$.

It is now clear that we can choose $\lambda > 0$ small to make $L_j[e^{\pm\lambda t}] < 0$ for all $t \in \mathbb{R}$.

Now fix such a $\lambda > 0$. We claim that if $\phi(t)$ is a bounded solution of $L_j[\phi] = 0$ on \mathbb{R}^\pm , then

$$|\phi(t)| \leq |\phi(0)|e^{-\lambda(\pm t)} \quad \text{for all } t \in \mathbb{R}^\pm,$$

which then implies (ii). This is because for any $\varepsilon > 0$, $|\phi(0)|e^{-\lambda(\pm t)} + \varepsilon e^{\lambda(\pm t)}$ is a supersolution of L_j on \mathbb{R}^\pm . So if $\phi(t)$ is a bounded solution of $L_j[\phi] = 0$ on \mathbb{R}^\pm , then by comparison principle,

$$|\phi(t)| \leq |\phi(0)|e^{-\lambda(\pm t)} + \varepsilon e^{\lambda(\pm t)} \quad \text{for all } t \in \mathbb{R}^\pm.$$

For any fixed $t \in \mathbb{R}^\pm$, since the above estimate holds for all $\varepsilon > 0$, we can send ε to 0 to verify our claim. (iii) now follows from (ii) and the maximum principle, and (iv) obviously is a direct corollary of (iii).

Next, any L_j has a pair of linearly independent solution basis $\{\phi_1(t), \phi_2(t)\}$ on \mathbb{R} . If both are bounded on \mathbb{R}^+ , and a and b are such that $a\phi_1(0) + b\phi_2(0) = 0$, then our claim implies that $a\phi_1(t) + b\phi_2(t) \equiv 0$ on \mathbb{R}^+ , contradicting their choice.

It remains to establish that there is a nontrivial solution of $L_j[\phi] = 0$ bounded on \mathbb{R}^+ . Since we have verified that L_j is uniformly elliptic on \mathbb{R}^+ and satisfies the maximum principle there, we can establish the desired existence by a convergence argument for solutions which are constructed on a sequence of finite intervals that exhaust \mathbb{R}^+ .

Finally we come back to verify (72). Since $1 \geq 1 - \xi_t^2 > 0$ by Theorem C (this is also true for $k = 1$), we see that, when $\lambda_j \geq 2n$, the coefficient of $\phi_j(t)$ in (66) is bounded from above by

$$\begin{aligned}
& -\lambda_j \left[\frac{n(k-1)}{k(n-1)} \frac{e^{-2k\xi}}{(1-\xi_t^2)^k} + \frac{(n-k)}{k(n-1)} \right] + \frac{ne^{-2k\xi(t)}}{(1-\xi_t^2(t))^{k-1}} \\
& \leq -2n \left[\frac{n(k-1)}{k(n-1)} \frac{e^{-2k\xi}}{(1-\xi_t^2)^k} + \frac{(n-k)}{k(n-1)} \right] + \frac{ne^{-2k\xi(t)}}{(1-\xi_t^2(t))^{k-1}} \\
& = -\frac{2ne^{-2k\xi(t)}}{(n-1)(1-\xi_t^2)^k} \left[\frac{n(k-1)}{k} - \frac{(n-1)(1-\xi_t^2)}{2} \right] - \frac{2n(n-k)}{k(n-1)} \\
& \leq -\frac{2ne^{-2k\xi(t)}}{(n-1)(1-\xi_t^2)^k} \left[\frac{n}{2} - \frac{n}{k} + \frac{1}{2} \right] - \frac{2n(n-k)}{k(n-1)} \\
& < -\frac{2n(n-k)}{k(n-1)} < 0,
\end{aligned}$$

when $n > k \geq 2$; when $k = 1$, the above estimate gives

$$\begin{aligned}
& -\lambda_j \left[\frac{n(k-1)}{k(n-1)} \frac{e^{-2k\xi}}{(1-\xi_t^2)^k} + \frac{(n-k)}{k(n-1)} \right] + \frac{ne^{-2k\xi(t)}}{(1-\xi_t^2(t))^{k-1}} \\
& \leq -\frac{2ne^{-2k\xi(t)}}{(n-1)(1-\xi_t^2)^k} \left[\frac{n(k-1)}{k} - \frac{(n-1)(1-\xi_t^2)}{2} \right] - \frac{2n(n-k)}{k(n-1)} \\
& = n \left[e^{-2\xi(t)} - 2 \right] \leq -n,
\end{aligned}$$

as $\xi(t) \geq 0$, which follows from the first integral

$$e^{-2\xi(t)} + he^{(n-2)\xi(t)} = 1 - \xi_t^2(t) \leq 1$$

with $h \geq 0$; while for $k = n$ and $h = 0$, the above estimate gives

$$\begin{aligned} & -\lambda_j \left[\frac{n(k-1)}{k(n-1)} \frac{e^{-2k\xi}}{(1-\xi_t^2)^k} + \frac{(n-k)}{k(n-1)} \right] + \frac{ne^{-2k\xi(t)}}{(1-\xi_t^2(t))^{k-1}} \\ & \leq -\frac{2ne^{-2n\xi(t)}}{(1-\xi_t^2)^n} \left[1 - \frac{1-\xi_t^2}{2} \right] \leq -\frac{ne^{-2n\xi(t)}}{(1-\xi_t^2(t))^n} \\ & = -n, \end{aligned}$$

from the first integral

$$e^{n\xi(t)}(1-\xi_t^2(t))^n - e^{-n\xi(t)} = h = 0.$$

□

Next is a proof for Proposition 3.

Proof for Proposition 3. Define

$$\widehat{\phi}(t, \theta) = \phi(t, \theta) - \sum_{j=0}^n \pi_j[\phi(t, \theta)]Y_j(\theta),$$

where $\phi_j(t) := \pi_j[\phi(t, \theta)]$ is the L^2 orthogonal projection of $\phi(t, \theta)$ onto $\text{span}\{Y_j(\theta)\}$. Then

$$(73) \quad \int_{\mathbb{S}^{n-1}} \widehat{\phi}(t, \theta) Y_j(\theta) d\theta = \int_{\mathbb{S}^{n-1}} \nabla \widehat{\phi}(t, \theta) \cdot \nabla Y_j(\theta) d\theta = \int_{\mathbb{S}^{n-1}} \Delta_\theta Y_j(\theta) \widehat{\phi}(t, \theta) d\theta = 0,$$

for $j = 0, \dots, n$. As a consequence,

$$(74) \quad \begin{cases} \int_{\mathbb{S}^{n-1}} \Delta_\theta \phi(t, \theta) \widehat{\phi}(t, \theta) d\theta = - \int_{\mathbb{S}^{n-1}} |\nabla_\theta \widehat{\phi}(t, \theta)|^2 d\theta, \\ \int_{\mathbb{S}^{n-1}} \phi_t(t, \theta) \widehat{\phi}(t, \theta) d\theta = \int_{\mathbb{S}^{n-1}} \widehat{\phi}_t(t, \theta) \widehat{\phi}(t, \theta) d\theta = \frac{1}{2} \frac{d}{dt} \int_{\mathbb{S}^{n-1}} |\widehat{\phi}(t, \theta)|^2 d\theta. \end{cases}$$

In the following we will prove separately the expected decays for $\widehat{\phi}(t, \theta)$ and $\phi_j(t) := \pi_j[\phi(t, \theta)]$, for $j = 0, 1, \dots, n$. We first estimate $\phi_j(t) = \pi_j[\phi(t, \theta)]$ for $j = 0, \dots, n$. Multiplying both sides of (68) by

$$\left\{ \frac{(1 - |\xi_t(t)|^2)^{k-2}}{2^{k-2}} \binom{n-1}{k-1} A(t) \right\}^{-1} Y_j(\theta)$$

and integrating over $\theta \in \mathbb{S}^{n-1}$, we obtain

$$(75) \quad \phi_j''(t) + \frac{B(t)}{A(t)} \phi_j'(t) + \left[\frac{ne^{-n\xi(t)}}{e^{-n\xi(t)} + h} (1 - \xi_t^2(t)) - \lambda_j \frac{C(t)}{A(t)} \right] \phi_j(t) = \widehat{r}_j(t)$$

where

$$\widehat{r}_j(t) = \int_{\mathbb{S}^{n-1}} \widehat{r}(t, \theta) Y_j(\theta) d\theta.$$

For $j = 1, \dots, n$, $\lambda_j = n - 1$, and $\phi_1^-(t) := e^{-t}(1 + \xi'(t))$, $\phi_1^+(t) := e^t(1 - \xi'(t))$ form a solution basis to the homogeneous equation

$$\phi''(t) + \frac{B(t)}{A(t)} \phi'(t) + \left[\frac{ne^{-n\xi(t)}}{e^{-n\xi(t)} + h} (1 - \xi_t^2(t)) - (n-1) \frac{C(t)}{A(t)} \right] \phi(t) = 0.$$

Since $\phi_j(t)$ is a solution to (75) and $\phi_j(t) \rightarrow 0$ as $t \rightarrow \infty$, by the variation of constant formula,

$$(76) \quad \phi_j(t) = c\phi_1^-(t) + \phi_1^-(t) \int_0^t \frac{\phi_1^+(s)\widehat{r}_j(s)}{W_1(s)} ds + \phi_1^+(t) \int_t^\infty \frac{\phi_1^-(s)\widehat{r}_j(s)}{W_1(s)} ds,$$

for some constant c , where

$$W_1(s) = \phi_1^+(t)\phi_1^-(t) - \phi_1^-(t)\phi_1^+(t)$$

is the Wronskian of $\{\phi_1^-(t), \phi_1^+(t)\}$, and satisfies

$$W_1'(s) = -\frac{B(s)}{A(s)} W_1(s).$$

Integrating this equation out, using

$$\frac{B(s)}{A(s)} = \left(\frac{2k-n}{k} - \frac{n(k-1)}{k} \frac{e^{-n\xi(s)}}{e^{-n\xi(s)}+h} \right) \xi'(s),$$

we find

$$W_1(s) = (\text{const.}) e^{\frac{n-2k}{k}\xi(s)} \left(e^{-n\xi(s)} + h \right)^{-\frac{k-1}{k}}$$

is a periodic function, having a positive upper and lower bound. According to our assumption on the decay rate of $r(t, \theta)$, we have

$$|r_j(s)| \leq C e^{-\beta t}.$$

Thus

$$\left| \int_t^\infty \frac{\phi_1^-(s) \widehat{r}_j(s)}{W_1(s)} ds \right| \lesssim \int_t^\infty e^{-(1+\beta)s} ds \lesssim e^{-(1+\beta)t},$$

from which we deduce that

$$\left| \phi_1^+(t) \int_t^\infty \frac{\phi_1^-(s) \widehat{r}_j(s)}{W_1(s)} ds \right| \lesssim e^{-\beta t}.$$

When $\beta \neq 1$, we also have

$$\left| \int_0^t \frac{\phi_1^+(s) \widehat{r}_j(s)}{W_1(s)} ds \right| \lesssim \int_0^t e^{(1-\beta)s} ds \lesssim e^{(1-\beta)t},$$

from which we deduce that

$$\left| \phi_1^-(t) \int_0^t \frac{\phi_1^+(s) \widehat{r}_j(s)}{W_1(s)} ds \right| \lesssim e^{-\beta t}.$$

Putting these estimates into (76), we have

$$\left| \phi_j(t) - c e^{-t} (1 + \xi'(t)) \right| \lesssim e^{-\beta t}.$$

When $\beta = 1$, (76) gives the modified estimate.

For $j = 0$, $\phi_0^+(t) := \xi_h'(t)$ and $\phi_0^-(t) := \partial_h \xi_h(t)$ also form a solution basis to the homogeneous equation

$$\phi''(t) + \frac{B(t)}{A(t)}\phi'(t) + \left[\frac{ne^{-n\xi(t)}}{e^{-n\xi(t)} + h}(1 - \xi_t^2(t)) \right] \phi(t) = 0.$$

Since $\phi_0(t)$ is a solution to (75) and $\phi_0(t) \rightarrow 0$ as $t \rightarrow \infty$, a variant of (76) gives:

$$(77) \quad \phi_0(t) = -\phi_0^-(t) \int_t^\infty \frac{\phi_0^+(s)\widehat{r}_0(s)}{W_0(s)} ds + \phi_0^+(t) \int_t^\infty \frac{\phi_0^-(s)\widehat{r}_0(s)}{W_0(s)} ds,$$

where

$$W_0(s) = \phi_0^+(t)\phi_0^-(t) - \phi_0^-(t)\phi_0^+(t)$$

is the Wronskian of $\{\phi_0^-(t), \phi_0^+(t)\}$, and also satisfies

$$W_0'(s) = -\frac{B(s)}{A(s)}W_0(s).$$

Thus, as for $W_1(s)$, $W_0(s)$ is a periodic function, having a positive upper and lower bound. Let $T(h)$ denotes the minimal period of the solution $\xi_h(t)$. Then $\xi_h(t + T(h)) = \xi_h(t)$. Differentiating in h , we obtain

$$(78) \quad \phi_0^-(t + T(h)) + T'(h)\xi_h'(t + T(h)) = \phi_0^-(t),$$

which implies that $\phi_0^-(t)$ grows in t at most linearly. Then (77) would imply that $|\phi_0(t)| \lesssim te^{-\beta t}$. This is not quite as claimed, but is good enough to be used in our iterative argument in proving (20). To obtain the more precise estimate (69), note that (78) implies that

$$p(t) := \phi_0^-(t) + \frac{T'(h)}{T(h)}t\xi_h'(t) = \phi_0^-(t) + \frac{T'(h)}{T(h)}t\phi_0^+(t)$$

is $T(h)$ periodic—such behavior can also be deduced from the application of Floquet theory to this case. Thus

we can express $\phi_0^-(t)$ as $p(t) - \frac{T'(h)}{T(h)}t\phi_0^+(t)$ in (77) to obtain

$$\begin{aligned}\phi_0(t) &= -\left(p(t) - \frac{T'(h)}{T(h)}t\phi_0^+(t)\right) \int_t^\infty \frac{\phi_0^+(s)\widehat{r}_0(s)}{W_0(s)} ds + \phi_0^+(t) \int_t^\infty \frac{\left(p(s) - \frac{T'(h)}{T(h)}s\phi_0^+(s)\right)\widehat{r}_0(s)}{W_0(s)} ds \\ &= -p(t) \int_t^\infty \frac{\phi_0^+(s)\widehat{r}_0(s)}{W_0(s)} ds + \phi_0^+(t) \int_t^\infty \frac{p(s)\widehat{r}_0(s)}{W_0(s)} ds \\ &\quad - \frac{T'(h)}{T(h)}\phi_0^+(t) \int_t^\infty \int_s^\infty \frac{\phi_0^+(\tau)\widehat{r}_0(\tau)}{W_0(\tau)} d\tau ds,\end{aligned}$$

from which follows $|\phi_0(t)| \lesssim e^{-\beta t}$.

Finally, we estimate the decay rate of $\widehat{\phi}(t, \theta)$. This part is analogous to an approach in [50]. Multiplying both sides of (68) by

$$\left\{ \frac{(1 - |\xi_r(t)|^2)^{k-2}}{2^{k-2}} \binom{n-1}{k-1} A(t) \right\}^{-1} \widehat{\phi}(t, \theta),$$

integrating over $\theta \in \mathbb{S}^{n-1}$ and using (73) and (74), we find

$$\begin{aligned}(79) \quad & \int_{\mathbb{S}^{n-1}} \left\{ \widehat{\phi}_t(t, \theta)\widehat{\phi}(t, \theta) + \frac{B(t)}{A(t)}\widehat{\phi}_t(t, \theta)\widehat{\phi}(t, \theta) + \frac{ne^{-n\xi(t)}}{e^{-n\xi(t)} + h}(1 - \xi_r^2(t))|\widehat{\phi}(t, \theta)|^2 \right\} d\theta \\ & - \frac{C(t)}{A(t)} \int_{\mathbb{S}^{n-1}} |\nabla_\theta \widehat{\phi}(t, \theta)|^2 d\theta = \int_{\mathbb{S}^{n-1}} \widehat{r}(t, \theta)\widehat{\phi}(t, \theta) d\theta,\end{aligned}$$

where

$$\widehat{r}(t, \theta) = \left\{ \frac{(1 - |\xi_r(t)|^2)^{k-2}}{2^{k-2}} \binom{n-1}{k-1} A(t) \right\}^{-1} r(t, \theta) \asymp r(t, \theta).$$

Defining

$$y(t) = \sqrt{\int_{\mathbb{S}^{n-1}} |\widehat{\phi}(t, \theta)|^2 d\theta},$$

then

$$y'(t) = \int_{\mathbb{S}^{n-1}} \widehat{\phi}_t(t, \theta) \widehat{\phi}(t, \theta) d\theta / y(t), \quad \text{whenever } y(t) > 0,$$

and

$$y(t)y''(t) = \int_{\mathbb{S}^{n-1}} \left\{ \widehat{\phi}_{tt}(t, \theta) \widehat{\phi}(t, \theta) + |\widehat{\phi}_t(t, \theta)|^2 \right\} d\theta - |y'(t)|^2.$$

Cauchy-Schwarz inequality implies that

$$|y'(t)|^2 \leq \int_{\mathbb{S}^{n-1}} |\widehat{\phi}_t(t, \theta)|^2 d\theta.$$

Using these relations and

$$\int_{\mathbb{S}^{n-1}} |\nabla_{\theta} \widehat{\phi}(t, \theta)|^2 d\theta \geq 2n \int_{\mathbb{S}^{n-1}} |\widehat{\phi}(t, \theta)|^2 d\theta$$

into (79), we obtain

$$y(t)y''(t) + \frac{B(t)}{A(t)}y(t)y'(t) + \left[\frac{ne^{-n\xi(t)}}{e^{-n\xi(t)} + h} (1 - \xi_t^2(t)) - 2n \frac{C(t)}{A(t)} \right] y^2(t) \geq -\|\widehat{r}(t, \cdot)\|_{L^2(\mathbb{S}^{n-1})} y(t),$$

whenever $y(t) > 0$, from which we deduce

$$(80) \quad y''(t) + \frac{B(t)}{A(t)}y'(t) + \left[\frac{ne^{-n\xi(t)}}{e^{-n\xi(t)} + h} (1 - \xi_t^2(t)) - 2n \frac{C(t)}{A(t)} \right] y(t) \geq -\|\widehat{r}(t, \cdot)\|_{L^2(\mathbb{S}^{n-1})},$$

whenever $y(t) > 0$. According to our assumption on $r(t, \theta)$, we have

$$\|\widehat{r}(t, \cdot)\|_{L^2(\mathbb{S}^{n-1})} \leq Ce^{-\beta t}$$

for some constant $C > 0$. By (67),

$$\left\{ \partial_{tt} + \frac{B(t)}{A(t)} \partial_t + \left[-2n \frac{C(t)}{A(t)} + \frac{ne^{-n\xi(t)}}{e^{-n\xi(t)} + h} (1 - \xi_t^2(t)) \right] \right\} (V^{-1}(t)e^{-\beta t})$$

$$\leq (\beta^2 + E) V^{-1}(t)e^{-\beta t} \leq -\varepsilon V^{-1}(t)e^{-\beta t},$$

for some $\varepsilon > 0$ when $\beta < \beta_*$. So $z(t) := C\varepsilon^{-1}(\max V)V^{-1}(t)e^{-\beta t}$ satisfies

$$(81) \quad \left\{ \partial_{tt} + \frac{B(t)}{A(t)} \partial_t + \left[-2n \frac{C(t)}{A(t)} + \frac{ne^{-n\xi(t)}}{e^{-n\xi(t)} + h} (1 - \xi_t^2(t)) \right] \right\} (z(t) - y(t)) \leq 0,$$

whenever $y(t) > 0$. We also know that $y(t) \rightarrow 0$ as $t \rightarrow \infty$. We may choose $C > 0$ large so that $z(0) \geq y(0)$. Then we claim that $z(t) - y(t) \geq 0$ for all $t \geq 0$, for, if not, $\min(z(t) - y(t)) < 0$ is finite, and is attained at some t_* , then $y(t_*) > z(t_*) > 0$, so (81) holds at $t = t_*$, and $\partial_t(z(t) - y(t))|_{t=t_*} = 0$, as well as $\partial_{tt}(z(t) - y(t))|_{t=t_*} \geq 0$. This contradicts (81). Thus we conclude

$$\sqrt{\int_{\mathbb{S}^{n-1}} |\widehat{\phi}(t, \theta)|^2 d\theta} = y(t) \leq C\varepsilon^{-1}(\max V)V^{-1}(t)e^{-\beta t}.$$

We can now bootstrap this integral estimate to obtain a pointwise decay estimate

$$|\widehat{\phi}(t, \theta)| \lesssim e^{-\beta t}.$$

When $\beta \geq \beta_*$, we can simply split those components ϕ_j of ϕ with $\lambda_j = 2n$ from $\widehat{\phi}(t, \theta)$, and estimate them as we

did for ϕ_j , $j = 0, \dots, n$, and estimate $\widehat{\phi}(t, \theta)$ with an improved exponential decay rate. \square

We now provide a proof for Theorem 2. Our proof is very much like the one in [32] for the $k = 1$ case, once we have obtained the needed linear analysis.

Proof of Theorem 2. Our starting point is still

$$L_{\xi_h(\cdot+\tau)}(\phi) + Q(\phi) + 2kce^{-2k\xi_h(t+\tau)}\phi(t, \theta) = 0,$$

and our premise is:

$$(82) \quad |Q(\phi)| \lesssim e^{-2\alpha t} \quad \text{whenever we have } |\phi, \partial\phi, \partial^2\phi| \lesssim e^{-\alpha t}.$$

We already established

Step 1. For some $\alpha_0 > 0$, $|\phi, \partial\phi, \partial^2\phi| \lesssim e^{-\alpha_0 t}$.

If $\alpha_0 \geq \rho_{n+1}$, we stop and have now proved $|w(t, \theta) - \xi_h(t + \tau)| = |\phi(t, \theta)| \lesssim e^{-\rho_{n+1}t}$, where $\rho_{n+1} > \sqrt{2}$; if $1 < \alpha_0 < \rho_{n+1}$, we jump to **Step 3**; if $\alpha_0 \leq 1$, we move onto

Step 2. Recall that we now have $|Q(\phi)| \lesssim e^{-2\alpha_0 t}$. If $2\alpha_0 > \rho_{n+1}$, then we can apply Proposition 3 directly to

conclude our proof; If $1 < 2\alpha_0 \leq \rho_{n+1}$, then we certainly still have $|Q(\phi)| \lesssim e^{-2\alpha t}$ for some $1 < 2\alpha < \rho_{n+1}$ and can apply Proposition 3 to imply that

$$(83) \quad |w(t, \theta) - \xi_h(t + \tau) - \sum_{j=1}^n a_j e^{-(t+\tau)} (1 + \xi'_h(t + \tau)) Y_j(\theta)| \lesssim e^{-2\alpha t},$$

for some constants a_j for $j = 1, \dots, n$, and jump to **Step 3**; if $2\alpha_0 \leq 1$, we may take α_0 to satisfy $2\alpha_0 < 1$ and apply Proposition 3 to imply that

$$|\phi(t, \theta) - \sum_{j=1}^n a_j e^{-(t+\tau)} (1 + \xi'_h(t + \tau)) Y_j(\theta)| \lesssim e^{-2\alpha_0 t}$$

for some constants a_j for $j = 1, \dots, n$. This certainly implies that

$$(84) \quad |\phi(t, \theta)| \lesssim e^{-2\alpha_0 t}.$$

Next we use higher derivative estimates for $w(t, \theta)$ and $\xi_h(t + \tau)$ and interpolation with (84) to obtain

$$|\phi, \partial\phi, \partial^2\phi| \lesssim e^{-2\alpha' t}$$

for any $\alpha' < \alpha_0$. Now we go back to the beginning of step 2 and repeat the process with a new $\alpha_1 > \alpha_0$ to replace the α_0 there, say, $\alpha_1 = 1.8\alpha_0$. After a finite number of steps, we will reach a stage where $2\alpha > 1$ and ready to move onto

Step 3. At this stage, we have $|\phi(t, \theta)| \lesssim e^{-t}$. Repeating the last part of **Step 2** involving the derivative estimates for $w(t, \theta)$ and $\xi_h(t + \tau)$ to bootstrap the estimate for $Q(\phi)$ to $|Q(\phi)| \lesssim e^{-\alpha t}$, with α can be as close to 2 as one needs. Then, depending on whether $\rho_{n+1} \geq 2$ or otherwise, one can apply Proposition 3 to obtain (69) or (70). In

the first case, we can continue the iteration until $2\alpha > 2$. But due to the presence of $e^{-(t+\tau)}(1 + \xi'_h(t + \tau))Y_j(\theta)$ in

the estimate for ϕ , the estimate for $Q(\phi)$ can not be better than e^{-2t} . This explains the appearance of $\min\{2, \rho_{n+1}\}$ in (20). \square

5. PROOF OF THEOREM 3

Remark. *First, some comments on the assumptions in Theorem 3.*

(a). Assumptions (32) and (33) imply that whenever $\beta'(\tau_j) = 0$ and $|\beta(\tau_j) - m| < \varepsilon_1$, then $|\beta(\tau_j) - m|$ is in

fact bounded above by $e_2(\tau_j)^{1/l}$.

(b). In the case that ψ is non-constant, it follows that

$$|f(0, m)| = a > 0.$$

Thus there exists $0 < \varepsilon_2 \leq \varepsilon_1$ such that

$$(85) \quad |f(x, y)| \geq 3a/4, \quad \text{for all } (x, y) \text{ with } |x| + |y - m| < \varepsilon_2.$$

Let T_0 be such that $|e_1(t)| < a/4$ for $t \geq T_0$. Then

$$(86) \quad |\beta''(t)| \geq a/2, \quad \text{whenever } |\beta'(t)| + |\beta(t) - m| < \varepsilon_2 \text{ and } t \geq T_0.$$

(c). By linearization at $\psi(t_{**} + \cdot)$, there exists $B > 0$ depending on f, T and the upper bound of $|\beta(\cdot)| + |\beta'(\cdot)|$ such that

$$(87) \quad \begin{aligned} & |\beta(t_* + \tau) - \psi(t_{**} + \tau)| + |\beta'(t_* + \tau) - \psi'(t_{**} + \tau)| \\ & \leq B \left(|\beta(t_*) - \psi(t_{**})| + |\beta'(t_*) - \psi'(t_{**})| + \max_{|t - t_*| \leq 2T} |e_1(t)| \right) \end{aligned}$$

for all $-2T \leq \tau \leq 2T$. One ingredient of our proof in case (ii) is to find a sequence of $\tau_j \rightarrow \infty$ with

$\tau_{j+1} - \tau_j \approx T$ as $j \rightarrow \infty$, such that $\beta'(\tau_j) = 0$ and $|\beta(\tau_j) - m| \rightarrow 0$ as $j \rightarrow \infty$. Then applying (87) to $\beta(\tau_j + \tau)$ and $\psi(\tau)$ would imply that

$$|\beta(\tau_j + \tau) - \psi(\tau)| + |\beta'(\tau_j + \tau) - \psi'(\tau)| \leq B \left(|\beta(\tau_j) - m| + \max_{0 \leq \tau \leq 2T} |e_1(\tau_j + \tau)| \right),$$

for $0 \leq \tau \leq 2T$.

Here is an outline of the main steps in our proof: we first use (29), (87) and (86) to deduce that $\beta(t)$ will have critical point for large t with its critical value close to m ; then use (a) to prove that the difference between the

critical value of $\beta(t)$ with m is actually bounded above by $e_2(t)^{1/l}$; then we can iterate this argument indefinitely and account for the possible time shift between consecutive times that $\beta(t)$ attains a critical value near m . We can now put the ingredients together to provide a complete proof.

Proof. First, for some $1/2 > \kappa > 0$ to be determined, by (29) and (87) there exists $t_{i_0} > T_0$ such that

$$(88) \quad |\beta(t_{i_0} + \tau) - \psi(-s + \tau)| + |\beta'(t_{i_0} + \tau) - \psi'(-s + \tau)| < \kappa \varepsilon_2 \text{ for } |\tau| \leq 2T.$$

First, we will dispose of case (i): when $\psi(t) \equiv m$ is a constant, (31) and (33) imply that

$$(89) \quad |\beta'(t)|^l + |\beta(t) - m|^l \leq A^{-1}e_2(t),$$

as long as $|\beta'(t)| + |\beta(t) - m| \leq \varepsilon_1$. This, together with (88), apparently implies that (89) continues to hold for

all $t \geq T_0$. So we are left to deal with the case that $\psi(t)$ is non-constant. Noting that $\psi(0) = m$ and $\psi'(0) = 0$, we have by (88) that

$$|\beta(t_{i_0} + s) - m| + |\beta'(t_{i_0} + s)| < \kappa\varepsilon_2.$$

We now prove that there exists δ_0 with $|\delta_0| \leq 2a^{-1}|\beta'(t_{i_0} + s)|$ such that

$$(90) \quad \beta'(t_{i_0} + s + \delta_0) = 0.$$

Let $\Lambda = \sup_{\mathbb{R}}\{|\psi''(t)|, |\psi'(t)|\}$. Then

$$(91) \quad |\psi(\tau) - m| + |\psi'(\tau)| \leq 2\Lambda|\tau| < \varepsilon_2/2,$$

for $|\tau| < \varepsilon_2/(4\Lambda)$. Together with (88) and (91), we know

$$|\beta(t_{i_0} + s + \tau) - m| + |\beta'(t_{i_0} + s + \tau)| < \varepsilon_2, \quad \text{for } |\tau| < \varepsilon_2/(4\Lambda).$$

Thus, by (86), we have $|\beta''(t_{i_0} + s + \tau)| \geq a/2$ for $|\tau| < \varepsilon_2/(4\Lambda)$. Then by elementary calculus, there exists δ_0

such that $\beta'(t_{i_0} + s + \delta_0) = 0$ and

$$(92) \quad |\delta_0| \leq 2a^{-1}|\beta'(t_{i_0} + s)| \leq 2a^{-1}\kappa\varepsilon_2 < \varepsilon_2/(4\Lambda),$$

provided κ is chosen to satisfy the last inequality above. We fix such a κ now. Set $\tau_0 = t_{i_0} + s + \delta_0$. Note that we have $|\beta(\tau_0) - m| < \varepsilon_2$. Thus from assumption (32), we have

$$\begin{aligned}
A|\beta(\tau_0) - m|^l &= A|\beta(\tau_0) - \psi(0)|^l \\
&\leq |H(0, \beta(\tau_0)) - H(0, \psi(0))| \\
&= |H(\beta'(\tau_0), \beta(\tau_0)) - H(\psi'(0), \psi(0))| \\
&= |H(\beta'(\tau_0), \beta(\tau_0)) - 0| \\
&\leq e_2(\tau_0),
\end{aligned}$$

which implies that

$$(93) \quad |\beta(\tau_0) - m| \leq \left(\frac{e_2(\tau_0)}{A} \right)^{1/l}.$$

Next we apply (87) to $\beta(\tau_0 + \tau)$ and $\psi(\tau)$ to obtain

$$\begin{aligned}
&|\beta(\tau_0 + \tau) - \psi(\tau)| + |\beta'(\tau_0 + \tau) - \psi'(\tau)| \\
(94) \quad &\leq B \left(|\beta(\tau_0) - m| + \max_{\tau_0 \leq t \leq \tau_0 + 2T} |e_1(t)| \right) \\
&\leq B \left(\left(\frac{e_2(\tau_0)}{A} \right)^{1/l} + \max_{\tau_0 \leq t \leq \tau_0 + 2T} |e_1(t)| \right)
\end{aligned}$$

for $0 \leq \tau \leq 2T$. Repeating the above argument, and in choosing T_0 also make sure that

$$B \left(\left(\frac{e_2(\tau)}{A} \right)^{1/l} + \max_{\tau \leq t \leq \tau + 2T} |e_1(t)| \right) < \kappa \varepsilon_2$$

for $\tau \geq T_0$, we obtain δ_1 such that $\tau_1 = \tau_0 + T + \delta_1$ satisfies

$$(95) \quad \beta'(\tau_1) = 0,$$

$$(96) \quad |\delta_1| \leq 2a^{-1}|\beta'(\tau_0 + T)| \leq 2a^{-1}B \left(\left(\frac{e_2(\tau_0)}{A} \right)^{1/l} + \max_{\tau_0 \leq t \leq \tau_0 + 2T} |e_1(t)| \right),$$

$$(97) \quad |\beta(\tau_1) - m| \leq \left(\frac{e_2(\tau_1)}{A} \right)^{1/l}.$$

We can now inductively find $\tau_j = \tau_{j-1} + T + \delta_j$ such that

$$(98) \quad \beta'(\tau_j) = 0,$$

$$(99) \quad |\delta_j| \leq 2a^{-1}|\beta'(\tau_{j-1} + T)| \leq 2a^{-1}B \left(\left(\frac{e_2(\tau_{j-1})}{A} \right)^{1/l} + \max_{\tau_{j-1} \leq t \leq \tau_{j-1} + 2T} |e_1(t)| \right),$$

$$(100) \quad |\beta(\tau_j) - m| \leq \left(\frac{e_2(\tau_j)}{A} \right)^{1/l},$$

$$(101) \quad \begin{aligned} & |\beta(\tau_j + \tau) - \psi(\tau)| + |\beta'(\tau_j + \tau) - \psi'(\tau)| \\ & \leq B \left(|\beta(\tau_j) - m| + \max_{\tau_j \leq t \leq \tau_j + 2T} |e_1(t)| \right) \\ & \leq B \left(\left(\frac{e_2(\tau_j)}{A} \right)^{1/l} + \max_{\tau_j \leq t \leq \tau_j + 2T} |e_1(t)| \right) \end{aligned}$$

for $0 \leq \tau \leq 2T$. Set $s_j = \tau_j - jT$. Then $s_j = s_{j-1} + \delta_j$, and due to estimate (99) and assumption (34) $s_\infty = \lim_{j \rightarrow \infty} s_j$ exists and equals $s_0 + \sum_{j=1}^{\infty} \delta_j$. (101) can be rewritten, with $t = \tau_j + \tau$, as

$$\begin{aligned} & |\beta(t) - \psi(t - s_j)| + |\beta'(t) - \psi'(t - s_j)| \\ &= |\beta(t) - \psi(t - \tau_j)| + |\beta'(t) - \psi'(t - \tau_j)| \\ &\leq B \left(\left(\frac{e_2(\tau_j)}{A} \right)^{1/l} + \max_{\tau_j \leq t \leq \tau_{j+1}} |e_1(t)| \right) \\ &\leq B \int_{\tau_{j-1}}^{\infty} \left(\left(\frac{e_2(t')}{A} \right)^{1/l} + \max_{t' \leq \tau} |e_1(\tau)| \right) dt' \end{aligned}$$

for $\tau_j \leq t \leq \tau_{j+1}$, which further implies that

$$\begin{aligned} & |\beta(t) - \psi(t - s_\infty)| + |\beta'(t) - \psi'(t - s_\infty)| \\ &\leq |\beta(t) - \psi(t - s_j)| + |\beta'(t) - \psi'(t - s_j)| + |\psi(t - s_\infty) - \psi(t - s_j)| + |\psi'(t - s_\infty) - \psi'(t - s_j)| \\ &\leq B \left(\left(\frac{e_2(\tau_j)}{A} \right)^{1/l} + \max_{\tau_j \leq t \leq \tau_{j+1}} |e_1(t)| \right) + \Lambda |s_\infty - s_j| \\ (102) \quad &\leq B \left(\left(\frac{e_2(\tau_j)}{A} \right)^{1/l} + \max_{\tau_j \leq t \leq \tau_{j+1}} |e_1(t)| \right) + \Lambda \sum_{k=j+1}^{\infty} |\delta_k| \\ &\leq C \int_{\tau_{j-1}}^{\infty} \left(\left(\frac{e_2(t')}{A} \right)^{1/l} + \max_{t' \leq \tau} |e_1(\tau)| \right) dt' \end{aligned}$$

for some constant $C > 0$ and $\tau_j \leq t \leq \tau_{j+1}$. (99) and our assumption (34) imply that the expression on the right side of the above inequality tends to 0 as $j \rightarrow \infty$, thus proving (35). \square

6. APPENDIX

Proof of Lemma 2. Introduce a new variable $\psi(t) = V(t)\phi(t)$ for some $V(t)$ to be chosen. Then

$$\begin{aligned} V(t)L_j[\phi] = & \psi_{tt}(t) + \left\{ \left[1 - (n-1)\frac{C(t)}{A(t)} \right] \xi_t(t) - 2\frac{V'(t)}{V(t)} \right\} \psi_t(t) + \left\{ 2\frac{|V'(t)|^2}{V(t)^2} - \frac{V''(t)}{V(t)} \right. \\ & \left. - \frac{V'(t)}{V(t)} \left[1 - (n-1)\frac{C(t)}{A(t)} \right] \xi_t(t) - \lambda_j \frac{C(t)}{A(t)} + \frac{ne^{-n\xi(t)}}{e^{-n\xi(t)}+h} (1 - \xi_t^2(t)) \right\} \psi(t). \end{aligned}$$

Choose $V(t)$ such that $\left[1 - (n-1)\frac{C(t)}{A(t)} \right] \xi_t(t) - 2\frac{V'(t)}{V(t)} = 0$. This amounts to

$$\begin{aligned} (2\ln V(t))_t &= \left[1 - \frac{n-k}{k} - \frac{n(k-1)e^{-n\xi(t)}}{k(e^{-n\xi(t)}+h)} \right] \xi_t(t) \\ &= \left[\left(2 - \frac{n}{k} \right) \xi_t(t) + \frac{k-1}{k} \ln \left(e^{-n\xi(t)} + h \right) \right]_t. \end{aligned}$$

Thus we can take $V(t) = e^{(1-\frac{n}{2k})\xi(t)} \left(e^{-n\xi(t)} + h \right)^{\frac{k-1}{2k}}$. Then

$$\begin{aligned} V(t)L_j[\phi] &= \psi_{tt}(t) + \left\{ -\frac{V''(t)}{V(t)} - \lambda_j \frac{C(t)}{A(t)} + \frac{ne^{-n\xi(t)}}{e^{-n\xi(t)}+h} (1 - \xi_t^2(t)) \right\} \psi(t) \\ &= \psi_{tt}(t) + E(t)\psi(t), \end{aligned}$$

where $E(t) = \left\{ -\frac{V''(t)}{V(t)} - \lambda_j \frac{C(t)}{A(t)} + \frac{ne^{-n\xi(t)}}{e^{-n\xi(t)}+h} (1 - \xi_t^2(t)) \right\}$, and

$$\begin{aligned} \frac{V''(t)}{V(t)} &= \left[1 - \frac{n}{2k} - \frac{n(k-1)e^{-n\xi(t)}}{2k(e^{-n\xi(t)}+h)} \right] \xi'' \\ &+ \frac{n^2(k-1)he^{-n\xi}|\xi'|^2}{2k(e^{-n\xi(t)}+h)^2} + \left[1 - \frac{n}{2k} - \frac{n(k-1)e^{-n\xi(t)}}{2k(e^{-n\xi(t)}+h)} \right]^2 |\xi'|^2. \end{aligned}$$

Using

$$\xi_{tt}(t) = \frac{n}{2k} e^{-2k\xi(t)} (1 - \xi_t^2(t))^{1-k} - \frac{n-2k}{2k} (1 - \xi_t^2(t)),$$

and

$$e^{(2k-n)\xi(t)} (1 - \xi_t^2(t))^k = e^{-n\xi(t)} + h,$$

we have

$$\begin{aligned} \frac{V''(t)}{V(t)} &= \left[\frac{(2k-n)^2}{4k^2} + \frac{n(n-2k)(k-2)e^{-n\xi(t)}}{4k^2(e^{-n\xi(t)}+h)} - \frac{n^2(k-1)}{4k^2} \left(\frac{e^{-n\xi(t)}}{e^{-n\xi(t)}+h} \right)^2 \right] (1 - |\xi'|^2) \\ &\quad + \frac{n^2(k-1)he^{-n\xi}}{2k(e^{-n\xi(t)}+h)^2} + \left[1 - \frac{n}{2k} - \frac{n(k-1)e^{-n\xi(t)}}{2k(e^{-n\xi(t)}+h)} \right]^2 |\xi'|^2 \\ &= \left[\frac{(2k-n)^2}{4k^2} + \frac{n(n-2k)(k-2)e^{-n\xi(t)}}{4k^2(e^{-n\xi(t)}+h)} - \frac{n^2(k-1)}{4k^2} \left(\frac{e^{-n\xi(t)}}{e^{-n\xi(t)}+h} \right)^2 \right] \\ &\quad + \left\{ -\frac{(2k-n)^2}{4k^2} - \frac{n(n-2k)(k-2)e^{-n\xi(t)}}{4k^2(e^{-n\xi(t)}+h)} + \frac{n^2(k-1)}{4k^2} \left(\frac{e^{-n\xi(t)}}{e^{-n\xi(t)}+h} \right)^2 \right. \\ &\quad \left. + \frac{n^2(k-1)he^{-n\xi}}{2k(e^{-n\xi(t)}+h)^2} + \left[1 - \frac{n}{2k} - \frac{n(k-1)e^{-n\xi(t)}}{2k(e^{-n\xi(t)}+h)} \right]^2 \right\} |\xi'|^2 \\ &= \left[\frac{(2k-n)^2}{4k^2} + \frac{n(n-2k)(k-2)e^{-n\xi(t)}}{4k^2(e^{-n\xi(t)}+h)} - \frac{n^2(k-1)}{4k^2} \left(\frac{e^{-n\xi(t)}}{e^{-n\xi(t)}+h} \right)^2 \right] \\ &\quad + \left[\frac{n^2(k-1)he^{-n\xi}}{2k(e^{-n\xi(t)}+h)^2} + \frac{n(n-2k)e^{-n\xi(t)}}{4k(e^{-n\xi(t)}+h)} + \frac{n^2(k-1)}{4k} \left(\frac{e^{-n\xi(t)}}{e^{-n\xi(t)}+h} \right)^2 \right] |\xi'|^2 \end{aligned}$$

Since we are interested in getting an upper bound for $E(t)$, and it's not easy to find more useful bound for terms of the form (negative factor) $|\xi'|^2$, so will drop such terms in our estimates and obtain, in the case $2k \leq n$ and

$\lambda_j \geq 2n$,

$$\begin{aligned}
E(t) &\leq -\frac{(2k-n)^2}{4k^2} - \frac{n(n-2k)(k-2)e^{-n\xi(t)}}{4k^2(e^{-n\xi(t)}+h)} + \frac{n^2(k-1)}{4k^2} \left(\frac{e^{-n\xi(t)}}{e^{-n\xi(t)}+h} \right)^2 \\
&\quad - \frac{2n(n-k)}{k(n-1)} - \frac{2n^2(k-1)e^{-n\xi(t)}}{k(n-1)(e^{-n\xi(t)}+h)} \\
&\leq -\frac{(2k-n)^2}{4k^2} - \frac{2n(n-k)}{k(n-1)} \\
&\quad - \left\{ \frac{n(n-2k)(k-2)}{4k^2} - \frac{n^2(k-1)}{4k^2} + \frac{2n^2(k-1)}{k(n-1)} \right\} \frac{e^{-n\xi(t)}}{e^{-n\xi(t)}+h} \\
&= -\frac{(2k-n)^2}{4k^2} - \frac{2n(n-k)}{k(n-1)} - \frac{2(n+3)k^2 - 4(n+1)k - n(n-1)}{4k^2(n-1)} \frac{ne^{-n\xi(t)}}{e^{-n\xi(t)}+h}.
\end{aligned}$$

When $2(n+3)k^2 - 4(n+1)k - n(n-1) \geq 0$, we obtain

$$E(t) \leq -\frac{(2k-n)^2}{4k^2} - \frac{2n(n-k)}{k(n-1)} = -\left(\frac{n}{2k} - 1\right)^2 - \frac{2n}{n-1} \left(\frac{n}{k} - 1\right) \leq -2 - \frac{2}{n-1},$$

provided $2k \leq n$; while if $2(n+3)k^2 - 4(n+1)k - n(n-1) \leq 0$, we obtain

$$\begin{aligned}
E(t) &\leq -\frac{(2k-n)^2}{4k^2} - \frac{2n(n-k)}{k(n-1)} + \frac{-2n(n+3)k^2 + 4n(n+1)k + n^2(n-1)}{4k^2(n-1)} \\
&= -\frac{n+1}{2}!
\end{aligned}$$

In all cases we conclude the proof of Lemma 2. □

Note added: After this paper was posted on arXiv, Lorenzo Mazzieri and Cheikh Birahim Ndiaye sent us a very interesting preprint, *Existence of solutions for the singular σ_k -Yamabe problem*, in which they show that, for a given finite set Λ of more than one point in \mathbb{S}^n , satisfying some additional assumptions involving their positions in the case $\text{card } \Lambda \geq 5$, there are complete Riemannian metrics on $\mathbb{S}^n \setminus \Lambda$, conformal to the standard metric $g_{\mathbb{S}^n}$ and having positive constant σ_k -curvature. Their construction complements our results in this current paper.

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