Asymptotic behavior of solutions to the Yamabe equation with an asymptotically flat metric *

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Abstract

We prove that any positive solution of the Yamabe equation on an asymptotically flat n-dimensional manifold of flatness order at least $\frac{n-2}{2}$ and $n \leq 24$ must converge at infinity either to a fundamental solution of the Laplace operator on the Euclidean space or to a radial Fowler solution defined on the entire Euclidean space. The flatness order $\frac{n-2}{2}$ is the minimal flatness order required to define ADM mass in general relativity; the dimension 24 is the dividing dimension of the validity of compactness of solutions to the Yamabe problem. We also prove such alternatives for bounded solutions when $n > 24$.

We prove these results by establishing appropriate asymptotic behavior near an isolated singularity of solutions to the Yamabe equation when the metric has a flatness order of at least $\frac{n-2}{2}$ at the singularity and $n \leq 24$, also when $n > 24$ and the solution grows no faster than the fundamental solution of the flat metric Laplacian at the singularity. These results extend earlier results of L. Caffarelli, B. Gidas and J. Spruck, also of N. Korevaar, R. Mazzeo, F. Pacard and R. Schoen, when the metric is conformally flat, and work of C.C. Chen and C. S. Lin when the scalar curvature is a non-constant function with appropriate flatness at the singular point, also work of F. Marques when the metric is not necessarily conformally flat but smooth, and the dimension of the manifold is three, four, or five, as well as recent similar results by the second and third authors in dimension six.

Keywords *asymptotic behavior, isolated singularity, Yamabe equation, asymptotically flat metric*

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1 Introduction

On a compact smooth Riemannian manifold (M, g) of dimension $n \geq 3$, the Yamabe problem, which concerns the existence of constant scalar curvature metrics in the conformal class of g, was solved affirmatively through Yamabe [61], Trudinger [59], Aubin [4] and Schoen [52]. The problem is equivalent to solving the Yamabe equation

$$-L_g u = n(n-2)\text{sign}(\lambda_1) u^{\frac{n+2}{n-2}} \quad \text{on } M, \quad u > 0,$$

where $L_g = \Delta_g - c(n)R_g$ is the conformal Laplacian with $c(n) = \frac{(n-2)}{4(n-1)}$, Δ_g is the Laplace-Beltrami operator and R_g is the scalar curvature of g, and $\text{sign}(\lambda_1) \in \{-1, 0, 1\}$ is the sign of the first eigenvalue of $-L_g$ on M.

Solutions of the Yamabe equation on the standard unit sphere $\mathbb{S}^n$ were classified by Obata [51]. Namely, they must be positive constants ($2^{-\frac{n-2}{2}}$ in our formulation) modulo Mobius transforms. The same conclusion was proved on $\mathbb{S}^n \setminus \{\mathcal{N}\}$ by Gidas-Ni-Nirenberg [20, 21] and Caffarelli-Gidas-Spruck [11], where $\mathcal{N}$ is the north pole. Equivalently, the theorem on $\mathbb{S}^n \setminus \{\mathcal{N}\}$ asserts that every positive solution of the Yamabe equation with the flat background metric

$$-\Delta u = n(n-2)u^{\frac{n+2}{n-2}} \tag{1}$$

in $\mathbb{R}^n$ has to be the form $\lambda^{\frac{n-2}{2}}(1 + \lambda^2|x - x_0|^2)^{\frac{2-n}{2}}$, where $\lambda > 0$ and $x_0 \in \mathbb{R}^n$. This Liouville type theorem implies that there is no positive solution of the Yamabe equation on $\mathbb{S}^n$ which is singular only at one point.

In the same paper [11], Caffarelli, Gidas and Spruck further studied the isolated singularities of positive solutions of (1). First, they classified all positive solutions of (1) in $\mathbb{R}^n \setminus \{0\}$ (or $\mathbb{S}^n \setminus \{\mathcal{N}, -\mathcal{N}\}$) with 0 being a non-removable singularity by proving that they are radially symmetric and solve an ODE studied by Fowler [22]. We refer these radial singular solutions on $\mathbb{R}^n \setminus \{0\}$ as *Fowler solutions*. The radial symmetry was also obtained by [21] under some decay assumption on the solution at infinity. Second, they proved that every positive solution of (1) in the punctured unit ball $B_1 \setminus \{0\}$ with 0 being a non-removable singularity must converge to a Fowler solution:

$$u(x) = u_0(x)(1 + o(1)) \quad \text{as } x \to 0, \tag{2}$$

where u_0 is a Fowler solution which can be written as

$$u_0(|x|) = |x|^{-\frac{n-2}{2}}\psi(\ln|x|)$$

and ψ is a positive periodic solution on $\mathbb{R}$ to $\psi''(t) - \frac{(n-2)^2}{4}\psi(t) + n(n-2)\phi(t)^{\frac{n+2}{n-2}} = 0$. A different proof and refinement of results in [11] were given by Korevaar-Mazzeo-Pacard-Schoen [30]; in particular, they improved the $o(1)$ remainder term to $O(|x|^\alpha)$ for some $\alpha > 0$. See also Chen-Lin [13] and Li [35] for alternative arguments and generalizations of [11] in establishing the upper bound of u, and Q. Han-Li-Li [25] for higher order expansion of $O(|x|^\alpha)$.

It is a natural problem to ask whether the theorem of Caffarelli, Gidas and Spruck [11] on a punctured ball still holds when the background metric is not conformally flat. This problem was solved positively, when the background metric is smooth, by Marques [45] for $3 \le n \le 5$ and by Xiong-Zhang [60] for $n = 6$ recently. In this paper, we prove that the theorem of [11], as described by (2), continues to hold for $3 \le n \le 6$ as well as for $7 \le n \le 24$, even if the metric g is not smooth across $\{0\}$, as long as it has an expansion at 0 of flatness of order $\tau \ge \frac{n-2}{2}$, as given by (7) below.

The problem is also formulated as the study of asymptotic behavior at ∞ of solutions of the Yamabe equation with an asymptotically flat background metric.

$$- L_g u = n(n-2)u^{\frac{n+2}{n-2}} \quad \text{in } \mathbb{R}^n \setminus B_1, \quad u > 0, \tag{3}$$

where g is a smooth Riemannian metric defined on $\mathbb{R}^n \setminus \bar{B}_1$ satisfying the standard asymptotically flat condition

$$\sum_{i,j=1}^{n} \Big|\nabla^m \Big(g_{ij}(x) - \delta_{ij}\Big)\Big| \le C|x|^{-\tau-m} \quad \text{for } x \in \mathbb{R}^n \setminus B_1, \tag{4}$$

where $\tau \ge \frac{n-2}{2}$, C is a positive constant and $m = 0, \ldots, n+2$. Note that $\tau > \frac{n-2}{2}$ is necessary to define the ADM mass in general relativity; see Denisov-Solov'ev [17] and Bartnik [6]. This exterior formulation (3) is equivalent to the punctured unit ball setting [11] if g is flat, since (1) is invariant under Kelvin transforms.

Our main results are as follows.

Theorem 1.1. *Let u be a solution of (3) with the metric g satisfying the asymptotic flatness* (4). *If $3 \le n \le 24$, then either there exists a positive constant a such that*

$$u(x) = a|x|^{2-n} + O(|x|^{1-n}) \quad \textit{as } x \to \infty$$

or there exist $\alpha \in (0, 1)$ and a Fowler solution u_0 such that

$$u(x) = u_0(|x|)(1 + O(|x|^{-\alpha}) \quad \textit{as } x \to \infty.$$

Both of these alternatives can happen. In higher dimensions, we have

Theorem 1.2. *Let u be a solution of (3) with the metric g satisfying the asymptotic flatness (4). If $n \ge 25$ and*

$$\limsup_{x \to \infty} u(x) < \infty, \tag{5}$$

then the conclusion of Theorem 1.1 still holds.

The existence of complete conformal metrics of positive constant scalar curvature on $\mathbb{S}^n \setminus \Lambda$ and related problems have been studied by Schoen [53], Schoen-Yau [55], Mazzeo-Smale [49], Mazzeo-Pollack-Uhlenbeck [48], Mazzeo-Pacard [47], Bettiol-Piccione-Santoro

[7] and others, where Λ is a closed set. It was proved in [55] that the Hausdorff dimension of Λ is necessary to be at most $\frac{n-2}{2}$. See Byde [10] and Silva Santos [56] on other manifolds.

When the scalar curvature is negative, the first study of this problem goes back to Loewner-Nirenberg [43], where the dimension of Λ was proved to be at least $\frac{n-2}{2}$. Later work on the negative scalar curvature case was done on general manifolds by Aviles-McOwen [5], Finn-McOwen [19], Andersson-Chrúsciel-Friedrich [3], Mazzeo [46] and etc. See also the recent paper Q. Han-Shen [26] and references therein for related work.

By performing a Kelvin transform, (3) is equivalent to

$$- L_{\hat{g}} u = n(n-2) u^{\frac{n+2}{n-2}} \quad \text{in } B_1 \setminus \{0\}, \quad u > 0, \tag{6}$$

and (4) translates into

$$|\nabla^m (\hat{g}_{ij}(x) - \delta_{ij})| \le C|x|^{\tau - m} \quad \text{in } B_1 \setminus \{0\} \text{ for } 0 \le m \le n+2. \tag{7}$$

Note that the scalar curvature $R_{\hat{g}}$ might not be continuous across 0 if $n \le 6$, but we still have $|R_{\hat{g}}(x)| \le C|x|^{\tau-2}$ in $B_1 \setminus \{0\}$, which implies that $|R_{\hat{g}}(x)| \in L^p(B_1)$ for some $p > n/2$. The general case of the De Giorgi-Nash-Moser theorem implies that any solution u of (6) which is *bounded* in $B_1 \setminus \{0\}$ must extend to a Hölder continuous solution in B_1, and as a supersolution of $L_{\hat{g}} u = 0$ in B_1, $u(0) = \lim_{x \to 0} u(x) > 0$, see, e.g., [57]. We refer to such a situation as $x = 0$ being a removable singularity, and the solution $u(x)$ as a classical solution. The assumption (5) in Theorem 1.2 for $n \ge 25$ translates into

$$\limsup_{x \to 0} |x|^{n-2} u(x) < \infty. \tag{8}$$

Theorem 1.3. *Suppose that u is a solution of* (6) *with $\hat{g}$ satisfying* (7). *If $n \ge 25$, we assume further that u satisfies* (8). *Then either 0 is a removable singularity, or there exists a constant $C > 0$ such that*

$$\frac{1}{C}|x|^{-\frac{n-2}{2}} \le u(x) \le C|x|^{-\frac{n-2}{2}} \quad \text{for } x \in B_{1/2} \setminus \{0\}. \tag{9}$$

Moreover, we have $P(u) \le 0$, and $P(u) = 0$ if and only if 0 is a removable singularity, where $P(u)$ is the Pohozaev limit of u defined in (92).

Theorem 1.4. *Under the same assumptions as in Theorem 1.3, if 0 is not a removable singularity, then there exist $\alpha \in (0,1)$ and a Fowler solution u_0 such that*

$$u(x) = u_0(|x|)(1 + O(|x|^{-\alpha}) \quad \text{as } x \to 0.$$

Based on previous work in this area, a key in proving Theorems 1.1, 1.2, 1.3, 1.4 is to prove, for an unbounded solution u of (6), (9) holds.

When $\hat{g}$ is conformally flat, the proofs of the upper bound in (9) of Caffarelli, Gidas and Spruck [11] and Korevaar-Mazzeo-Pacard-Schoen [30] rely on inversion symmetries of (1), using different variants of the Alexandrov reflection. When the background metric is not conformally flat, that approach is no longer directly applicable, as $u - u_{reflection}$ now satisfies an inhomogeneous linear elliptic equation and thus a key ingredient of the reflection argument (maximum principle) is not directly available.

Chen-Lin [14] introduced an idea of constructing suitable auxiliary functions, when employing the moving planes method, to compensate for the loss of invariance under Mobius transformations of the prescribing scalar curvature equation

$$-\Delta u = K u^{\frac{n+2}{n-2}} \quad \text{in } B_1$$

when K is a positive non-constant continuous function—they used the method to deal with solutions with or without isolated singularities. For the case of a solution with an isolated singularity at 0, they obtained a sharp flatness criterion of $K(x)$ in another paper [15] to have the asymptotic behavior (2) for isolated singularities—their criterion turns out to be $\frac{1}{C}|x|^{l-1} \leq |\nabla K(x)| \leq C|x|^{l-1}$ for some constants $C \geq 1$ and $l \geq \frac{n-2}{2}$. A counterexample was constructed when the flatness is less than $\frac{n-2}{2}$. See also Zhang [62], Taliaferro-Zhang [58] and Lin-Prajapat [41, 42]. This idea of constructing auxiliary functions was adapted and developed to study the compactness of solutions to the Yamabe equation by Li-Zhang [37, 38, 39] via the moving spheres method, and to study isolated singularities of the Yamabe equation by previously mentioned paper Marques [45] and Xiong-Zhang [60] for $n \leq 6$. The situation in these papers is somewhat different from that of the prescribing scalar curvature equation in that one can not impose some condition analogous to the positive lower bound $|\nabla K(x)| \geq \frac{1}{C}|x|^{l-1}$ as in [15].

The auxiliary functions in [45] and [60] are radially symmetric. In our situation, we need to construct non-radial auxiliary functions and prove some needed quantitative estimates. A major part of these estimates is proved by applying and refining the blow up analysis developed in the studies of compactness of solutions to the Yamabe problem by [54, 40, 18, 44, 38, 39, 33] up to dimension 24. In particular, for $n \geq 7$, we adapt some arguments from Li-Zhang [38, 39] and Khuri-Marques-Schoen [33], where the spectral analysis of the linearized Yamabe equation at the spherical solutions played an important role. However, our situation is different from theirs, as the solutions and the metric contain singular points and the blow up analysis is implemented near those points. If $n \geq 25$, the assumption (8) is used to ensure the desired sign of the Pohozaev integral in the notation of [33] for some specific blowing up sequence of solutions. Note that the compactness of solutions to the Yamabe problem on smooth compact manifolds which are not conformal diffeomorphic to the unit sphere fails in dimension $n \geq 25$, see Brendle [8] and Brendle-Marques [9].

Once we have the upper bound, the lower bound in (9) is proved via the method of [14] and [45], based on an analysis of behavior of solutions of an ordinary differential inequality satisfied by the spherical average of the solution and the Pohozaev integral. However, there is a non-trivial linear term in the differential inequality of the spherical average of solutions in our case, which causes a technical issue when $\tau = \frac{n-2}{2}$. When $n = 6$ and the background metric is smooth, this issue was solved by finding a good conformal metric in [60]. Here we prove a refined ODE type estimate to prove the lower bound in (9).

The results of [11] and [30] have been extended to some fully nonlinear Yamabe equations or higher order conformally invariant equations, see, for example, Li-Li [34], Li [36], Chang-Han-Yang [12], Han-Li-Teixeira [28], Jin-Xiong [29], and the references therein.

The paper is organized as follows. In Section 2, we reduce the exterior problem to (6) and outline a proof of the upper bound of solutions with an isolated singularity in Theorem 2.1, deferring the details of the construction of auxiliary functions used in the proof to Section 4. In Section 3, we recall some local blow up analysis for smooth solutions of the Yamabe equation up to dimension 24, and set up conformal normal like coordinates to be used later. In Section 4, we provide details in proving the upper bound in Theorem 2.1. We divide the construction of the auxiliary function used in the moving spheres argument into the cases $3 \leq n \leq 6$ and $n \geq 7$. The difficulty of former case lies in the singularity of the metric while that of the latter lies in high dimensional effect. In Section 5, we prove the lower bound and give a criterion of removability in terms of the sign of the Pohozaev integral. In Section 6, we provide details of some improved ODE type estimates for the spherical average of the solution, which are used in Section 5.

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2 Reduction to an isolated singularity and outline proof of the upper bound of solutions with an isolated singularity

Let u be a positive solution of (3) with the metric g satisfying (4). We shall use an inversion to transform the problem into one with an isolated singularity on the punctured unit ball. For any $x \in \mathbb{R}^n \setminus B_1$, let $x = \frac{z}{|z|^2}$ for $z \in B_1 \setminus \{0\}$. Then we see that

$$\begin{aligned} u(x)^{\frac{4}{n-2}} g_{ij}(x) \mathrm{d}x^i \mathrm{d}x^j &= \Big(\frac{1}{|z|^{n-2}} u(\frac{z}{|z|^2}) \Big)^{\frac{4}{n-2}} |z|^4 g_{ij}(\frac{z}{|z|^2}) \mathrm{d}(\frac{z^i}{|z|^2}) \mathrm{d}(\frac{z^j}{|z|^2}) \\ &=: v(z)^{\frac{4}{n-2}} \hat{g}_{kl}(z) \mathrm{d}z^k \mathrm{d}z^l, \end{aligned}$$

where $v(z) = \frac{1}{|z|^{n-2}} u(\frac{z}{|z|^2})$, and

$$\hat{g}(z) := \hat{g}_{kl}(z) \mathrm{d}z^k \mathrm{d}z^l = \sum_{i,j} g_{ij}(\frac{z}{|z|^2})(\delta_{ik} - 2z^i z^k)(\delta_{il} - 2z^i z^l) \mathrm{d}z^k \mathrm{d}z^l. \tag{10}$$

By (4), we have

$$\sum_{k,l=1}^{n} \Big| \nabla^m \Big(\hat{g}_{kl}(z) - \delta_{kl} \Big) \Big| \leq C_0 |z|^{\tau - m} \quad \text{for } z \in B_1 \setminus \{0\},\ m = 0, 1, ..., n+2,$$

which is the same as (7). By the conformal invariance of L_g,

$$-L_{\hat{g}}v = n(n-2)v^{\frac{n+2}{n-2}} \quad \text{in } B_1 \setminus \{0\}. \tag{11}$$

Hence, the study of solutions of (3) with the metric g satisfying (4) has been reduced to the study of solutions of (11) with the metric $\hat{g}$ satisfying (7). If the assumption (5) in Theorem 1.2 holds, then it implies the upper bound (8) for v:

$$\limsup_{z\to 0} |z|^{n-2}v(z) < \infty.$$

To avoid using too many variables, from now on we will rename the z variable in $B_1 \setminus \{0\}$ as x, $\hat{g}$ as g, and $v(z)$ as $u(x)$, namely, we will study $u(x)$, which satisfies (6).

Theorem 2.1. *Let u be a solution of* (6) *with the metric g satisfying* (7)*. When $n \geq 25$, suppose that* (8) *holds. Then*

$$\limsup_{x\to 0} |x|^{\frac{n-2}{2}}u(x) < \infty. \tag{12}$$

When g is a smooth metric in B_1, Theorem 2.1 was proved by Marques [45] for $n \leq 5$ and by Xiong-Zhang [60] for $n = 6$.

Our proof of Theorem 2.1 follows the classical approach initiated by Schoen, further developed by many authors over the last three decades as described earlier; we do need to overcome difficulties caused by the potential singularity of the metric at 0 when $3 \leq n \leq 6$ and the high dimensional effect when $n \geq 7$. We will discuss the technical aspects of this analysis a bit later, much of which will follow the approach in [60].

For now, we will first outline the setup for proving Theorem 2.1. Clearly, it suffices to consider

$$\tau = \frac{n-2}{2}.$$

If (12) were invalid, there would exist a sequence $x_k \to 0$ such that

$$|x_k|^{\frac{n-2}{2}}u(x_k) \to \infty \quad \text{as } k \to \infty. \tag{13}$$

Lemma 2.2. *The sequence x_k in* (13) *can be selected to be local maximum points of u. Moreover, there exists $0 < \alpha_k < |x_k|/2$ such that*

$$u(x) \leq 2^{\frac{n-2}{2}}u(x_k) \quad \forall\, x \in B_{\alpha_k}(x_k)$$

and

$$\lim_{k\to\infty} u(x_k)\alpha_k^{\frac{n-2}{2}} = \infty.$$

Proof. The proof is standard by now. See page 3 of [60] for details. $\square$

We will do a blow up analysis of $u(x)$ near x_k. To carry out a more refined analysis needed later on, we will work with a conformal normal like coordinate system $\{z\}$ centered at x_k. The precise formulation will be given as Lemma 3.4 in section 3. The key is to set up this coordinate system on a common ball centered at $z = 0$ for a certain neighborhood in metric g "centered at" x_k which includes $B_{\frac{1}{2}}(x_k)$, via a coordinate map $\phi_{x_k}(z)$ with appropriate control, and with $\phi_{x_k}(0) = x_k$, $\phi_{x_k}(-x_k) = 0$.

We will work with $u_k(z) = (\kappa_{x_k} u) \circ \phi_{x_k}(z)$ for $z \in B_{1/2} \setminus \{-x_k\}$, where κ_{x_k}, ϕ_{x_k}, and $g_k(z)$ are defined as in Lemma 3.4, with x_k as $\bar{z}$ there, (46) holds for $g_k(z)$, and

$$\det g_k(z) = 1 \quad \text{in } B_{\sigma |x_k|}. \tag{14}$$

As usual, the analysis in this work uses multiple rescalings and multiple coordinate systems. To avoid excessive notations, we may reuse the same variable names in different contexts. For example x_k is selected in the x coordinate in $B_1 \setminus \{0\}$ for u according to Lemma 2.2, but in the conformal normal like coordinate z, the $x = 0$ point corresponds to $z = -x_k$, so we are using x_k in two different coordinate system.

According to Lemma 3.4, $u_k(z)$ is well defined for $z \in B_{1/2} \setminus \{-x_k\}$ for large k, and

$$- L_{g_k} u_k(z) = n(n-2) u_k(z)^{\frac{n+2}{n-2}} \quad \text{in } B_{1/2} \setminus \{-x_k\}. \tag{15}$$

Let

$$M_k = u_k(0), \quad l_k = M_k^{\frac{2}{n-2}}, \quad S_k = -l_k x_k,$$

and

$$v_k(y) = M_k^{-1} u_k(l_k^{-1} y), \quad y \in B_{\frac{l_k}{2}} \setminus \{S_k\}.$$

By Lemma 2.2, $\lim_{k \to \infty} |S_k| = \infty$. By equation (15) of u_k, we have

$$- L_{\bar{g}_k} v_k(y) = n(n-2) v_k(y)^{\frac{n+2}{n-2}} \quad \text{in } B_{\frac{l_k}{2}} \setminus \{S_k\}, \tag{16}$$

where $(\bar{g}_k)_{ij}(y) = (g_k)_{ij}(l_k^{-1} y)$.

By Lemma 2.2 and Lemma 3.4, we know that $x_k = \phi_{x_k}(0)$ is a critical point of u and κ_{x_k}. Thus $\nabla v_k(0) = 0$. It follows from local estimates for solutions of linear elliptic equations that, up to passing to a subsequence,

$$v_k(y) \to U(y) \quad \text{in } C^2_{loc}(\mathbb{R}^n) \quad \text{as } k \to \infty$$

for some nonnegative function $U \in C^2(\mathbb{R}^n)$ satisfying

$$\Delta U(y) + n(n-2) U(y)^{\frac{n+2}{n-2}} = 0 \quad \text{in } \mathbb{R}^n,$$

$$U(0) = 1, \quad \nabla U(0) = 0.$$

By the classification theorem of Caffarelli-Gidas-Spruck [11], we have

$$U(y) = (1 + |y|^2)^{\frac{2-n}{2}}. \tag{17}$$

Later on, we may write $U(|y|) = U(y)$ without causing confusion.

Recall that the Kelvin transform of a function f defined on some measurable set $\Omega \subset \mathbb{R}^n$ with respect to the sphere ∂B_λ, $\lambda > 0$, is defined as

$$f^\lambda(y) := (\frac{\lambda}{|y|})^{n-2} f(y^\lambda), \quad \text{for } y \text{ such that } y^\lambda := \frac{\lambda^2 y}{|y|^2} \in \Omega.$$

Applying the Kelvin transform to $U(y)$, we note that $U^\lambda(y) = U(y)$ for all y when $\lambda = 1$, and

$$\begin{cases} U^\lambda(y) < U(y) & \text{for } 0 < \lambda < 1 \text{ and } y \text{ with } |y| > \lambda, \text{ and} \\ U^\lambda(y) > U(y) & \text{for } \lambda > 1 \text{ and } y \text{ with } |y| > \lambda. \end{cases}$$

In fact we have a more precise estimate

$$U(y) - U^\lambda(y) \begin{cases} > c_1(n)(1-\lambda)(|y|-\lambda)|y|^{1-n}, \quad \text{if } 0 < \lambda < 1 \text{ and } |y| > \lambda, \\ \equiv 0, \quad \text{if } \lambda = 1, \\ < -c_1(n)(\lambda-1)(|y|-\lambda)|y|^{1-n}, \quad \text{if } \lambda > 1 \text{ and } |y| > \lambda, \end{cases} \tag{18}$$

where $c_1(n) > 0$ is a dimensional constant.

Using (18) and $v_k(y) \to U(y) = (1+|y|^2)^{\frac{2-n}{2}}$ in $C^2_{loc}(\mathbb{R}^n)$, we see that for $\lambda > 1$, y with $|y| > \lambda$,

$$v_k(y) - v_k^\lambda(y) < -c_1(n)(\lambda-1)(|y|-\lambda)|y|^{1-n} \text{ for sufficiently large } k. \tag{19}$$

For any fixed $\lambda' > \lambda > 1$ and $R > \lambda'$, we can choose a common k' such that the above holds for any $k \geq k'$ and $\lambda' \leq |y| \leq R$; but for our purpose it suffices that for each fixed $|y| > \lambda > 1$, there exists some k' such that the above holds for $k \geq k'$.

On the other hand we shall apply the method of moving spheres in $\Sigma_\lambda^k \backslash \{S_k\}$ to $w_\lambda(y) = v_k(y) - v_k^\lambda(y)$ (for simplicity we omit the subscript k in w_λ) to show that

$$v_k(y) - v_k^\lambda(y) + f_{\lambda,k}(y) \geq 0 \quad \text{for } y \in \Sigma_\lambda^k \setminus \{S_k\}, \tag{20}$$

for $1 - \delta_1 \leq \lambda \leq 1 + \delta_1$, and for sufficiently large k, where

$$\Sigma_\lambda^k := B_{Q_1 l_k^{1/2}} \setminus \bar{B}_\lambda,$$

$f_{\lambda,k}(y)$ is an auxiliary function constructed to make $v_k(y) - v_k^\lambda(y) + f_{\lambda,k}(y)$ a supersolution of a linear operator (30) related to $L_{\bar{g}_k}$ and $f_{\lambda,k}(y) \to 0$ uniformly over $y \in \Sigma_\lambda^k$ as $k \to \infty$, and for the $3 \leq n \leq 6$ cases δ_1 can be taken as $\frac{1}{2}$, and for the $n \geq 7$ cases $\delta_1 \in (0, 1/2)$ is determined to make it possible to construct the auxiliary function $f_\lambda^{(3)}(y)$ for $1 - \delta_1 \leq \lambda \leq 1 + \delta_1$ from (84). Furthermore, according to Proposition 4.3 to be proved later, $Q_1 > 0$ can be taken such that

$$S_k \in B_{Q_1 l_k^{1/2}}.$$

The construction of $f_{\lambda,k}(y)$ with the desired properties is carried out in Lemmas 4.6, 4.7, and Proposition 4.8 for $\frac{1}{2} \leq \lambda \leq 2$ in the $3 \leq n \leq 6$ cases, and respectively, in Lemma 4.11, (85), and Proposition 4.12 for $1-\delta_1 \leq \lambda \leq 1+\delta_1$ in the $n \geq 7$ cases; and the moving spheres argument for the $n \geq 7$ requires a minor modification to Σ_λ^k, as done in Proposition 4.12.

The two opposing estimates (19) and (20) on $v_k(y) - v_k^\lambda(y)$ are obtained under the assumption (13). We now fix some $1 < \lambda \leq 1+\delta_1$ and some y with $|y| > \lambda$: on the one hand, for sufficiently large k, we have (19); on the other hand, $f_{\lambda,k}(y) \to 0$ implies that for sufficiently large k, $c_1(n)(\lambda-1)(|y|-\lambda)|y|^{1-n} > f_{\lambda,k}(y)$, which, together with $v_k(y) - v_k^\lambda(y) + f_{\lambda,k}(y) \geq 0$, leads to $v_k(y) - v_k^\lambda(y) + c_1(n)(\lambda-1)(|y|-\lambda)|y|^{1-n} > 0$. This contradiction shows that the scenario of (13) can't happen.

To apply the method of moving spheres, we first need to work out the equations satisfied by w_λ. We first record the expansions for the coefficients in L_{g_k} based on Lemma 3.4,

$$\begin{aligned} L_{g_k} &= \Delta_{g_k} - c(n)R_{g_k}(z) \\ &= \Delta + \frac{1}{\sqrt{\det g_k(z)}}\partial_j\left(g_k^{ij}(z)\sqrt{\det g_k(z)}\right)\partial_i + (g_k^{ij}(z)-\delta_{ij})\partial_{ij} - c(n)R_{g_k}(z) \\ &=: \Delta + b_i(z)\partial_i + d_{ij}(z)\partial_{ij} - c(z), \end{aligned} \tag{21}$$

with

$$b_i(z) = \begin{cases} O(|x_k|^{\tau-2})|z|, & |z| < \sigma|x_k|, \\ O(|z+x_k|^{\tau-1}), & |z| \geq \sigma|x_k|, \end{cases}$$

$$d_{ij}(z) = \begin{cases} O(|x_k|^{\tau-2})|z|^2, & |z| < \sigma|x_k|, \\ O(|z+x_k|^{\tau}), & |z| \geq \sigma|x_k|, \end{cases}$$

$$c(z) = \begin{cases} O(|x_k|^{\tau-4})|z|^2, & |z| < \sigma|x_k|, \\ O(|z+x_k|^{\tau-2}), & |z| \geq \sigma|x_k|, \end{cases}$$

where $\sigma > 0$ is independent of k, and we drop the subscript k of b_i, d_{ij} and c for brevity.

After the scaling $y = l_k z$, by (21) and the definition of $\bar{g}_k$ after (16), we have

$$L_{\bar{g}_k} = \Delta + \bar{b}_i(y)\partial_i + \bar{d}_{ij}(y)\partial_{ij} - \bar{c}(y) \tag{22}$$

where the differentiations are with respect to y and

$$\begin{aligned} \bar{b}_i(y) =& l_k^{-1} b_i(l_k^{-1} y) \\ =& \begin{cases} O(l_k^{-2}|x_k|^{\tau-2}|y|), & |y| < \sigma|S_k|, \\ O(l_k^{-\tau}|y-S_k|^{\tau-1}), & |y| \geq \sigma|S_k|, \end{cases} \end{aligned} \tag{23}$$

$$\bar{d}_{ij}(y) = d_{ij}(l_k^{-1}y)$$
$$= \begin{cases} O(l_k^{-2}|x_k|^{\tau-2}|y|^2), & |y| < \sigma|S_k|, \\ O(l_k^{-\tau}|y - S_k|^{\tau}), & |y| \geq \sigma|S_k|, \end{cases} \tag{24}$$

and

$$\bar{c}(y) = c(n)l_k^{-2}R_{g_k}(l_k^{-1}y)$$
$$= \begin{cases} O(l_k^{-4}|x_k|^{\tau-4}|y|^2), & |y| < \sigma|S_k|, \\ O(l_k^{-\tau}|y - S_k|^{\tau-2}), & |y| \geq \sigma|S_k|. \end{cases} \tag{25}$$

Unlike in the locally conformally flat case, here, $v_k^\lambda(y)$ no longer satisfies an equation of the exact same form as $v_k(y)$ so we can't directly apply the moving spheres method to $w_\lambda(y)$. But a direct computation, using the equation for $v_k(y)$ at y and y^λ, as well as $\Delta v_k^\lambda(y) = \left(\frac{\lambda}{|y|}\right)^{n+2} \Delta v_k(y^\lambda)$, yields

$$\left[\Delta + \bar{b}_i(y)\partial_i + \bar{d}_{ij}(y)\partial_{ij} - \bar{c}(y)\right] w_\lambda(y) + \xi_\lambda(y)w_\lambda(y) = E_\lambda(y), \quad y \in \Sigma_\lambda^k \setminus \{S_k\}, \tag{26}$$

where

$$\xi_\lambda(y) = \begin{cases} n(n-2)\dfrac{v_k(y)^{\frac{n+2}{n-2}} - v_k^\lambda(y)^{\frac{n+2}{n-2}}}{v_k(y) - v_k^\lambda(y)}, & \text{if} \quad v_k(y) \neq v_k^\lambda(y), \\ n(n+2)v_k^{\frac{4}{n-2}}(y), \quad \text{if} \quad v_k(y) = v_k^\lambda(y), \end{cases} \tag{27}$$

and

$$E_\lambda(y) \;=\; \left(\bar{c}(y)v_k^\lambda(y) - (\frac{\lambda}{|y|})^{n+2}\bar{c}(y^\lambda)v_k(y^\lambda)\right) - \left(\bar{b}_i(y)\partial_i v_k^\lambda(y) + \bar{d}_{ij}(y)\partial_{ij}v_k^\lambda(y)\right)$$
$$+(\frac{\lambda}{|y|})^{n+2}\left(\bar{b}_i(y^\lambda)\partial_i v_k(y^\lambda) + \bar{d}_{ij}(y^\lambda)\partial_{ij}v_k(y^\lambda)\right). \tag{28}$$

What is going to make the moving spheres method work to prove Theorem 2.1 is that we are able to construct some $f_\lambda(y)$ (we have dropped the index k for $f_\lambda(y)$) such that

$$\left[\Delta + \bar{b}_i(y)\partial_i + \bar{d}_{ij}(y)\partial_{ij} - \bar{c}(y)\right] f_\lambda(y) + \xi_\lambda(y)f_\lambda(y) \leq -|E_\lambda(y)| \quad \text{in } \Sigma_\lambda^k$$

with control, including $f_\lambda(y) = 0$ for $y \in \partial B_\lambda$ and

$$|f_\lambda(y)| + |\nabla f_\lambda(y)| = o(1)|y|^{2-n} \text{ uniformly for } y \in \Sigma_\lambda^k. \tag{29}$$

We remark that (29) for the $3 \leq n \leq 6$ cases follows from Lemma 4.6 and Lemma 4.7, and for the $n \geq 7$ cases follows from Proposition 4.12. In fact, we need to modify Σ_λ^k slightly into $\tilde{\Sigma}_\lambda^k$ for the $n \geq 7$ cases—see Proposition 4.12 for the construction of $f_\lambda(y)$ for the

$n \geq 7$ cases and the definition of $\tilde{\Sigma}_\lambda^k$, and Proposition 4.8 for the $3 \leq n \leq 6$ cases. Using $f_\lambda(y)$, we have

$$\left[\Delta + \bar{b}_i(y)\partial_i + \bar{d}_{ij}(y)\partial_{ij} - \bar{c}(y)\right][w_\lambda(y) + f_\lambda(y)] + \xi_\lambda[w_\lambda(y) + f_\lambda(y)] \leq 0 \quad \text{in } \Sigma_\lambda^k \setminus \{S_k\}. \tag{30}$$

Our construction allows us to show that the moving spheres process can be started:

$$w_\lambda(y) + f_\lambda(y) > 0 \quad \text{in} \quad \Sigma_\lambda^k \setminus \{S_k\} \quad \text{for } \lambda \in [1 - \delta_1, 1 - \delta_1/2], \tag{31}$$

Indeed, for any $\lambda \in [1 - \delta_1, 1 - \delta_1/2]$ (the arithmetic below is worked out assuming $1 - \delta_1/2 \leq 3/5$ to get a clean constant in the estimate), by (18) we have

$$U(y) - U^\lambda(y) > \frac{2c_1(n)}{5}(|y| - \lambda)|y|^{1-n} \quad \text{for } |y| > \lambda;$$

in addition, $\frac{y}{|y|} \cdot \nabla\left(U(y) - U^\lambda(y)\right) \geq \frac{2c_1(n)\lambda^{1-n}}{5}$ for y with $|y| = \lambda$. Since $v_k(y) \to U(y)$ in $C^2_{loc}(\mathbb{R}^n)$ as $k \to \infty$, for any fixed $R >> 1$ we have

$$v_k(y) - v_k^\lambda(y) > \frac{c_1(n)}{5}(|y| - \lambda)|y|^{1-n}, \quad \lambda < |y| < R, \tag{32}$$

provided k is sufficiently large. We also have

$$v_k^\lambda(y) \leq (1 - 3\epsilon_0)|y|^{2-n}, \quad |y| \geq R,$$

where $\varepsilon_0 > 0$ is some constant. By Proposition 4.2 to be proved later,

$$v_k(y) \geq (1 - \epsilon_0)|y|^{2-n}, \quad |y| \geq R.$$

Thus

$$v_k(y) - v_k^\lambda(y) > \begin{cases} \frac{c_1(n)}{5}(|y| - \lambda)|y|^{1-n}, & \lambda < |y| < R, \\ 2\epsilon_0|y|^{2-n}, & R \leq |y| \leq Q_1 l_k^{1/2}. \end{cases} \tag{33}$$

Our estimate (29) implies that for all sufficiently large k we have $v_k(y) - v_k^\lambda(y) + f_\lambda(y) > 0$ on $\lambda < |y| < Q_1 l_k^{1/2}$. Therefore, (31) follows.

The critical position in the moving sphere method is defined by

$$\bar{\lambda} := \sup\{\lambda \leq 1 + \delta_1 : v_k(y) - v_k^\mu(y) + f_\mu(y) > 0, \quad \forall\, y \in \Sigma_\mu^k \setminus \{S_k\} \text{ and } 1 - \delta_1 < \mu < \lambda\}.$$

By (31), $\bar{\lambda}$ is well-defined. In order to reach to the final contradiction we claim that $\bar{\lambda} = 1 + \delta_1$.

If $\bar{\lambda} < 1 + \delta_1$, by the definition (56) of Q_1 and $|f_\lambda| = o(1)M_k^{-1}$ on $\partial B_{Q_1 l_k^{1/2}}$, we still have $v_k - v_k^{\bar{\lambda}} + f_{\bar{\lambda}} > 0$ on $\partial B_{Q_1 l_k^{1/2}}$. By the maximum principle, $v_k - v_k^{\bar{\lambda}} + f_{\bar{\lambda}}$ is strictly positive in $\Sigma_{\bar{\lambda}}^k$ and $\frac{\partial}{\partial r}(v_k - v_k^{\bar{\lambda}} + f_{\bar{\lambda}}) > 0$ on $\partial B_{\bar{\lambda}}$, therefore we can move spheres a little further than $\bar{\lambda}$ by a standard argument in the moving spheres method—the presence of a potential singularity of v_k at S_k does not create any issues in applying this method, as was done in [15, 36, 45, 60]. This contradicts the definition of $\bar\lambda$. Therefore, the claim is proved.

Sending k to ∞ in the inequality

$$v_k(y) - v_k^{\bar\lambda}(y) + f_{\bar\lambda}(y) \geq 0 \quad \text{for } 1+\delta_1 = \bar\lambda < |y| < Q_1 l_k^{1/2},$$

we have

$$U(y) \geq U^{\bar\lambda}(y) \quad \text{for all } \ |y| \geq \bar\lambda = 1+\delta_1,$$

which is a clear violation of (18). This contradiction concludes the proof of Theorem 2.1.

3 Blow up analysis for local solutions to the Yamabe equation

In this section, we summarize a few key facts in Khuri-Marques-Schoen [33] needed for our analysis of the behavior of $u(x)$ near $x = x_k$ (namely for $u_k(z)$ near $z = 0$) for the $n \geq 7$ cases.

Suppose that g_k are smooth metrics defined in B_1 satisfying

$$\|g_k\|_{C^{n+2}(B_1)} \leq C, \quad \det g_k = 1 \quad \text{in } B_1, \tag{34}$$

where $n \geq 3$ is the dimension and $k = 1, 2, \ldots$, and B_1 is a normal coordinates chart of $g_k(x) = \exp(h_{ij}(x))$, where we dropped the subscript k of h_{ij}, and in addition,

$$\sum_j h_{ij}(x) x^j = 0 \quad \text{and} \quad \mathrm{trace}(h_{ij}(x)) = 0.$$

Define, when $n \geq 6$,

$$H_{ij}(x) = \sum_{2 \leq |\alpha| \leq n-4} h_{ij\alpha} x^\alpha,$$

$$H_{ij}^{(l)}(x) = \sum_{|\alpha|=l} h_{ij\alpha} x^\alpha, \quad |H_{ij}^{(l)}|^2 = \sum_{|\alpha|=l} |h_{ij\alpha}|^2,$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\alpha_i \geq 0$ are integers, $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and

$$h_{ij\alpha} = \frac{\partial^\alpha h_{ij}(0)}{\alpha!} = \frac{\partial_{x^1}^{\alpha_1} \partial_{x^2}^{\alpha_2} \ldots \partial_{x^n}^{\alpha_n} h_{ij}(0)}{\alpha_1! \cdots \alpha_n!}.$$

Then $H_{ij}(x) = H_{ji}(x)$, $H_{ij}(x) x^j = 0$ and $trac(H_{ij}(x)) = 0$. By the Taylor expansion,

$$\begin{aligned} &\left| R_{g_k} - \partial_i \partial_j h_{ij} + \partial_l (H_{ij} \partial_l H_{il}) - \frac{1}{2} \partial_j H_{ij} \partial_l H_{il} + \frac{1}{4} \partial_l H_{ij} \partial_l H_{ij} \right| \\ &\leq C \sum_{|\alpha|=2}^{d} \sum_{i,j} |h_{ij\alpha}|^2 |x|^{2|\alpha|} + C|x|^{n-2} \end{aligned} \tag{35}$$

and

$$|R_{g_k} - \partial_i\partial_j h_{ij}| \\ \leq C\sum_{|\alpha|=2}^{d}\sum_{i,j}|h_{ij\alpha}|^2|x|^{2|\alpha|-2} + C|x|^{n-2}, \tag{36}$$

where $d = [\frac{n-2}{2}]$, see Proposition 4.3 of Khuri-Marques-Schoen [33] (see also Ambrosetti-Malchiodi [2], Brendle [8]). For $\varepsilon > 0$, let

$$\tilde{H}_{ij}(y) = H_{ij}(\varepsilon y).$$

If $n \geq 8$, it was proved in section 4 of [33] that there exists a solution $\tilde{Z}_\varepsilon(y)$ of

$$\Delta\tilde{Z}_\varepsilon(y) + n(n+2)U^{\frac{4}{n-2}}(y)\tilde{Z}_\varepsilon(y) = c(n)\sum_{l=4}^{n-4}\sum_{i,j}\partial_i\partial_j\tilde{H}_{ij}^{(l)}(y)U(y), \tag{37}$$

satisfying that $\tilde{Z}_\varepsilon(0) = 0$, $\nabla\tilde{Z}_\varepsilon(0) = 0$,

$$\int_{\partial B_r}\tilde{Z}_\varepsilon(y)\,\mathrm{d}S = \int_{\partial B_r}\tilde{Z}_\varepsilon(y)y^i\,\mathrm{d}S = 0, \quad r > 0,\ i = 1, \dots, n$$

and

$$|\nabla^m\tilde{Z}_\varepsilon(y)| \leq C\sum_{|\alpha|=4}^{n-4}\sum_{i,j}\varepsilon^{|\alpha|}|h_{ij\alpha}|(1+|y|)^{|\alpha|+2-n-m}, \tag{38}$$

where $U(y) = (1+|y|^2)^{-\frac{n-2}{2}}$, $c(n) = \frac{(n-2)}{4(n-1)}$, $C > 0$ is independent of ε and H_{ij}, and $m = 0, 1, 2$. Note that if $Z_\varepsilon(x) = \varepsilon^{-\frac{n-2}{2}}\tilde{Z}_\varepsilon(\frac{x}{\varepsilon})$ and $U_\varepsilon(x) = \varepsilon^{\frac{n-2}{2}}(\varepsilon^2 + |x|^2)^{-\frac{n-2}{2}}$, we have

$$\Delta Z_\varepsilon(x) + n(n+2)U_\varepsilon^{\frac{4}{n-2}}(x)Z_\varepsilon(x) = c(n)\sum_{l=4}^{n-4}\sum_{i,j}\partial_i\partial_j H_{ij}^{(l)}(x)U_\varepsilon(x), \tag{39}$$

and

$$|\nabla^m Z_\varepsilon(x)| \leq C\varepsilon^{\frac{n-2}{2}}\sum_{|\alpha|=4}^{n-4}\sum_{i,j}|h_{ij\alpha}|(\varepsilon+|x|)^{|\alpha|+2-n-m}, \tag{40}$$

where $C > 0$ is independent of ε and H_{ij}, and $m = 0, 1, 2$.

Suppose that $\{u_k\}_{k=1}^\infty$ is a sequence of solutions of

$$-L_{g_k}u_k = n(n-2)u_k^{\frac{n+2}{n-2}} \quad \text{in } B_1, \quad u_k > 0 \tag{41}$$

with g_k satisfying (34).

We say 0 is an isolated blow up point of u_k if $\lim_{k\to\infty} u_k(0) = \infty$, 0 is a local maximum point of u_k, and

$$u_k(x) \le A_1 |x|^{-\frac{n-2}{2}} \quad \text{in } B_{\rho_0},$$

where A_1, ρ_0 are positive constants independent of k.

We say 0 is an isolated simple blow up point of u_k, if 0 is an isolated blow up point and

$$r^{\frac{n-2}{2}} \bar{u}_k(r) \text{ has exactly one critical point in } (0,\rho)$$

for some constant $\rho \in (0, \rho_0]$ independent of k, where

$$\bar{u}_k(r) = \fint_{\partial B_r} u_k \, \mathrm{d}S.$$

Define $\varepsilon_k = u_k(0)^{-\frac{2}{n-2}}$ and

$$v_k(y) = \varepsilon_k^{\frac{n-2}{2}} u_k(\varepsilon_k y) \quad \text{for } y \in B_{1/\varepsilon_k}.$$

In the following proposition, we take $h_{ij\alpha} = 0$ and $\tilde{Z}_{\varepsilon_k} = 0$ when $n \le 5$.

Proposition 3.1. *Let 0 be an isolated simple blow up point of u_k. Then, for $|y| \le \rho\varepsilon_k^{-1}$,*

$$\begin{aligned}\left|\nabla^m (v_k - U - \tilde{Z}_{\varepsilon_k})(y)\right| \le & C \sum_{|\alpha|=2}^{d-1} \sum_{i,j} |h_{ij\alpha}|^2 \varepsilon_k^{2|\alpha|} |\ln \varepsilon_k|^{\theta_{|\alpha|}} (1+|y|)^{2|\alpha|+2-n-m} \\ & + C\varepsilon_k^{n-3}(1+|y|)^{-1-m}, \quad \text{for } m = 0,1,2,\end{aligned}$$

where $C > 0$ depends only on the upper bound of $\|g_k\|_{C^{n+2}(B_1)}$, A_1 and ρ_0, $\theta_{|\alpha|} = 1$ if $|\alpha| = \frac{n-2}{2}$ while $\theta_{|\alpha|} = 0$ otherwise.

If $6 \le n \le 24$, then

$$\sum_{|\alpha|=2}^{d} \sum_{i,j} |h_{ij\alpha}|^2 \varepsilon_k^{2|\alpha|} |\ln \varepsilon_k|^{\theta_{|\alpha|}} \le C \varepsilon_k^{n-2}, \tag{42}$$

where $C > 0$ depends only on the upper bound of $\|g_k\|_{C^{n+2}(B_1)}$, A_1 and ρ_0.

Proof. The first part can be found in [40, 18, 44, 38] for $n \le 5$, and in [33] for $n \ge 6$. The second part can be found in the proof of Theorem 6.1 of [33].

□

The dimension restriction $n \le 24$ for (42) of Proposition 3.1 is necessary to guarantee a positive lower bound of the Pohozaev quadratic form; see Theorem A.4 and Theorem A.8 of [33]. We note that when (42) holds for u_k and g_k, then a corresponding version holds for the rescaled $v_k(y) = \tau_k^{\frac{n-2}{2}} u_k(\tau_k y)$ and $\hat{g}_k(y) = g_k(\tau_k y)$ for $\tau_k \to 0$ such that $v_k(0) \to \infty$, as in the set up for the proof of Lemma 8.2 of [33], with ϵ_k replaced by $v_k(0)^{-\frac{2}{n-2}}$, thus Lemma 8.2 of [33] continues to hold without the dimension restriction $n \le 24$, as long as (42) holds. Therefore we have

Proposition 3.2. *Let 0 be an isolated blow up point of u_k. Suppose that either $n \le 24$ or* (42) *holds for some constant C independent of k. Then 0 is an isolated simple blow up point of u_k, and*

$$\left|\nabla^m (v_k - U - \tilde{Z}_{\varepsilon_k})(y)\right| \le C\varepsilon_k^{n-3}(1+|y|)^{-1-m}$$

for every $|y| \le \rho\varepsilon_k^{-1}$ and $m = 0, 1, 2$, where $C > 0$ is independent of k. The latter estimate is equivalent to

$$|\nabla^m (u_k - U_{\varepsilon_k} - Z_{\varepsilon_k})(x)| \le C\varepsilon_k^{\frac{n-2}{2}}(\varepsilon_k + |x|)^{-1-m} \tag{43}$$

for $|x| \le \rho$.

Proof. It suffices to show that 0 is an isolated simple blow up point of u_k, since the estimates will follow from Proposition 3.1 and (42). If $n \le 24$, this is what Lemma 8.2 of [33] asserts. If $n \ge 25$ and (42) is assumed, as discussed above, the conclusion of Theorem 7.1 of [33] (the local sign restriction of Pohozaev integral) still holds and the same proof of Lemma 8.2 of [33] applies.

Therefore, we complete the proof. □

The reason we are willing to assume (42) in Proposition 3.2 for $n \ge 25$ is that, in the setting of (13), assumptions (7) and (8) imply that a relevant version of (42) holds for appropriately rescaled u; see (80).

We formulate a version of (42) for the more general situation: For any local maximum point of u_k in $\bar{x} \in B_{1/2}$ with $u_k(\bar{x}) \ge 1$, there exists a conformal normal coordinates system centered at $\bar{x}$ such that

$$\sum_{|\alpha|=2}^{d} \sum_{i,j} |\partial^\alpha (g_k)_{ij}(0)|^2 \varepsilon_{\bar{x},k}^{2|\alpha|} |\ln \varepsilon_{\bar{x},k}|^{\theta_{|\alpha|}} \le C\varepsilon_{\bar{x},k}^{n-2} \tag{44}$$

for some constant C independent of k, where $\varepsilon_{\bar{x},k} = u_k(\bar{x})^{-\frac{2}{n-2}}$.

Proposition 3.3. *Suppose that 0 is a local maximum point of u_k and $\lim_{k\to\infty} u_k(0) = \infty$. If $n \ge 25$, suppose further that for any local maximum point of u_k in $\bar{x} \in B_{1/2}$ with $u_k(\bar{x}) \ge 1$, there exists a conformal normal coordinates system centered at $\bar{x}$ such that* (44) *holds. Then 0 is an isolated simple blow up point of u_k in some ball B_ρ with $0 < \rho < 1$.*

Proof. If $n \le 24$, the proof amounts to localizing the arguments in section 8 of [33] on compact manifolds. There are only two places which need some modification. First, by considering the function $(1/2 - |x|)^{\frac{n-2}{2}} u_k(x)$ in $\bar{B}_{1/2}$, we can obtain a local bubbles decomposition in $B_{1/2}$ to replace Proposition 8.1 of [33]. In fact, this was done by Lemma 1.1 of Han-Li [27]. Second, to prove a positive lower bound for the distance between centers of bubbles (Proposition 8.3 of [33]), we can use a selecting process to find two almost closest two bubbles and scale them apart; see Lemma 2.1 of Niu-Peng-Xiong [50] or the proof of Proposition 8.2 of Almaraz [1]. The rest of section 8 of [33], in particular Lemma 8.2, can be applied identically and Proposition 3.3 follows.

If $n \ge 25$, (44) implies (42). Given Proposition 3.2, the proof is similar as above. □

We next formulate and prove the properties of the conformal normal coordinates in our set up.

Lemma 3.4. *Let g be a smooth Riemannian metric defined in $B_1 \setminus \{0\}$ and satisfy* (7)*. Then there exist constants $0 < \sigma < \bar{\sigma} < 1/16 < \Lambda$, depending only n, g and C_0 in* (7)*, such that for any $\bar{z} \in B(0, \frac{1}{10}) \setminus \{0\}$ one can find a function $\kappa_{\bar{z}} \in C^\infty(B_1)$ satisfying*

$$\frac{1}{\Lambda} \leq \kappa_{\bar{z}} \leq \Lambda, \quad \kappa_{\bar{z}}(\bar{z}) = 1, \quad |\nabla \kappa_{\bar{z}}(\bar{z})| = 0 \tag{45}$$

such that the conformal metric $\varkappa = \kappa_{\bar{z}}^{-\frac{4}{n-2}} g$ has the following properties. There exists a smooth bijection $\phi_{\bar{z}} : B_{1/2} \to B_{1/2} + \{\bar{z}\}$ satisfying

(i) $\phi_{\bar{z}}(0) = \bar{z}$, $\nabla \phi_{\bar{z}}(0)$ is the identity matrix, $\phi_{\bar{z}}(-\bar{z}) = 0$,

$$\Lambda^{-1} \leq |\nabla \phi_{\bar{z}}| \leq \Lambda \quad \textit{and} \quad |\nabla^m \phi_{\bar{z}}| \leq \Lambda |\bar{z}|^{-(m-1)} \quad \textit{in } B_{1/2}$$

for $m = 2, \ldots, n$;

(ii) In this coordinates system $(B_{1/2}, \phi_{\bar{z}})$, write $\varkappa(\phi_{\bar{z}}(x)) = \varkappa_{ij}(x) \mathrm{d}x^i \mathrm{d}x^j$. We have

$$\det \varkappa_{ij}(x) = 1 \quad \textit{for } |x| \leq \sigma |\bar{z}|$$

and for $m = 0, 1, 2, \ldots, n$

$$\sum_{i,j=1}^{n} |\nabla^m (\varkappa_{ij}(x) - \delta_{ij})| \leq \begin{cases} \Lambda |\bar{z}|^{\tau - m} (\frac{|x|}{|\bar{z}|})^{\max\{2-m, 0\}} & \textit{if } |x| \leq 2\sigma |\bar{z}|, \\ \Lambda |x + \bar{z}|^{\tau - m} & \textit{if } |x| > 2\sigma |\bar{z}|. \end{cases} \tag{46}$$

Proof. For each $\bar{z} \in B_{1/10} \setminus \{0\}$, we set $\bar{r} = |\bar{z}|$ and define

$$g^{\bar{r}}(y) := g^{\bar{r}}_{ij}(y) \mathrm{d}y^i \mathrm{d}y^j \quad \text{for } |y| < 1/2,$$

where $g^{\bar{r}}_{ij}(y) = g_{ij}(\bar{z} + \bar{r} y)$. By (7), we have, for $|y| < 1/2$,

$$|\nabla^l (g^{\bar{r}}_{ij}(y) - \delta_{ij})| \leq C \bar{r}^\tau \quad \text{for } l = 0, 1, \ldots, n+2. \tag{47}$$

Hence, there exists a constant $\delta_0 > 0$ independent of $\bar{z}$ such that the exponential maps $\exp_0^{g^{\bar{r}}}(x)$ centered at 0 of $(B_{1/2}, g^{\bar{r}}(y))$ is well defined for $|x| \leq \delta_0$, namely, x, $|x| < \delta_0$, provides a geodesic normal coordinates for $y = \exp_0^{g^{\bar{r}}}(x)$ in the metric $g^{\bar{r}}(y)$.

By Günther [24], there exist a positive function $\kappa(y) \in C^\infty(B_{1/4})$ and $\delta_1 < \frac{\delta_0}{2}$ such that the metric $h(y) = \kappa^{-\frac{4}{n-2}}(y) g^{\bar{r}}(y)$, when expressed in terms of x via $y = \exp_0^h(x)$, satisfies

$$\det(h_{ij}(\exp_0^h(x))) = 1 \quad \text{for } |x| < 2\delta_1. \tag{48}$$

Moreover, $\kappa(\exp_0^h(0)) = 1$, $\nabla\kappa(\exp_0^h(0)) = 0$,

$$\Lambda^{-1} \le \kappa(\exp_0^h(x)) \le \Lambda, \quad |\nabla^m \kappa(\exp_0^h(x))| \le \Lambda \bar{r}^\tau \quad \text{for } |x| \le 4\delta_1,\ m = 1, \dots, n,$$

where Λ and δ_1 depend only on C_0 and n. Since

$$h_{ij}(y) = \sum_{k,l=1}^{n} \frac{\partial x_k}{\partial y_i} h(\frac{\partial}{\partial x_k}, \frac{\partial}{\partial x_l}) \frac{\partial x_l}{\partial y_j},$$

and $h(\frac{\partial}{\partial x_k}, \frac{\partial}{\partial x_l}) = \delta_{kl} + O(|x|^2)$ near $x = 0$, it follows from (47) that

$$\left| (\frac{\partial y}{\partial x})\Big|_{x=0} - I \right| \le \Lambda \bar{r}^\tau,$$

where $(\frac{\partial y}{\partial x})\Big|_{x=0}$ is the Jacobi matrix at $x = 0$ and I is the identity matrix. Hence, we can find $\bar{\sigma} > 0$ such that for $\bar{r} < \bar{\sigma}$, there exists a smooth bijection map ϕ from $\mathbb{R}^n \to \mathbb{R}^n$ extending $y = \exp_0^h(x)$ for $|x| < 2\delta_1$ such that

$$\phi(x) = x \quad \text{for } |x| > 4\delta_1.$$

Set $\sigma := 2\delta_1$ and

$$\phi_{\bar{z}}(x) := \bar{z} + \bar{r}\phi(\frac{x}{\bar{r}}) \quad \text{for } x \in B_{1/2}. \tag{49}$$

We can extend and modify $\kappa(y) = \kappa(\exp_0^h(x))$ for $2\delta_1 \le |x| \le 4\delta_1$ so that

$$\kappa(\phi(x)) = \kappa(0) \quad \text{for } |x| \ge 4\delta_1,$$

and

$$(C\Lambda)^{-1} \le \kappa(\phi(x)) \le C\Lambda, \quad |\nabla^m \kappa(\phi(x))| \le C\Lambda |\bar{z}|^\tau \quad \text{for } |x| \in \mathbb{R}^n,\ m = 1, \dots, n.$$

Set

$$\kappa_{\bar{z}}(\phi_{\bar{z}}(x)) := \kappa(\phi(\frac{x}{|\bar{z}|})). \tag{50}$$

Then

$$\varkappa_{ij}(x) = \kappa(\phi(\frac{x}{\bar{r}}))^{-\frac{4}{n-2}} \sum_{k,l=1}^{n} \frac{\partial \phi_k}{\partial x_i}\left(\frac{x}{\bar{r}}\right) g_{kl}\left(\bar{z} + \bar{r}\phi(\frac{x}{\bar{r}})\right) \frac{\partial \phi_l}{\partial x_j}\left(\frac{x}{\bar{r}}\right),$$

and it is easy to check that $\kappa_{\bar{z}}(z)$, $\varkappa_{ij}(x)$, and $\phi_{\bar{z}}(x)$ satisfy all the conclusions in the lemma. Therefore, we complete the proof.

□

4 Details in proving the upper bound in Theorem 2.1

We now furnish details for the construction of f_λ satisfying (29) and (30). The construction has some differences between the $3 \le n \le 6$ and $n \ge 7$ cases. We first summarize lower bounds of u_k and v_k in Propositions 4.1 and 4.2, and then an upper bound of $|x_k|$ in terms of $l_k^{1/2}$ in Proposition 4.3.

By Lemma 3.4, g is at least Hölder continuous in $B_{1/2}$ and $|R_g(z)| \le C|z|^{\tau-2}$. By Hardy inequality, there exists a $\delta > 0$ such that

$$\int_{B_\delta} \left(|\nabla_g \phi|^2 - c(n)|R_g|\varphi^2 \right) \mathrm{d}vol_g \ge \int_{B_\delta} |\varphi|^2 \,\mathrm{d}vol_g, \quad \forall\, \varphi \in H_0^1(B_\delta). \tag{51}$$

Without loss of generality, we assume $\delta = 1/2$.

Proposition 4.1. *There exists $c_0 > 0$ independent of k such that $u_k(z) > c_0$ in $B_{1/2}$.*

Proof. Since the conformal factors κ_{x_k} are uniformly controlled, it suffices to prove that $u(x) \ge c > 0$ on $B_{1/2} \setminus \{0\}$ for some $c > 0$. Since $u(x)$ is a positive solution of (11) in $B_{1/2} \setminus \{0\}$, it is a positive supersolution of L_g there. The Hardy inequality for g on B_δ (we have taken $\delta = 1/2$) implies that there exists a classical solution $v(x)$ on $B_{1/2}$ of $L_g v(x) = 0$ with $v(x) = u(x)$ on $\partial B_{1/2}$, and that $v(x) > 0$ in $B_{1/2}$—we have used the De Giorgi-Nash-Moser theory here as explained in the introduction. The maximum principle holds for L_g on $B_{1/2}$, and just as in proving Bocher's theorem for harmonic functions, we conclude that $u(x) \ge v(x)$ on $B_{1/2} \setminus \{0\}$, it then follows that $u(x) \ge \min_{\overline{B}_{1/2}} v > 0$. $\square$

Proposition 4.2. *For any given $\epsilon_0 > 0$, there exists $R > 0$ such that for all sufficiently large k*

$$v_k(y) \ge (1-\epsilon_0)|y|^{2-n}, \quad R < |y| < \frac{l_k}{2}. \tag{52}$$

Proof. By Proposition 4.1, we have $v_k(y) \ge c_0/M_k$. Hence, (52) holds if $|y| \ge l_k^{3/4} = M_k^{\frac{3}{2(n-2)}}$.

Next, we consider $|y| < l_k^{3/4}$. Since $v_k \to U$ in $C^2_{loc}(\mathbb{R}^n)$ as $k \to \infty$, for any $\epsilon_0 > 0$ small and R large, we have

$$v_k(y) \ge (1-\frac{\epsilon_0}{8})(1+|y|)^{2-n}, \quad |y| \le R, \tag{53}$$

when k is large. Let $G_k \in C^2(B_{1/2} \setminus \{0\})$ be a nonnegative solution of

$$-L_{g_k} G_k = 0 \quad \text{in } B_{1/2} \setminus \{0\}, \quad G_k = 0 \quad \text{on } \partial B_{1/2}$$

and

$$\lim_{z \to 0} |z|^{n-2} G_k(z) = 1.$$

Making use of the standard local estimates of linear elliptic equations, we have

$$G_k(z) = |z|^{2-n} + a^k(z), \tag{54}$$

where

$$|a^k(z)| \le C|z|^{2-n+\gamma}$$

for some constants $C > 0$ and $0 < \gamma \le 1$ independent of k. By (53) and (54), we have $u_k \ge (1 - \frac{\epsilon_0}{4})M_k^{-1}G_k$ on $\partial B_{Rl_k^{-1}}$ when k is large. Applying the maximum principle to $u_k - (1 - \frac{\epsilon_0}{4})M_k^{-1}G_k$, we obtain

$$u_k(z) - (1 - \frac{\epsilon_0}{4})M_k^{-1}G_k(z) \ge 0 \quad \text{in } B_{1/2} \setminus B_{Rl_k^{-1}},$$

where we have used $-L_{g_k}$ is coercive in $H_0^1(B_{1/2})$, i.e., (51). It follows that

$$v_k(y) \ge (1 - \frac{\epsilon_0}{4})|y|^{2-n}\Big(1 - C(l_k^{-1}|y|)^\gamma\Big) \quad \text{for } R < |y| < \frac{1}{2}l_k.$$

Hence, if further $|y| < l_k^{3/4}$, we have

$$v_k(y) \ge (1 - \frac{\epsilon_0}{4})(1 - Cl_k^{-\frac{\gamma}{4}})|y|^{2-n} \ge (1 - \epsilon_0)|y|^{2-n}, \tag{55}$$

provided k is sufficiently large. This completes our proof. □

Proposition 4.3. *Under the assumptions of Theorem 2.1 and Lemma 2.2, we have*

$$|x_k| \le \bar{C}l_k^{-1/2}$$

for some $\bar{C} > 0$ independent of k.

Proof. If $n \ge 25$, the proposition follows immediately from the assumption (8). Now, we assume $n \le 24$, and define $\tilde{u}_k(y) = |x_k|^{\frac{n-2}{2}}u_k(|x_k|y)$ and $\tilde{g}_k(y) = g_k(|x_k|y)$. Then $\tilde{u}_k(y)$ is a smooth solution of

$$-L_{\tilde{g}_k}\tilde{u}_k = n(n-2)\tilde{u}_k^{\frac{n+2}{n-2}} \quad \text{in } B_1,$$

and 0 is a local maximum point of $\tilde{u}_k$, $\tilde{u}_k(0) = |x_k|^{\frac{n-2}{2}}M_k \to \infty$ as $k \to \infty$, and

$$\det \tilde{g}_k = 1 \quad \text{in } B_\sigma, \quad \|\tilde{g}_k\|_{C^{n+2}(B_1)} \le C$$

for some $C > 0$ independent of k. By Proposition 3.3, 0 must be an isolated simple blow up point of $\tilde{u}_k$ for some $0 < \rho < \sigma$. As a consequence of the last estimate of Proposition 3.2, $\tilde{u}_k(0)\tilde{u}_k(y) \le C$ for $|y| = \rho$ for some $C > 0$ depending on ρ and all sufficiently large k. However, for $|y| = \rho$,

$$\tilde{u}_k(0)\tilde{u}_k(y) = |x_k|^{\frac{n-2}{2}}M_k|x_k|^{\frac{n-2}{2}}u_k(|x_k|y) = |x_k|^{n-2}M_ku_k(|x_k|y) \ge c_0|x_k|^{n-2}M_k$$

where $c_0 > 0$ is the constant in Proposition 4.1. So we obtain an upper bound for $|x_k| \le \bar{C}l_k^{\frac{1}{2}}$.

□

Let $Q_1 > 0$ such that for all $\lambda \leq 2$

$$2U^\lambda(y) < \frac{c_0}{2} M_k^{-1} \quad \text{for } |y| \geq \frac{Q_1 l_k^{1/2}}{4}, \tag{56}$$

where $c_0 > 0$ is the constant in Proposition 4.1. This choice of Q_1 can guarantee that $v_k^\lambda < v_k$ near the boundary of $B_{Q_1 l_k^{1/2}}$ due to (52).

Remark 4.4. *We now summarize the relations among $|x_k|, l_k^{1/2}$ and $|S_k|$ for future references:*

(i). $|x_k| \to 0$, $M_k \to \infty$, $|S_k| = l_k|x_k| \to \infty$ *as* $k \to \infty$,

(ii). $|x_k| \leq \bar{C} l_k^{-1/2}$, $|S_k| \leq \bar{C} l_k^{1/2} < Q_1 l_k^{1/2}$,

(iii). $l_k^\tau = M_k$, $l_k^{-2}|x_k|^{\tau-2} = l_k^{-\tau}|S_k|^{\tau-2} = M_k^{-1}|S_k|^{\frac{n-6}{2}}$ *and* $l_k^{-4}|x_k|^{\tau-4} = l_k^{-\tau}|S_k|^{\tau-4} = M_k^{-1}|S_k|^{\frac{n-10}{2}}$, *using* $\tau = \frac{n-2}{2}$.

4.1 Case of $3 \leq n \leq 6$.

Here we first provide an upper bound for E_λ as defined in (28) in terms of some powers of $|y|$ and $|y - S_k|$, then construct $f_\lambda(y)$ with respect to these power functions in Lemmas 4.6 and 4.7.

Let $\chi_k \in C_c^\infty(B_{|S_k|/2}(S_k))$ be a cutoff function satisfying $0 \leq \chi_k \leq 1$ and $\chi_k = 1$ in $B_{|S_k|/4}(S_k)$. Let

$$\sigma_k := \|v_k - U\|_{C^2(B_2)}. \tag{57}$$

We have $\sigma_k \to 0$ as $k \to \infty$. Note also that $\frac{1}{2} \leq \tau \leq 2$ here.

Proposition 4.5. *Suppose $n \leq 6$. For $\lambda \in [1/2, 2]$ and $\lambda \leq |y| \leq Q_1 l_k^{1/2}$, we have*

$$E_\lambda(y) \leq C_1\Big(E_\lambda^{(1)}(|y|) + E_\lambda^{(2)}(y)\Big), \tag{58}$$

where $C_1 > 0$ is independent of k,

$$E_\lambda^{(1)}(|y|) = \begin{cases} l_k^{-\tau}|S_k|^{\tau-4}|y|^{4-n} + \sigma_k l_k^{-\tau}|S_k|^{\tau-2}|y|^{-n}, & |y| < \sigma|S_k|, \\ l_k^{-\tau}|y|^{\tau-n}, & |y| \geq \sigma|S_k|, \end{cases}$$

and

$$E_\lambda^{(2)}(y) = l_k^{-\tau}|S_k|^{2-n}|y - S_k|^{\tau-2}\chi_k(y).$$

Proof. If $|y| < \sigma|S_k|$, the proof is identical to that of Proposition 2.3 of [60]; see also Proposition 2.1 of [38]. We include the proof here for reader's convenience. We start from the second term of E_λ:

$$I := (\bar{b}_j(y)\partial_j v_k^\lambda(y) + \bar{d}_{ij}(y)\partial_{ij} v_k^\lambda(y)).$$

Since y is conformal normal for $g_k(y)$ in $|y| \le \sigma|S_k|$, we have

$$0 = (\Delta_{g_k} - \Delta)V(y) = (\bar{b}_j(y)\partial_j + \bar{d}_{ij}(y)\partial_{ij})V(y) \tag{59}$$

for any smooth radial function $V(y) = V(|y|)$ and $|y| \le \sigma|S_k|$. It follows that, for $|y| \le \sigma|S_k|$,

$$I = (\bar{b}_j(y)\partial_j + \bar{d}_{ij}(y)\partial_{ij})[(v_k - U)^\lambda(y)].$$

By a direct computation,

$$\partial_j\Big\{(\frac{\lambda}{|y|})^{n-2}(v_k - U)(y^\lambda)\Big\} = \partial_j\Big\{(\frac{\lambda}{|y|})^{n-2}\Big\}(v_k - U)(y^\lambda) + (\frac{\lambda}{|y|})^{n-2}\partial_j\Big\{(v_k - U)(y^\lambda)\Big\},$$

$$\begin{aligned}
&\partial_{ij}\Big\{(\frac{\lambda}{|y|})^{n-2}(v_k - U)(y^\lambda)\Big\}\\
&= \partial_{ij}\Big\{(\frac{\lambda}{|y|})^{n-2}\Big\}(v_k - U)(y^\lambda) + \partial_i\Big\{(\frac{\lambda}{|y|})^{n-2}\Big\}\partial_j\Big\{(v_k - U)(y^\lambda)\Big\}\\
&\quad + \partial_j\Big\{(\frac{\lambda}{|y|})^{n-2}\Big\}\partial_i\Big\{(v_k - U)(y^\lambda)\Big\} + (\frac{\lambda}{|y|})^{n-2}\partial_{ij}\Big\{(v_k - U)(y^\lambda)\Big\}.
\end{aligned}$$

Since $\bar{d}_{ij} \equiv \bar{d}_{ji}$, using (59) with $V(y) = (\frac{\lambda}{|y|})^{n-2}$, we have

$$\begin{aligned}
I =&(\frac{\lambda}{|y|})^{n-2}\bar{b}_j(y)\partial_j\Big\{(v_k - U)(y^\lambda)\Big\} + 2\bar{d}_{ij}(y)\partial_i\Big\{(\frac{\lambda}{|y|})^{n-2}\Big\}\partial_j\Big\{(v_k - U)(y^\lambda)\Big\}\\
&+ (\frac{\lambda}{|y|})^{n-2}\bar{d}_{ij}(y)\partial_{ij}\Big\{(v_k - U)(y^\lambda)\Big\},
\end{aligned}$$

for $|y| \le \sigma|S_k|$. To evaluate terms in I, we observe that for $z \in B_2$,

$$\begin{aligned}
(v_k - U)(z) &= O(\sigma_k)|z|^2,\\
|\nabla_z(v_k - U)(z)| &= O(\sigma_k)|z|,\\
|\nabla_z^2(v_k - U)(z)| &= O(\sigma_k),
\end{aligned} \tag{60}$$

where we have used $(v_k - U)(0) = |\nabla(v_k - U)(0)| = 0$. Here we recall that $\sigma_k = \|v_k - U\|_{C^2(B_1)} \to 0$. It follows from the first components of (23), (24) and (25) that

$$\begin{aligned}
I &= O(|x_k|^{\tau-2})\sigma_k l_k^{-2}\Big(|y|^{2-n}|y||y^\lambda||\nabla_y y^\lambda| + |y|^2|y|^{1-n}|y^\lambda||\nabla_y y^\lambda|\\
&\quad + |y|^{2-n}|y|^2(|y^\lambda||\nabla_y^2 y^\lambda| + |\nabla_y y^\lambda|^2)\Big)\\
&= \sigma_k l_k^{-2}O(|x_k|^{\tau-2})|y|^{-n} = \sigma_k l_k^{-\tau}O(|S_k|^{\tau-2})|y|^{-n}, \quad |y| < \sigma|S_k|.
\end{aligned} \tag{61}$$

Similarly,

$$\begin{aligned}
&(\frac{\lambda}{|y|})^{n+2}\left(\bar{b}_j(y^\lambda)\partial_j v_k(y^\lambda) + \bar{d}_{ij}(y^\lambda)\partial_{ij}v_k(y^\lambda)\right)\\
=&\sigma_k l_k^{-2}|y|^{-n}O(|x_k|^{\tau-2})|y|^{-n} = \sigma_k l_k^{-\tau}O(|S_k|^{\tau-2})|y|^{-n}, \quad |y| < \sigma|S_k|
\end{aligned}$$

and

$$
\begin{aligned}
&|\bar{c}(y)||v_k^\lambda(y) - U^\lambda(y)| + (\frac{\lambda}{|y|})^{n+2}|\bar{c}(y^\lambda)||v_k(y^\lambda) - U(y^\lambda)| \\
=&\sigma_k l_k^{-4} O(|x_k|^{\tau-4})|y|^{-n} = \sigma_k l_k^{-\tau} O(|S_k|^{\tau-4})|y|^{-n} \quad \text{for} \quad |y| < \sigma|S_k|.
\end{aligned}
$$

Finally, using the estimates of $\bar{c}$ we have

$$
\bar{c}(y)U^\lambda(y) - (\frac{\lambda}{|y|})^{n+2}\bar{c}(y^\lambda)U(y^\lambda) = l_k^{-4}O(|x_k|^{\tau-4})|y|^{4-n} = l_k^{-\tau}O(|S_k|^{\tau-4})|y|^{4-n}.
$$

This finishes the proof for the case $|y| \leq \sigma|S_k|$.

If $|y| \geq \sigma|S_k|$, by the estimates in (23), (24) and (25), we have

$$
\begin{aligned}
E_\lambda(y) &\leq Cl_k^{-\tau}|y - S_k|^{\tau-2}|y|^{2-n} + Cl_k^{-2}|x_k|^{\tau-2}|y|^{-2-n} \\
&= Cl_k^{-\tau}(|y - S_k|^{\tau-2}|y|^{2-n} + |S_k|^{\tau-2}|y|^{-2-n}) \\
&\leq Cl_k^{-\tau}(|y - S_k|^{\tau-2}|S_k|^{2-n}\chi_k(y) + |y|^{\tau-n}).
\end{aligned}
$$

This finishes our proof. □

We next construct supersolutions of the linearized operator (26). Since $E_\lambda^{(1)}(|y|)$ is radial, we let $f_\lambda^{(1)}(r)$ be the radial solution of

$$
\begin{aligned}
\Delta f_\lambda^{(1)} = \frac{\mathrm{d}^2}{\mathrm{d}r^2}f_\lambda^{(1)} + \frac{n-1}{r}\frac{\mathrm{d}}{\mathrm{d}r}f_\lambda^{(1)} = -NE_\lambda^{(1)}(r), \quad r \in (\lambda, Q_1 l_k^{1/2}), \\
f_\lambda^{(1)}(\lambda) = \frac{\mathrm{d}}{\mathrm{d}r}f_\lambda^{(1)}(\lambda) = 0,
\end{aligned}
\tag{62}
$$

where $N > 2C_1 + 2$ is a constant.

Lemma 4.6. *Suppose that* $3 \leq n \leq 6$*. Let* Q_1 *be a constant defined in* (56) *and* $\lambda \in [1/2, 2]$*. Then for* $\lambda < |y| < Q_1 l_k^{1/2}$*, there hold*

$$
f_\lambda^{(1)}(|y|) < 0, \quad f_\lambda^{(1)}(|y|) = o(1)M_k^{-1} \tag{63}
$$

and

$$
\left[\Delta + \bar{b}_i(y)\partial_i + \bar{d}_{ij}(y)\partial_{ij} - \bar{c}(y)\right] f_\lambda^{(1)}(y) + (N-1)E_\lambda^{(1)}(y) \leq E_\lambda^{(2)}(y). \tag{64}
$$

Proof. By solving the Cauchy problem of (62), we obtain

$$
f_\lambda^{(1)}(r) = -N\int_\lambda^r \frac{1}{s^{n-1}}\left(\int_\lambda^s t^{n-1}E_\lambda^{(1)}(t)\,\mathrm{d}t\right)\mathrm{d}s.
$$

Since $E^{(1)}_\lambda$ is positive, $f^{(1)}_\lambda(r) < 0$ for $r > \lambda \geq \frac{1}{2}$. Note that for $p \in \mathbb{R}$ and $\frac{1}{2} \leq \lambda \leq 2$,

$$\int_\lambda^r \frac{1}{s^{n-1}} \left(\int_\lambda^s t^{n-1} t^{p-n} \, \mathrm{d}t \right) \mathrm{d}s = \begin{cases} \frac{1}{p} \int_\lambda^r \frac{s^p - \lambda^p}{s^{n-1}} \, \mathrm{d}s & \text{if } p \neq 0, \\ \int_\lambda^r s^{1-n} \ln \frac{s}{\lambda} \, \mathrm{d}s & \text{if } p = 0, \end{cases}$$

$$\leq C \begin{cases} 1, & \text{if } p < n-2, \\ \ln \frac{r}{\lambda}, & \text{if } p = n-2, \\ r^{p+2-n}, & \text{if } p > n-2, \end{cases}$$

where $C > 0$ depends only on n and p.

By the expression of $E^{(1)}_\lambda$ and using $p = 4$ or 0 as well as the relation $|x_k|^\tau = |S_k|^{\frac{n-2}{2}} l_k^{-\frac{n-2}{2}} = |S_k|^{\frac{n-2}{2}} M_k^{-1}$ as recorded in (iii) of Remark 4.4, we obtain, for $|y| \leq \sigma |S_k|$ and when $3 \leq n \leq 5$,

$$\begin{aligned} |f^{(1)}_\lambda(y)| &\leq CN \left(l_k^{-4} |x_k|^{\tau-4} |y|^{6-n} + \sigma_k l_k^{-2} |x_k|^{\tau-2} \right) \\ &\leq CN M_k^{-1} (|S_k|^{-\frac{n-2}{2}} + \sigma_k |S_k|^{-\frac{6-n}{2}}) \\ &= o(1) M_k^{-1} \quad \text{for } |y| \leq \sigma |S_k|, \end{aligned} \tag{65}$$

where $o(1)$ is with respect to $k \to \infty$; for the $n = 6$ case, we only need to modify the $|y|^{6-n}$ term in the first line into $\ln(\frac{|y|}{\lambda})$ and the $|S_k|^{-\frac{n-2}{2}}$ term in the second line into $|S_k|^{-2} \ln(\frac{|S_k|}{\lambda})$.

For $|y| \geq \sigma |S_k|$ and $3 \leq n \leq 6$—with the same modifications as above for the $n = 6$ case, we have

$$\begin{aligned} &|f^{(1)}_\lambda(y)| \\ &= |f^{(1)}_\lambda(\sigma |S_k|)| + N \int_{\sigma |S_k|}^r \frac{1}{s^{n-1}} \left(\int_\lambda^{\sigma |S_k|} t^{n-1} |E^{(1)}_\lambda(t)| \, \mathrm{d}t + \int_{\sigma |S_k|}^s t^{n-1} |E^{(1)}_\lambda(t)| dt \right) \mathrm{d}s \\ &\leq CN \left(l_k^{-4} |x_k|^{\tau-4} |S_k|^{6-n} + \sigma_k l_k^{-2} |x_k|^{\tau-2} \right) \\ &\quad + CN \Big(l_k^{-4} |x_k|^{\tau-4} |S_k|^{6-n} + \sigma_k l_k^{-2} |x_k|^{\tau-2} |S_k|^{2-n} \ln |S_k| \Big) \\ &\quad + CN l_k^{-\tau} |S_k|^{\tau+2-n} \\ &\leq CN M_k^{-1} \Big(|S_k|^{-\frac{n-2}{2}} + \sigma_k |S_k|^{-\frac{6-n}{2}} \Big) = o(1) M_k^{-1}. \end{aligned} \tag{66}$$

Hence, (63) is proved.

Next, we shall prove (64).

i). If $|y| < \sigma |S_k|$, we have $\Delta_{\bar{g}_k} f^{(1)}_\lambda(y) = \Delta f^{(1)}_\lambda(y) = -N E^{(1)}_\lambda$ and

$$\begin{aligned} |\bar{c}(y) f^{(1)}_\lambda(y)| &\leq CN l_k^{-4} |x_k|^{\tau-4} |y|^2 \left(l_k^{-4} |x_k|^{\tau-4} |y|^{6-n} + \sigma_k l_k^{-2} |x_k|^{\tau-2} \right) \\ &= CN l_k^{-\tau} |S_k|^{\tau-4} |y|^{4-n} \left(l_k^{-\tau} |S_k|^{\tau-4} |y|^4 + \sigma_k l_k^{-\tau} |S_k|^{\tau-2} |y|^{n-2} \right). \end{aligned}$$

Recall that

$$\begin{aligned}E_\lambda^{(1)}(|y|) &= l_k^{-4}|x_k|^{\tau-4}|y|^{4-n} + \sigma_k l_k^{-2}|x_k|^{\tau-2}|y|^{-n}\\ &= l_k^{-\tau}|S_k|^{\tau-4}|y|^{4-n}\left(1+\sigma_k|S_k|^2|y|^{-4}\right)\\ &\geq l_k^{-\tau}|S_k|^{\tau-4}|y|^{4-n}\end{aligned}$$

for $|y| < \sigma|S_k|$. We shall show that $|\bar{c}(y)f_\lambda^{(1)}(y)| = o(1)E_\lambda^{(1)}(y)$.

First we note that for $|y| < \sigma|S_k|$,

$$l_k^{-\tau}|S_k|^{\tau-4}|y|^4 \leq \sigma^4 l_k^{-\tau}|S_k|^\tau = \sigma^4|x_k|^\tau \to 0.$$

Furthermore,

$$\begin{aligned}\sigma_k l_k^{-\tau}|S_k|^{\tau-2}|y|^{n-2} &\leq \sigma_k\sigma^{n-2}l_k^{-\tau}|S_k|^{\tau-2+n-2}\\ &\leq \sigma_k\sigma^{n-2}Q_1^{\frac{3n-10}{2}}l_k^{-\tau+\frac{3n-10}{4}}\\ &= \sigma_k\sigma^{n-2}Q_1^{\frac{3n-10}{2}}l_k^{\frac{n-6}{4}} \to 0,\end{aligned}$$

when $n \leq 6$, where we have used $|y| \leq |S_k| \leq Q_1 l_k^{1/2}$. Therefore, $|\bar{c}(y)f_\lambda^{(1)}(y)| = o(1)E_\lambda^{(1)}(y)$ for $|y| \leq \sigma|S_k|$. In conclusion,

$$\left[\Delta + \bar{b}_i(y)\partial_i + \bar{d}_{ij}(y)\partial_{ij} - \bar{c}(y)\right]f_\lambda^{(1)}(y) = (-N+o(1))E_\lambda^{(1)}(y) \quad \text{for } |y| \leq \sigma|S_k|. \tag{67}$$

ii). If $\sigma|S_k| \leq |y| \leq Q_1 l_k^{1/2}$, then $l_k^{-\tau}|S_k|^{2-n} \geq \sigma^{n-2}Q_1^{2-n}l_k^{2-n}$ and

$$E_\lambda^{(2)}(y) = l_k^{-\tau}|S_k|^{2-n}|y-S_k|^{\tau-2}\chi_k \geq \sigma^{n-2}Q_1^{2-n}l_k^{2-n}|y-S_k|^{\tau-2} = \sigma^{n-2}Q_1^{2-n}M_k^{-2}|y-S_k|^{\tau-2}$$

when $|y-S_k| \leq |S_k|/4$; when $|y-S_k| \geq |S_k|/4$, $|y-S_k| \geq |y|-|S_k| \geq |y|-4|y-S_k|$, so $5|y-S_k| \geq |y|$ and due to $\tau \leq 2$, we have $|y-S_k|^{\tau-2}/|y|^{\tau-n} \leq 5^{2-\tau}|y|^{n-2} \leq Cl_k^{\frac{n-2}{2}}$ for some $C > 1$, and

$$\begin{aligned}E_\lambda^{(1)}(|y|) + E_\lambda^{(2)}(y) &= l_k^{-\tau}|y|^{\tau-n} + l_k^{-\tau}|S_k|^{2-n}|y-S_k|^{\tau-2}\chi_k\\ &\geq \frac{1}{C}l_k^{-\tau}|y-S_k|^{\tau-2}l_k^{\frac{2-n}{2}} = \frac{1}{C}M_k^{-2}|y-S_k|^{\tau-2},\end{aligned} \tag{68}$$

for some $C > 1$—this certainly holds as well when $|y-S_k| \leq |S_k|/4$.

In this region, we may no longer have $(\bar{b}_i(y)\partial_i + d_{ij}(y)\partial_{ij})f_\lambda^{(1)}(|y|) = 0$, but by the estimates (23), (24) and (25) of $\bar{b}_i, \bar{d}_{ij}$ and $\bar{c}$, we have

$$\begin{aligned}&|\bar{b}_i(y)\partial_i f_\lambda^{(1)}(|y|) + d_{ij}(y)\partial_{ij}f_\lambda^{(1)}(|y|) - \bar{c}(y)f_\lambda^{(1)}(|y|)|\\ &\leq Cl_k^{-\tau}\Big(|y-S_k|^{\tau-1}|\frac{\mathrm{d}}{\mathrm{d}r}f_\lambda^{(1)}(|y|)| + |y-S_k|^\tau|\frac{\mathrm{d}^2}{\mathrm{d}r^2}f_\lambda^{(1)}(|y|)| + |y-S_k|^{\tau-2}|f_\lambda^{(1)}(|y|)|\Big)\\ &= o(1)M_k^{-2}|y-S_k|^{\tau-2} = o(1)(E_\lambda^{(1)}(|y|) + E_\lambda^{(2)}(y)),\end{aligned}$$

where we have used (66), and

$$|\frac{\mathrm{d}}{\mathrm{d}r}f_\lambda^{(1)}(r)| = \frac{N}{r^{n-1}}\int_\lambda^r t^{n-1}E_\lambda^{(1)}(t)\,\mathrm{d}t \le CNM_k^{-1}r^{-\frac{n-2}{2}-1},$$

$$|\frac{\mathrm{d}^2}{\mathrm{d}r^2}f_\lambda^{(1)}(r)| \le CE_\lambda^{(1)}(r) + \frac{n-1}{r}|\frac{\mathrm{d}}{\mathrm{d}r}f_\lambda^{(1)}(r)| \le CNM_k^{-1}r^{-\frac{n-2}{2}-2},$$

when $r \ge \sigma|S_k|$.

It follows from (68) that, for $\sigma|S_k| \le |y| \le Q_1 l_k^{1/2}$,

$$\begin{aligned}
&\left(\Delta + \bar{b}_i(y)\partial_i + d_{ij}\partial_{ij} - \bar{c}(y)\right) f_\lambda^{(1)}(y) + (N-1)E_\lambda^{(1)}(|y|)\\
\le&\Delta f_\lambda^{(1)}(y) + |\bar{b}_i(y)\partial_i f_\lambda^{(1)}(y) + d_{ij}(y)\partial_{ij}f_\lambda^{(1)}(y) - \bar{c}(y)f_\lambda^{(1)}(y)| + (N-1)E_\lambda^{(1)}(|y|)\\
=&-E_\lambda^{(1)}(|y|) + o(1)(E_\lambda^{(1)}(|y|) + E_\lambda^{(2)}(y))\\
\le&E_\lambda^{(2)}(y).
\end{aligned}$$

This completes our proof of Lemma 4.6. □

Next we construct auxiliary functions to handle the $E_\lambda^{(2)}(y)$ term. To exploit the conformal normal property of $\bar{g}_k(y)$ inside $B_{\sigma|S_k|/2}$ so that we can use radial auxiliary functions as much as possible, we modify $\bar{g}_k(y)$ outside of $B_{\sigma|S_k|/2}$ into $\hat{g}_k(y)$ as a smooth Riemann metric defined on $\mathbb{R}^n$ satisfying

$$\hat{g}_k(y) = \bar{g}_k(y) \quad \text{in } B_{\sigma|S_k|/2}, \quad (\hat{g}_k)_{ij}(y) = \delta_{ij} \quad \text{in } \mathbb{R}^n \setminus B_{\sigma|S_k|},$$

$$\det \hat{g}_k(y) = 1 \quad \text{in } \mathbb{R}^n,$$

and

$$|\nabla^m((\hat{g}_k)_{ij}(y) - \delta_{ij})| \le Cl_k^{-2}|x_k|^{\tau-2}|y|^{2-m} \quad \text{for } y \in B_{\sigma|S_k|} \text{ and } m = 0,1,2,$$

where this last estimate is due to (46) of Lemma 3.4 applied to $g_k(x)$ for $|x| < \sigma$ with $\bar{z} = x_k$, and $\bar{g}_k(y) = g_k(l_k^{-1}y)$. In particular, $|(\hat{g}_k)_{ij}(y) - \delta_{ij}| \le C|x_k|^\tau \to 0$ uniformly over $\mathbb{R}^n$ as $k \to \infty$.

We will also require that y be a geodesic coordinate system for $\hat{g}_k$ on $\mathbb{R}^n$. This can be done by first expressing $\bar{g}_k(y) = \exp((\bar{h}_k)_{ij}(y))$ for $y \in B_{\sigma|S_k|}$, where $(\bar{h}_k)_{ij}(y) = (h_k)_{ij}(l_k^{-1}y)$ satisfies (a) $\sum_{j=1}^n(\bar{h}_k)_{ij}(y)y_j = 0$, and (b) trace$((\bar{h}_k)_{ij}(y)) = 0$ in $B_{\sigma|S_k|}$. Then it is trivial to modify $(\bar{h}_k)_{ij}(y)$ outside of $B_{\sigma|S_k|/2}$ and extend $(\bar{h}_k)_{ij}(y)$ to $y \in R^n$ such that $(\bar{h}_k)_{ij}(y) = 0$ for $y \in \mathbb{R}^n \setminus B_{\sigma|S_k|}$ while maintaining the two conditions (a) and (b), which guarantee that y is a geodesic coordinate system for $\hat{g}_k$ and its determinant $\equiv 1$ on $\mathbb{R}^n$.

Let $\eta_k^{(1)}$ be the smooth solution of

$$-\Delta_{\hat{g}}\eta_{k,\lambda}^{(1)}(y) = |S_k|^{2-n}|y - S_k|^{-\frac{6-n}{2}}\chi_k(y) \quad \text{in } \mathbb{R}^n \setminus B_\lambda,$$

with the boundary condition

$$\eta_{k,\lambda}^{(1)} = 0 \quad \text{on } \partial B_\lambda, \quad \text{and} \quad \lim_{|y|\to\infty} \eta_{k,\lambda}^{(1)}(y) = 0.$$

By the upper bound estimate on the Green's function $\hat{G}_k(y,z)$ of $\Delta_{\hat{g}}$ due to Littman, Stampacchia and Weinberger [32] (see also Grüter and Widman [23]),

$$\hat{G}_k(y,z) \le C|y-z|^{2-n} \quad \text{for some } C>0 \text{ independent of } k, \text{ and all } y,z \text{ with } \lambda < |y|, |z|,$$

we have

$$\begin{aligned} 0 \le \eta_{k,\lambda}^{(1)}(y) \le & C|S_k|^{2-n} \int_{B_{|S_k|/2}(S_k)} |y-z|^{2-n} |z-S_k|^{-\frac{6-n}{2}} \, \mathrm{d}z \\ \le & C_2 \begin{cases} |S_k|^{-\frac{n-2}{2}} & \text{for } |y| < 2|S_k|, \\ |y|^{2-n} |S_k|^{\frac{n-2}{2}} & \text{for } |y| \ge 2|S_k|. \end{cases} \end{aligned}$$

Furthermore, $0 \le \partial_r \eta_{k,\lambda}^{(1)}(y) \le C_2 |S_k|^{-\frac{n-2}{2}}$ on ∂B_λ, where $C_2 > 0$ is independent of λ and k (if k is large), and

$$\eta_{k,\lambda}^{(1)}(y) \le 2C_2 |S_k|^{-\frac{n-2}{2}} (|y| - \lambda) \quad \text{for } \lambda \le |y| \le \lambda + 2.$$

Let

$$\eta_{k,\lambda}^{(2)}(|y|) = Q \cdot C_2 |S_k|^{-\frac{n-2}{2}} (\lambda^{n-2} |y|^{2-n} - 1),$$

where $Q > 0$ is depending only on n such that $\eta_{k,\lambda} := \eta_{k,\lambda}^{(1)} + \eta_{k,\lambda}^{(2)} \le 0$ in $\mathbb{R}^n \setminus B_\lambda$. Since y is a conformal normal coordinate for $\hat{g}_k$ on $\mathbb{R}^n$, $\Delta_{\hat{g}_k} \eta_{k,\lambda}^{(2)} = \Delta \eta_{k,\lambda}^{(2)} = 0$. Hence,

$$-\Delta_{\hat{g}_k} \eta_{k,\lambda}(y) = |S_k|^{2-n} |y - S_k|^{-\frac{6-n}{2}} \chi_k(y) \ge 0 \quad \text{in } \mathbb{R}^n \setminus B_\lambda,$$

$$\eta_{k,\lambda} = 0, \quad \partial_r \eta_{k,\lambda} < 0 \quad \text{on } \partial B_\lambda.$$

Moreover,

$$|\nabla^m \eta_{k,\lambda}(y)| \le C \begin{cases} |S_k|^{-\frac{n-2}{2}} |y - S_k|^{-m}, & \frac{1}{2}\sigma |S_k| \le |y| \le 2|S_k|, \\ |S_k|^{-\frac{n-2}{2}} |y|^{2-n-m}, & |y| \ge 2|S_k|, \end{cases} \tag{69}$$

for $m = 1, 2$. Let

$$f_\lambda^{(2)}(y) = (2C_1 + 1) l_k^{-\tau} \eta_{k,\lambda}(y).$$

We have dropped the subscript k of $f_\lambda^{(2)}(y)$ here.

Lemma 4.7. *Let $f_\lambda^{(2)}(y)$ be defined above. Then we have*

$$f_\lambda^{(2)}(y) \le 0 \quad \text{in } \mathbb{R}^n \setminus B_\lambda, \quad f_\lambda^{(2)}(y) = 0 \quad \text{on } \partial B_\lambda, \tag{70}$$

and for $\lambda < |y| < Q_1 l_k^{1/2}$,

$$\left[\Delta + \bar{b}_i(y)\partial_i + \bar{d}_{ij}(y)\partial_{ij} - \bar{c}(y)\right] f_\lambda^{(2)}(y) + 2C_1 E_\lambda^{(2)}(y) \le o(1) E_\lambda^{(1)}(y), \tag{71}$$

$$|f_\lambda^{(2)}(y)| = o(1) M_k^{-1} \quad \text{uniformly over } \lambda < |y| < Q_1 l_k^{1/2}. \tag{72}$$

Proof. By the definition of $f_\lambda^{(2)}$ and its construction, (70) and (72) hold. It remains to show (71).

If $|y| \le \frac{1}{2}\sigma|S_k|$, we have $\hat{g}_k = \bar{g}_k$, and thus

$$\begin{aligned} &| \left[\Delta + \bar{b}_i(y)\partial_i + \bar{d}_{ij}(y)\partial_{ij} - \bar{c}(y)\right] f_\lambda^{(2)}(y)| \\ &= |\bar{c}(y) f_\lambda^{(2)}(y)| \\ &\le C l_k^{-4}|x_k|^{\tau-4}|y|^2 \cdot l_k^{-\tau}|S_k|^{\frac{2-n}{2}}. \end{aligned}$$

It follows that

$$\frac{| \left[\Delta + \bar{b}_i(y)\partial_i + \bar{d}_{ij}(y)\partial_{ij} - \bar{c}(y)\right] f_\lambda^{(2)}(y)|}{E_\lambda^{(1)}(y)} \le C|y|^{n-2} l_k^{-\tau}|S_k|^{\frac{2-n}{2}} \to 0$$

uniformly over $|y| \le \frac{1}{2}\sigma|S_k|$ as $k \to \infty$.

Let us note that

$$E_\lambda^{(1)}(|y|) \ge l_k^{-4}|x_k|^{\tau-4}|y|^{4-n} \ge \frac{1}{C}|x_k|^{\tau}|S_k|^{-(n-2)-2} \ge |x_k|^{-\tau} M_k^{-2}|S_k|^{-2}$$

for $\frac{1}{2}\sigma|S_k| \le |y| \le \sigma|S_k|$, and recall (68)

$$E_\lambda^{(1)}(|y|) + E_\lambda^{(2)}(y) \ge \frac{1}{C} l_k^{-\tau}|y - S_k|^{\tau-2}|l_k^{1/2}|^{2-n}$$

for $\sigma|S_k| \le |y| \le Q_1 l_k^{1/2}$. Making use of (69), the properties of $\hat{g}_k$, and the coefficients estimates (23), (24) and (25), we obtain

$$| (\Delta - \Delta_{\hat{g}_k}) f_\lambda^{(2)}(y)| \le C M_k^{-2}|S_k|^{-2} = o(1) E_\lambda^{(1)}(|y|) \quad \text{if } \frac{1}{2}\sigma|S_k| \le |y| \le \sigma|S_k|,$$

$$|(\Delta - \Delta_{\hat{g}_k}) f_\lambda^{(2)}(y)| = 0 \quad \text{if } |y| > \sigma|S_k|,$$

and

$$\begin{aligned} &|(\bar{b}_i(y)\partial_i + \bar{d}_{ij}(y)\partial_{ij} - \bar{c}(y)) f_\lambda^{(2)}(y)| \\ &\le C \begin{cases} M_k^{-2}|S_k|^{-2}, & \frac{1}{2}\sigma|S_k| \le |y| \le \sigma|S_k| \\ l_k^{-\tau}|y - S_k|^{\tau-2} \cdot l_k^{-\tau}|S_k|^{\tau+2-n}, & \sigma|S_k| \le |y| \le Q_1 l_k^{1/2} \end{cases} \\ &= o(1)\left(E_\lambda^{(1)}(|y|) + E_\lambda^{(2)}(y)\right). \end{aligned}$$

Hence,

$$\left[\Delta + \bar{b}_i(y)\partial_i + \bar{d}_{ij}(y)\partial_{ij} - \bar{c}(y)\right] f_\lambda^{(2)}(y) = -2(C_1+1)E_\lambda^{(2)}(y)+o(1)\left(E_\lambda^{(1)}(|y|) + E_\lambda^{(2)}(y)\right),$$

which implies (71). This completes our proof of Lemma 4.7.

□

Proposition 4.8. *Let $f_\lambda = f_\lambda^{(1)} + f_\lambda^{(2)}$. By taking $N > 2C_1 + 2$ in* (62)*, we have*

$$\left[\Delta + \bar{b}_i(y)\partial_i + \bar{d}_{ij}(y)\partial_{ij} - \bar{c}(y)\right] f_\lambda(y) + \xi_\lambda(y) f_\lambda(y) \le -|E_\lambda(y)| \quad \text{in } \Sigma_\lambda^k,$$

$$f_\lambda(y) = 0 \quad \text{on } \partial B_\lambda, \quad f_\lambda(y) < 0 \quad \text{in } \Sigma_\lambda^k, \tag{73}$$

$$|f_\lambda(y)| + |\nabla f_\lambda(y)| = o(1)M_k^{-1} \quad \text{uniformly in } \Sigma_\lambda^k \text{ as } k \to \infty. \tag{74}$$

We remark that when $y \in \Sigma_\lambda^k$, $|y| \le Q_1 l_k^{1/2}$, so $|y|^{2-n} \ge Q_1^{2-n} l_k^{\frac{2-n}{2}}$, and the estimate $|f_\lambda(y)| = o(1)M_k^{-1}$ certainly implies (29), namely, $|f_\lambda(y)| + |\nabla f_\lambda(y)| = o(1)|y|^{2-n}$ on Σ_λ^k.

Proof. By Lemma 4.6 and Lemma 4.7, we have

$$\begin{aligned}
&\left[\Delta + \bar{b}_i(y)\partial_i + \bar{d}_{ij}(y)\partial_{ij} - \bar{c}(y)\right] f_\lambda(y) + \xi_\lambda(y) f_\lambda(y) \\
&\le -(N-1)E_\lambda^{(1)}(y) + E_\lambda^{(2)}(y) - (2C_1+1)E_\lambda^{(2)}(y) + o(1)E_\lambda^{(1)}(y) \\
&= -C_1(E_\lambda^{(1)}(y) + E_\lambda^{(2)}(y)) \le -|E_\lambda(y)|,
\end{aligned}$$

where we have used $\xi_\lambda(y) \ge 0$.

(73) and (74) follow from our construction of $f_\lambda^{(1)}$ and $f_\lambda^{(2)}$ in Lemmas 4.6 and 4.7 respectively. This completes our proof. □

4.2 Case of $n \ge 7$.

The way we handled the term $\bar{c}(y)v_k^\lambda(y) - (\frac{\lambda}{|y|})^{n+2}\bar{c}(y^\lambda)v_k(y^\lambda)$ for $|y| \le \frac{\sigma}{2}|S_k|$ in constructing $f_\lambda^{(1)}(r)$ when $3 \le n \le 6$ is no longer adequate for the $n \ge 7$ cases. We will need to modify the construction of $f_\lambda^{(1)}(r)$ here, which uses more delicate estimates. The key is to identify leading order terms as given in (83) and construct the auxiliary function $f_\lambda^{(3)}$ with respect to these leading order terms as in (84). In this process we need to obtain refined estimates for $u_k(z)$ near $z = 0$, or equivalently for $v_k(y) - U(y)$ in terms of $|x_k|$ and $|S_k|^{-1}$, as given by Lemma 4.9.

Since $\tau = \frac{n-2}{2} > 2$ if $n \ge 7$, we can expand the conformal Laplacian as

$$L_{\bar{g}_k} = \Delta + \bar{b}_i(y)\partial_i + \bar{d}_{ij}(y)\partial_{ij} - \bar{c}(y)$$

with

$$\bar{b}_i(y) = l_k^{-1} b_i(l_k^{-1} y)$$
$$= \begin{cases} O(l_k^{-2}|x_k|^{\tau-2}|y|), & |y| < \sigma|S_k|, \\ O(M_k^{-1}|y|^{\tau-1}), & |y| > \sigma|S_k|, \end{cases} \tag{75}$$

$$\bar{d}_{ij}(y) = d_{ij}(l_k^{-1} y)$$
$$= \begin{cases} O(l_k^{-2}|x_k|^{\tau-2}|y|^2), & |y| < \sigma|S_k|, \\ O(M_k^{-1}|y|^{\tau}), & |y| > \sigma|S_k|, \end{cases} \tag{76}$$

and

$$\bar{c}(y) = c(n) R_{\bar{g}_k}(y) = c(n) l_k^{-2} R_{g_k}(l_k^{-1} y)$$
$$= \begin{cases} O(l_k^{-4}|x_k|^{\tau-4}|y|^2), & |y| < \sigma|S_k|, \\ O(M_k^{-1}|y|^{\tau-2}), & |y| > \sigma|S_k|. \end{cases} \tag{77}$$

Recall that

$$\begin{aligned} E_\lambda(y) &= \left(\bar{c}(y) v_k^\lambda(y) - (\frac{\lambda}{|y|})^{n+2} \bar{c}(y^\lambda) v_k(y^\lambda)\right) - \left(\bar{b}_i(y)\partial_i v_k^\lambda(y) + \bar{d}_{ij}(y)\partial_{ij} v_k^\lambda(y)\right) \\ &\quad + (\frac{\lambda}{|y|})^{n+2} \left(\bar{b}_i(y^\lambda)\partial_i v_k(y^\lambda) + \bar{d}_{ij}(y^\lambda)\partial_{ij} v_k(y^\lambda)\right). \end{aligned}$$

We rescale g_k and u_k so that the local maximum of u_k at $z = 0$ and its singular point at $z = -x_k$ are of unit distance. Let $(\varpi_k)_{ij}(z) = (g_k)_{ij}(|x_k|z)$ for $|z| \le 1$. By (14), we have

$$\det \varpi_k = 1 \quad \text{in } B_\sigma.$$

Moreover, $\|\varpi_k\|_{C^{n+2}(B_\sigma)} \le C$. We shall apply the results in Section 3 to $(\varpi_k)_{ij}(z)$. Write $(\varpi_k)_{ij}(z) = e^{h_{ij}(z)}$, where we dropped the subscript k of h_{ij}. Define H_{ij}, $\tilde{H}_{ij}$, $\tilde{Z}_\varepsilon$ and Z_ε etc. as there. Since (46) holds for $g_k(x)$ with $\bar{z} = x_k$, it follows that, for $z \in B_\sigma$,

$$|\nabla^m h_{ij}(z)| \le C|x_k|^\tau, \quad m = 0, \ldots, n+2, \tag{78}$$

for some C independent of k. Furthermore, it follows from (36) and (78) that

$$\begin{aligned} |R_{\varpi_k}(z) - \partial_i\partial_j H_{ij}(z)| &\le C \sum_{|\alpha|=2}^{d} \sum_{i,j} |h_{ij\alpha}|^2 |z|^{2|\alpha|-2} + C\|h_{ij}\|_{C^{n-2}(B_\sigma)}|z|^{n-5} \\ &\le C(|x_k|^{2\tau} + |x_k|^\tau |z|^{n-5}), \end{aligned} \tag{79}$$

where $H_{ij} = \sum_{l=4}^{n-4} \partial_i \partial_j H_{ij}^{(l)}$ if $n \geq 8$ while $H_{ij} = 0$ otherwise. Define

$$\psi_k(z) = |x_k|^{\frac{n-2}{2}} u_k(|x_k|z), \quad |z| \leq \frac{1}{2|x_k|}.$$

Then

$$-L_{\varpi_k}\psi_k = n(n-2)\psi_k^{\frac{n+2}{n-2}} \quad \text{in } B_{\frac{1}{2|x_k|}} \setminus \{-\frac{x_k}{|x_k|}\},$$

0 is a local maximum point of ψ_k and $\psi_k(0) = |S_k|^{\frac{n-2}{2}} \to \infty$ as $k \to \infty$.

If $n \geq 25$, we shall verify (44) in the current setting:

$$\sum_{l=1}^{d} |\nabla^l \varpi_k(\bar{z})|^2 \varepsilon_{\bar{z},k}^{2l} |\ln \varepsilon_{\bar{z},k}|^{\theta_l} = o(1)\varepsilon_{\bar{z},k}^{n-2} \tag{80}$$

for any $\bar{z} \in B_\sigma$ with $\psi_k(\bar{z}) \geq 1$, where $\varepsilon_{\bar{z},k} = \psi_k(\bar{z})^{-\frac{2}{n-2}}$. By (78) and (8), we have, for any point $\bar{z} \in B_\sigma$,

$$|\nabla^l \varpi_k(\bar{z})|^2 \leq C|x_k|^{2\tau} \leq Cu_k(|x_k|\bar{z})^{-1}, \quad l = 1, 2 \ldots, n+2,$$

here, we have used $|x_k| \leq C|\phi_{x_k}(|x_k|\bar{z})|$ for $\bar{z} \in B_\sigma$. Furthermore, using

$$\varepsilon_{\bar{z},k} = |x_k|^{-1} u_k(|x_k|\bar{z})^{-\frac{2}{n-2}},$$

we get

$$u_k(|x_k|\bar{z})^{-1} = \varepsilon_{\bar{z},k}^{\frac{n-2}{2}} |x_k|^{\frac{n-2}{2}} \leq C\varepsilon_{\bar{z},k}^{\frac{n-2}{2}} u_k(|x_k|\bar{z})^{-1/2},$$

from which we get

$$|x_k|^{n-2} \leq Cu_k(|x_k|\bar{z})^{-1} \leq C\varepsilon_{\bar{z},k}^{n-2}.$$

Then, for $1 \leq l \leq d$,

$$|\nabla^l \varpi_k(\bar{z})|^2 \varepsilon_{\bar{z},k}^{2l} \leq C|x_k|^{n-2} \varepsilon_{\bar{z},k}^{2l} \leq C\varepsilon_{\bar{z},k}^{2l+n-2},$$

and for $l = d$,

$$|\nabla^d \varpi_k(\bar{z})|^2 \varepsilon_{\bar{z},k}^{2d} |\ln \varepsilon_{\bar{z},k}| \leq C\varepsilon_{\bar{z},k}^{2(n-2)} |\ln \varepsilon_{\bar{z},k}| = o(1)\varepsilon_{\bar{z},k}^{n-2}.$$

It is now clear that (80) holds.

We are now ready to deal with the case of $n \geq 7$. It follows from Proposition 3.3 that 0 is an isolated simple blow up point of ψ_k with some $\rho > 0$ independent of k. We may take $\rho = \sigma/2$ without loss of generality. Note that

$$v_k(y) = |S_k|^{-\frac{n-2}{2}} \psi_k(|S_k|^{-1}y)$$

and $\bar{g}_k(y) = \varpi_k(|S_k|^{-1}y)$.

Lemma 4.9. *Let $\varepsilon_k = |S_k|^{-1}$. We have*

$$|\nabla^m (v_k - U - \tilde{Z}_{\varepsilon_k})(y)| \le C\varepsilon_k^{n-3}(1+|y|)^{-1-m}, \quad |y| \le \frac{\sigma}{2}|S_k|,$$

where $C > 0$, $m = 0, 1, 2$, and $\tilde{Z}_{\varepsilon_k}$ solves (37) *with the bound*

$$|\nabla^m \tilde{Z}_{\varepsilon_k}(y)| \le C \min\{|x_k|^\tau \varepsilon_k^4, \varepsilon_k^{\frac{n-2}{2}}\}(1+|y|)^{6-n-m}.$$

Proof. It follows by applying Proposition 3.3 and Proposition 3.2 to the solution $\psi_k(x)$ with respect to the metric ϖ_k—note that $v_k(y)$ is the normalization of $\psi_k(x)$ by $|S_k|$. □

Let

$$V_\lambda(r) = \begin{cases} n(n-2)\frac{U(r)^{\frac{n+2}{n-2}} - U^\lambda(r)^{\frac{n+2}{n-2}}}{U(r) - U^\lambda(r)} & \text{if } \lambda \neq 1, \\ n(n+2)U(r)^{\frac{4}{n-2}} & \text{if } \lambda = 1. \end{cases}$$

Define

$$\mathcal{O}_\lambda = \{y \in B_{Q_1 l_k^{1/2}} \setminus B_\lambda : v_k(y) \le 2v_k^\lambda(y)\},$$

where $Q_1 > 0$ is the constant defined in (56). It is easy to see that $\mathcal{O}_\lambda \subset\subset B_{Q_1 l_k^{1/2}}$. $V_\lambda(y)$ provides a good approximation for $\xi_\lambda(y)$, as given below.

Lemma 4.10. *Let ξ_λ be the function defined in* (27)*. Then we have*

$$|\xi_\lambda(y) - V_\lambda(y)| \le C|S_k|^{-\frac{n-2}{2}}|y|^{n-6} \quad \text{for } \lambda \le |y| \le \frac{\sigma}{2}|S_k|$$

and

$$|\xi_\lambda(y) - V_\lambda(y)| \le C|y|^{-4} \quad \text{for } y \in \mathcal{O}_\lambda.$$

Proof. By Lemma 4.9, we have

$$a_k(y) := v_k(y) - U(y) = O(\varepsilon_k^{\frac{n-2}{2}}) \quad \text{and} \quad b_k(y) := v_k^\lambda(y) - U^\lambda(y) = O(\varepsilon_k^{\frac{n-2}{2}})$$

if $|y| \le \frac{\sigma}{2}|S_k|$. By direct calculus, we have

$$\begin{aligned}
&\frac{(U(|y|) + a_k(y))^{\frac{n+2}{n-2}} - (U^\lambda(|y|) + b_k(y))^{\frac{n+2}{n-2}}}{(U(|y|) + a_k(y)) - (U^\lambda(|y|) + b_k(y))} \\
&= \frac{n+2}{n-2}\int_0^1 \Big(tU(|y|) + (1-t)U^\lambda(|y|)\Big)^{\frac{4}{n-2}} \mathrm{d}t + O(1)(|a_k(y)| + |b_k(y)|)|y|^{n-6} \\
&= \frac{1}{n(n-2)}V_\lambda + O(1)\varepsilon_k^{\frac{n-2}{2}}|y|^{n-6}.
\end{aligned}$$

This proves the first inequality of Lemma 4.10. The second inequality is obvious.

□

We now work out the leading order terms as described at the beginning of this subsection. First, by (79) and Proposition 4.3, we have

$$\begin{aligned}\left|R_{\bar{g}_k}(y) - |S_k|^{-2}\partial_i\partial_j H_{ij}(|S_k|^{-1}y)\right| &\le C(|x_k|^{2\tau}|S_k|^{-2} + |x_k|^{\tau}|S_k|^{3-n}|y|^{n-5}) \\ &\le CM_k^{-1}(|S_k|^{-2} + |S_k|^{\frac{4-n}{2}}|y|^{n-5}).\end{aligned} \tag{81}$$

It follows that for $\lambda \le |y| \le \frac{\sigma}{2}|S_k|$, $\lambda \in [1/2, 2]$,

$$\begin{aligned}&R_{\bar{g}_k}(y)U^{\lambda}(y) - (\frac{\lambda}{|y|})^{n+2}R_{\bar{g}_k}(y^{\lambda})U(y^{\lambda}) \\ &= |S_k|^{-2}(\partial_i\partial_j H_{ij}(|S_k|^{-1}y)U^{\lambda}(y) - (\frac{\lambda}{|y|})^{n+2}\partial_i\partial_j H_{ij}(|S_k|^{-1}y^{\lambda})U(y^{\lambda})) \\ &\quad + O(1)M_k^{-1}(|S_k|^{-2}|y|^{2-n} + |S_k|^{\frac{4-n}{2}}|y|^{-3}). \\ &= \lambda^{n-2}\sum_{l=4}^{n-4}|S_k|^{-l}|y|^{l-n}(1-(\frac{\lambda}{|y|})^{2l})\partial_i\partial_j H_{ij}^{(l)}(\frac{y}{|y|})U(y^{\lambda}) + O(1)M_k^{-1}|S_k|^{-\frac{3}{2}}|y|^{-3}.\end{aligned} \tag{82}$$

By (77), for $|y| \ge \frac{\sigma}{2}|S_k|$

$$\begin{aligned}&R_{\bar{g}_k}(y)U^{\lambda}(y) - (\frac{\lambda}{|y|})^{n+2}R_{\bar{g}_k}(y^{\lambda})U(y^{\lambda}) \\ &\le C(M_k^{-1}|y|^{\tau-n} + l_k^{-4}|x_k|^{\tau-4}|y|^{-4-n}) \le CM_k^{-1}|y|^{-\frac{n+2}{2}}.\end{aligned}$$

Let

$$T_k(y) = \begin{cases}\lambda^{n-2}\sum_{l=4}^{n-4}|S_k|^{-l}|y|^{l-n}(1-(\frac{\lambda}{|y|})^{2l})\partial_i\partial_j H_{ij}^{(l)}(\frac{y}{|y|})U(y^{\lambda}), & \lambda \le |y| < \frac{\sigma}{2}|S_k|, \\ \lambda^{n-2}\sum_{l=4}^{n-4}(\frac{\sigma}{2})^{l}|y|^{-n}(1-(\frac{\lambda}{|y|})^{2l})\partial_i\partial_j H_{ij}^{(l)}(\frac{y}{|y|})U(y^{\lambda}), & |y| \ge \frac{\sigma}{2}|S_k|.\end{cases}$$

Hence,

$$\begin{aligned}&R_{\bar{g}_k}(y)U^{\lambda}(y) - (\frac{\lambda}{|y|})^{n+2}R_{\bar{g}_k}(y^{\lambda})U(y^{\lambda}) \\ &= T_k(y) + O(1)\begin{cases}M_k^{-1}|S_k|^{-\frac{3}{2}}|y|^{-3}, & \lambda \le |y| < \frac{\sigma}{2}|S_k|, \\ M_k^{-1}|y|^{-\frac{n+2}{2}}, & |y| \ge \frac{\sigma}{2}|S_k|.\end{cases}\end{aligned} \tag{83}$$

Since

$$\int_{|y|=1}\partial_i\partial_j H_{ij}^{(l)}(y) = \int_{|y|=1}y_a\partial_i\partial_j H_{ij}^{(l)}(y) = 0, \quad a = 1, \dots, n,$$

we have

$$\partial_i\partial_j H_{ij}^{(l)}(\frac{y}{|y|}) = \sum_{s=2}^{l-2}Y_{l,s}(\frac{y}{|y|}),$$

where $Y_{l,s}$ are some spherical harmonics of degree s on $\mathbb{S}^{n-1}$, orthogonal to each other in $L^2(\mathbb{S}^{n-1})$. Let

$$T_k^{(l)}(r) = \begin{cases} \lambda^{n-2}|S_k|^{-l} r^{l-n}(1-(\frac{\lambda}{r})^{2l})U(\frac{\lambda^2}{r}), & \lambda \le r < \frac{\sigma}{2}|S_k|, \\ \lambda^{n-2}(\frac{\sigma}{2})^l |r|^{-n}(1-(\frac{\lambda}{r})^{2l})U(\frac{\lambda^2}{r}), & r \ge \frac{\sigma}{2}|S_k|. \end{cases}$$

By Proposition 6.1 of [38], there exists a small constant $\delta_1 > 0$ depending only on n, such that, for every $\lambda \in [1-\delta_1, 1+\delta_1]$, the boundary value problem

$$f_{l,s}''(r) + \frac{n-1}{r} f_{l,s}'(r) + (V_\lambda(r) - \frac{s(s+n-2)}{r^2}) f_{l,s} = -c(n) T_k^{(l)}(r), \quad \lambda < r < Q_1 l_k^{1/2}$$

with

$$f_{l,s}(\lambda) = f_{l,s}(Q_1 l_k^{1/2}) = 0,$$

has a unique solution. Moreover,

$$|f_{l,s}(r)| \le \begin{cases} C|S_k|^{-4}(1+r)^{6-n}, & \lambda \le r \le |S_k|, \\ Cr^{2-n}, & |S_k| \le r \le Q_1 l_k^{1/2}. \end{cases}$$

We remark that the restriction on $\lambda \in [1-\delta_1, 1+\delta_1]$ is to ensure that $V_\lambda(|y|)$ is sufficiently close to $V_1(r) = n(n+2)U(r)^{\frac{4}{n-2}}$. Let

$$f_\lambda^{(3)}(y) = \sum_{l=4}^{n-4} \sum_{s=2}^{l-2} f_{l,s}(|y|) Y_{l,s}(\frac{y}{|y|}).$$

Then

$$\begin{cases} \Delta f_\lambda^{(3)}(y) + V_\lambda(|y|) f_\lambda^{(3)}(y) = -c(n) T_k(y) & \text{in } B_{Q_1 l_k^{1/2}} \setminus B_\lambda, \\ f_\lambda^{(3)}(y) = 0 & \text{on } \partial(B_{Q_1 l_k^{1/2}} \setminus B_\lambda). \end{cases} \tag{84}$$

Using (78), we have

$$\max_{|e|=1} |\partial_i \partial_j H_{ij}^{(l)}(\frac{y}{|y|})| \le C|x_k|^\tau \quad \text{and} \quad |Y_{l,s}(\frac{y}{|y|})| \le C|x_k|^\tau,$$

which leads to the following crucial estimate for $f_\lambda^{(3)}(y)$:

$$|\nabla^m f_\lambda^{(3)}(y)| \le \begin{cases} C|S_k|^{-4}|x_k|^\tau (1+|y|)^{6-n-m}, & |y| < \frac{\sigma}{2}|S_k|, \\ C|x_k|^\tau (1+|y|)^{2-n-m}, & |y| \ge \frac{\sigma}{2}|S_k|, \end{cases} \tag{85}$$

for $m = 0, 1, \cdots, n+2$. Next we have

$$|(\bar{b}_i(y)\partial_i + \bar{d}_{ij}(y)\partial_{ij} - \bar{c}(y)) f_\lambda^{(3)}(y)| \le C \begin{cases} |S_k|^{-6}|x_k|^{2\tau}|y|^{6-n}, & |y| < \frac{\sigma}{2}|S_k|, \\ M_k^{-1}|x_k|^\tau |y|^{\tau-n}, & |y| > \frac{\sigma}{2}|S_k|. \end{cases}$$

By Lemma 4.10,

$$|(V_\lambda(y)-\xi_\lambda(y))f_\lambda^{(3)}(y)| \le C \begin{cases} |S_k|^{-\frac{n+6}{2}}|x_k|^\tau, & |y|<\frac{\sigma}{2}|S_k|, \\ |x_k|^\tau |y|^{-2-n}, & |y|>\frac{\sigma}{2}|S_k|,\ y\in\mathcal{O}_\lambda. \end{cases}$$

Since $|x_k| \le C l_k^{-1/2}$ and $|S_k| = l_k|x_k|$, it follows that

$$\begin{aligned} &\left[\Delta+\bar b_i(y)\partial_i+\bar d_{ij}(y)\partial_{ij}-\bar c(y)\right] f_\lambda^{(3)}(y)+\xi_\lambda(y)f_\lambda^{(3)}(y) \\ &=(\Delta+V_\lambda(y))f_\lambda^{(3)}(y)+|(\bar b_i(y)\partial_i+\bar d_{ij}(y)\partial_{ij}-\bar c(y))f_\lambda^{(3)}(y)|+|V_\lambda(y)-\xi_\lambda(y)||f_\lambda^{(3)}(y)| \\ &=(\Delta+V_\lambda(y))f_\lambda^{(3)}(y)+O(1)\begin{cases} M_k^{-1}|S_k|^{-4}, & |y|<\frac{\sigma}{2}|S_k|, \\ M_k^{-1}|y|^{-\frac{n+6}{2}}, & |y|>\frac{\sigma}{2}|S_k|,\ y\in\mathcal{O}_\lambda. \end{cases} \end{aligned} \tag{86}$$

Let

$$\hat E_\lambda(y)=\left[\Delta+\bar b_i(y)\partial_i+\bar d_{ij}(y)\partial_{ij}-\bar c(y)+\xi_\lambda(y)\right]\left(w_\lambda+f_\lambda^{(3)}\right)(y), \quad |y|\ge\lambda,$$

where $w_\lambda(y)=v_k(y)-v_k^\lambda(y)$. Recall that $\left[\Delta+\bar b_i(y)\partial_i+\bar d_{ij}(y)\partial_{ij}-\bar c(y)+\xi_\lambda(y)\right] w_\lambda(y) = E_\lambda(y)$. We have

$$|\hat E_\lambda(y)| \le C_2 \begin{cases} M_k^{-1}|S_k|^{-1}|y|^{-3}, & \lambda<|y|<\frac{\sigma}{2}|S_k|, \\ M_k^{-1}|y|^{-\frac{n+2}{2}}, & |y|>\frac{\sigma}{2}|S_k|,\ y\in\mathcal{O}_\lambda. \end{cases} \tag{87}$$

Proof of (87). By Lemma 4.9, we have

$$|\nabla^m(v_k-U)(y)| \le C|S_k|^{\frac{2-n}{2}} \quad \text{for } |y|<\frac{\sigma}{2}|S_k|. \tag{88}$$

By (83) and (77), we have

$$\begin{aligned} &\bar c(y)v_k^\lambda(y)-(\frac{\lambda}{|y|})^{n+2}\bar c(y^\lambda)v_k(y^\lambda) \\ &=\bar c(y)U^\lambda(y)-(\frac{\lambda}{|y|})^{n+2}\bar c(y^\lambda)U(y^\lambda)+\bar c(y)(v_k^\lambda(y)-U^\lambda(y))-(\frac{\lambda}{|y|})^{n+2}\bar c(y^\lambda)(v_k(y^\lambda)-U(y^\lambda)) \\ &=(\Delta+V_\lambda)f_\lambda^{(3)}(y)+O(1)\begin{cases} M_k^{-1}(|S_k|^{-\frac{3}{2}}|y|^{-3}+|S_k|^{-4}|y|^{4-n}), & \lambda\le|y|<\frac{\sigma}{2}|S_k|, \\ M_k^{-1}|y|^{-\frac{n+2}{2}}, & |y|\ge\frac{\sigma}{2}|S_k|. \end{cases} \end{aligned}$$

From the proof of Proposition 4.5, and using $\sigma_k \le C|S_k|^{-\frac{n-2}{2}}$, which is based on Lemma 4.9, we know that

$$\begin{aligned} |\bar b_j(y)\partial_j v_k^\lambda(y)+\bar d_{ij}(y)\partial_{ij}v_k^\lambda(y)| &\le C\sigma_k|x_k|^{\tau-2}l_k^{-2}|y|^{-n} \\ &\le C|S_k|^{-\frac{n-2}{2}}M_k^{-1}|S_k|^{\frac{n-6}{2}}|y|^{-n}=CM_k^{-1}|S_k|^{-2}|y|^{-n} \end{aligned}$$

when $\lambda \le |y| \le \frac{\sigma}{2}|S_k|$. By (75) and (76), we have

$$|\bar{b}_j(y)\partial_j v_k^\lambda(y) + \bar{d}_{ij}(y)\partial_{ij} v_k^\lambda(y)| \le CM_k^{-1}|y|^{-\frac{n+2}{2}}$$

when $|y| \ge \frac{\sigma}{2}|S_k|$. Similarly, we have

$$\begin{aligned}
&(\frac{\lambda}{|y|})^{n+2}\left(\bar{b}_i(y^\lambda)\partial_i v_k(y^\lambda) + \bar{d}_{ij}(y^\lambda)\partial_{ij} v_k(y^\lambda)\right) \\
&\le C\begin{cases} M_k^{-1}|S_k|^{-2}|y|^{-2-n}, & \lambda \le |y| < \frac{\sigma}{2}|S_k|, \\ M_k^{-1}|y|^{-\frac{n+6}{2}}, & |y| \ge \frac{\sigma}{2}|S_k|. \end{cases}
\end{aligned}$$

Making use of (86), we complete the proof.

□

Let $f_\lambda^{(4)}(r)$ be the radial solution of

$$-\Delta f_\lambda^{(4)} = -\frac{\mathrm{d}^2}{\mathrm{d}r^2} f_\lambda^{(4)} - \frac{n-1}{r}\frac{\mathrm{d}}{\mathrm{d}r} f_\lambda^{(4)} = QM_k^{-1}|S_k|^{-1}r^{-3}, \quad r > \lambda, \tag{89}$$

$$f_\lambda^{(4)}(\lambda) = \frac{\mathrm{d}}{\mathrm{d}r} f_\lambda^{(4)}(\lambda) = 0,$$

where $Q > C_2 + 2$ is a constant to be fixed. In fact,

$$f_\lambda^{(4)}(r) = QM_k^{-1}|S_k|^{-1}\lambda^{-1}\left(\frac{1}{n-3}\frac{\lambda}{r} - \frac{1}{(n-2)(n-3)}(\frac{\lambda}{r})^{n-2} - \frac{1}{n-2}\right) < 0.$$

Lemma 4.11. *By taking a large Q independent of k, we have*

$$\left[\Delta + \bar{b}_i(y)\partial_i + \bar{d}_{ij}(y)\partial_{ij} - \bar{c}(y) + \xi_\lambda(y)\right] f_\lambda^{(4)}(|y|) \le -(C_2+1)M_k^{-1}|S_k|^{-1}|y|^{-3}$$

for $\lambda \le |y| \le Q_1 l_k^{1/2}$.

Proof. If $\lambda \le |y| \le \sigma|S_k|$, by (77) we have

$$\begin{aligned}
&\left[\Delta + \bar{b}_i(y)\partial_i + \bar{d}_{ij}(y)\partial_{ij} - \bar{c}(y)\right] f_\lambda^{(4)}(|y|) \\
&= \Delta f_\lambda^{(4)}(|y|) - \bar{c} f_\lambda^{(4)}(|y|) \\
&\le -QM_k^{-1}|S_k|^{-1}|y|^{-3} + C(|S_k|^{-4}|x_k^\tau||y|^2)M_k^{-1}|S_k|^{-1}|y|^{-3} \\
&\le -(Q-1)M_k^{-1}|S_k|^{-1}|y|^{-3}.
\end{aligned}$$

If $\sigma|S_k| \le |y| \le Q_1 l_k^{1/2}$, by (75), (76) and (77) we have

$$\begin{aligned}
&\left[\Delta + \bar{b}_i\partial_i + \bar{d}_{ij}\partial_{ij} - \bar{c}\right] f_\lambda^{(4)}(|y|) \\
&\le -QM_k^{-1}|S_k|^{-1}|y|^{-3} + C(M_k^{-1}|y|^\tau)M_k^{-1}|S_k|^{-1}|y|^{-3} \\
&\le -(Q-1)M_k^{-1}|S_k|^{-1}|y|^{-3}.
\end{aligned}$$

Since $\xi_\lambda > 0$ and $f_\lambda^{(4)}(|y|) < 0$, and $Q > 2C_2 + 2$, the lemma follows immediately.

□

Proposition 4.12. *Let* $f_\lambda(y) = f_\lambda^{(3)}(y) + f_\lambda^{(4)}(|y|)$*, where* $\lambda \in [1-\delta_1, 1+\delta_1]$ *and* $\delta_1 \in (0, 1/2)$ *is determined to make it possible to construct* $f_\lambda^{(3)}(y)$ *from* (84)*. Then*

$$\left[\Delta + \bar{b}_i(y)\partial_i + \bar{d}_{ij}(y)\partial_{ij} - \bar{c}(y) + \xi_\lambda(y)\right](w_\lambda + f_\lambda)(y) \le 0$$

for $y \in \tilde{\Sigma}^k := \mathcal{O}_\lambda \cup \{y : \lambda \le |y| \le \frac{1}{2}\sigma|S_k|\}$*,*

$$f_\lambda(y) = 0 \quad \textit{on } \partial B_\lambda, \quad |f_\lambda(y)| + |\nabla f_\lambda(y)| = o(1)|y|^{2-n} \textit{ for } y \in \tilde{\Sigma}^k_\lambda,$$

In particular, $|f_\lambda(y)| = o(1)M_k^{-1}$ *on* $\partial B_{Q_1 l_k^{1/2}}$*.*

Proof. The differential inequality follows from (87) and Lemma 4.11. The boundary condition $f_\lambda(y) = 0$ on ∂B_λ follows from the construction of $f_\lambda(y)$. To obtain the estimate for $|f_\lambda(y)|+|\nabla f_\lambda(y)|$, we apply (85) to obtain $|f_\lambda^{(3)}(y)|+|\nabla f_\lambda^{(3)}(y)| \le C|x_k|^\tau(1+|y|)^{2-n}$. Furthermore, the estimate for $|f_\lambda^{(4)}(y)|$ implies that $|f_\lambda^{(4)}(y)| + |\nabla f_\lambda^{(4)}(y)| = O(|S_k|^{-1})M_k^{-1} = O(|S_k|^{-1})l_k^{-\frac{n-2}{2}}$, but for $\lambda \le |y| \le Q_1 l_k^{1/2}$, we certainly have $l_k^{-\frac{n-2}{2}} = O(|y|^{2-n})$. Since $|x_k|, |S_k|^{-1} \to 0$ as $k \to \infty$, we thus conclude that $|f_\lambda^{(4)}(y)| + |\nabla f_\lambda^{(4)}(y)| = o(1)|y|^{2-n}$ for $\lambda \le |y| \le Q_1 l_k^{1/2}$. Finally, on $\partial B_{Q_1 l_k^{1/2}}$, $|y|^{2-n} = Q_1^{2-n} l_k^{-\frac{n-2}{2}}$, we see that $|f_\lambda(y)| = o(1)M_k^{-1}$. □

5 The lower bound and removability of the singularity

Suppose that u is a solution of (6) with g satisfying (7). If $n \ge 25$, we assume further that (8) holds. It follows from Theorem 2.1 that

$$u(x) \le C|x|^{-\frac{n-2}{2}} \quad \text{for } x \in B_1, \tag{90}$$

where $C > 0$ is independent of x.

Lemma 5.1. *Suppose that* u *is a solution of* (6) *with* g *satisfying* (7)*. If* $n \ge 25$*, we assume further that* (8) *holds. Then*

$$\max_{r/2 \le |x| \le 2r} u(x) \le C_3 \min_{r/2 \le |x| \le 2r} u(x)$$

and

$$|\nabla u(x)| + |x||\nabla^2 u(x)| \le C_3 |x|^{-1} u(x)$$

for every $0 < |x| = r < 1/4$*, where* C_3 *is independent of* r*.*

Proof. For any $\bar{x} \in B_{1/4} \setminus \{0\}$, let $r = |\bar{x}|$ and

$$v_r(y) = r^{\frac{n-2}{2}} u(ry).$$

By (6), we have

$$-L_{\bar g} v_r = n(n-2) v_r(y)^{\frac{n+2}{n-2}} = 0 \quad \text{ in } B_{1/r} \setminus \{0\},$$

where $\bar g_{ij}(y) = g_{ij}(ry)$. Since g satisfies (7), we have $|\nabla^l(\bar g_{ij}(y) - \delta_{ij})| \le r^\tau$ for $\frac14 \le |y| \le 4$, $l = 0, \dots, n+2$. By (90), $v_r(y) \le C$ for $y \in B_4 \setminus \bar B_{1/4}$. Applying the standard local estimates and Harnack inequality to v_r in the annulus $B_4 \setminus \bar B_{1/4}$ and scaling back to u, the lemma follows immediately.

□

Recall that the Pohozaev identity for u is

$$P(r,u) - P(s,u) = -\int_{B_r \setminus B_s} (\frac{n-2}{2} u + x \cdot \nabla u)(L_g - \Delta) u \,\mathrm{d}x, \tag{91}$$

where $0 < s \le r < 1$,

$$P(r,u) = \int_{\partial B_r} \left(\frac{n-2}{2} u \frac{\partial u}{\partial r} - \frac12 r |\nabla u|^2 + r|\frac{\partial u}{\partial r}|^2 + \frac{(n-2)^2}{2} r u^{\frac{2n}{n-2}} \right) \mathrm{d}S_r,$$

and $\mathrm{d}S_r$ is the standard area measure on ∂B_r. By Lemma 5.1 and the flatness condition (7) on g, we have

$$|(\frac{n-2}{2} u + x \cdot \nabla u)(L_g - \Delta) u| \le C|x|^{\tau - n}.$$

As in previous sections, we take $\tau = \frac{n-2}{2}$. It follows that $|P(r,u) - P(s,u)| \le C r^\tau$ for any $0 < s < r$ and hence the limit

$$\lim_{r \to 0} P(r,u) =: P(u) \tag{92}$$

exists.

Lemma 5.2. *Assume as in Lemma 5.1. Then we have*

$$P(u) \le 0.$$

Moreover, $P(u) = 0$ *if and only if* $\liminf_{x \to 0} |x|^{\frac{n-2}{2}} u(x) = 0$.

Proof. Let $\{r_k\}$ be any sequence of positive numbers satisfying $\lim_{k\to\infty} r_k = 0$, and let

$$v_k(y) = r_k^{\frac{n-2}{2}} u(r_k y).$$

By the proof of Lemma 5.1, we see that, up to passing to a subsequence,

$$v_k \to v \quad \text{in } C^2_{loc}(\mathbb{R}^n \setminus \{0\}),$$

where

$$-\Delta v = n(n-2) v^{\frac{n+2}{n-2}} \quad \text{in } \mathbb{R}^n \setminus \{0\}, \quad v \ge 0.$$

It follows that

$$P(1,v)=\lim_{k\to\infty}P(1,v_k)=\lim_{k\to\infty}P(r_k,u)=P(u).$$

By [11], $P(r,v)$ is a nonpositive constant independent of r and $P(1,v)=0$ if and only if v is smooth cross $\{0\}$. Clearly, if $P(u)=0$ then $\liminf_{x\to 0}|x|^{\frac{n-2}{2}}u(x)=0$; otherwise $v_k(y)\geq \frac{1}{C}|y|^{-\frac{n-2}{2}}$ for some $C>0$ independent of k and thus v is singular at 0.

On the other hand, if $\liminf_{x\to 0}|x|^{\frac{n-2}{2}}u(x)=0$, by Lemma 5.1 we can find $r_k\to 0$ such that v_k defined as above has trivial limit $v\equiv 0$. Hence $P(1,v)=0=P(u)$.

□

Let $\bar{u}(r)=\fint_{\partial B_r}u\,\mathrm{d}S_r$ be the average of u on ∂B_r. Let $t=-\ln r$ and $\bar{u}(r)=e^{\frac{n-2}{2}t}w(t)$. By a direct computation,

$$\bar{u}_r=-e^{\frac{n}{2}t}\left(\frac{n-2}{2}w+w_t\right),$$

$$\bar{u}_{rr}=e^{\frac{n+2}{2}t}\left(\frac{n(n-2)}{4}w+(n-1)w_t+w_{tt}\right).$$

Therefore, we have

$$\bar{u}_{rr}+\frac{n-1}{r}\bar{u}_r=e^{\frac{n+2}{2}t}\left(w_{tt}-(\frac{n-2}{2})^2w\right).$$

By Lemma 5.1 and the flatness condition (7) on g, we have

$$-c_1w^{\frac{n+2}{n-2}}-c_3e^{-\tau t}w\leq w_{tt}-(\frac{n-2}{2})^2w\leq -c_2w^{\frac{n+2}{n-2}}+c_3e^{-\tau t}w. \tag{93}$$

Proposition 5.3. *Assume as in Lemma 5.1. If* $\liminf_{x\to 0}|x|^{\frac{n-2}{2}}u(x)=0$*, then*

$$\lim_{x\to 0}|x|^{\frac{n-2}{2}}u(x)=0 \tag{94}$$

and 0 *is a removable singularity.*

Proof. Our proof will largely follow the approach initiated by Chen-Lin [15], but we need to prove some necessary bounds in our context, as given later in Lemma 6.2 and Lemma 6.3. We argue by contradiction. Since $\liminf_{x\to 0}|x|^{\frac{n-2}{2}}u(x)=0$, by Lemma 5.1, if the conclusion of the Proposition does not hold, we would have $\limsup_{x\to 0}u(x)|x|^{\frac{n-2}{2}}>0$, so

$$\limsup_{t\to\infty}w(t)>\liminf_{t\to\infty}w(t)=0.$$

Making use of (93), w is convex ($w''>0$) when w small and t is large. It follows that there exist

$$\bar{t}_i<t_i<t_i^*\quad\text{with}\quad \lim_{i\to\infty}\bar{t}_i=\infty,$$

such that

$$w(\bar{t}_i) = w(t_i^*) = \epsilon_0, \quad \lim_{i\to\infty} w(t_i) = 0$$

t_i is the unique minimum point of w in $(\bar{t}_i, t_i^*)$,

where $\epsilon_0 > 0$ is a small constant. Hence, w is decreasing in $(\bar{t}_i, t_i)$ and increasing in (t_i, t_i^*).

Using (93), we will prove the following estimates

$$\frac{2}{n-2}\ln\frac{w(t)}{w(t_i)} - C \le t - t_i \le \left(\frac{2}{n-2} + Ce^{-\tau t_i}\right)\ln\frac{w(t)}{w(t_i)} + C, \quad t_i \le t \le t_i^*, \tag{95}$$

and

$$\frac{2}{n-2}\ln\frac{w(t)}{w(t_i)} - C \le t_i - t \le \left(\frac{2}{n-2}\right)\ln\frac{w(t)}{w(t_i)} + C', \quad \bar{t}_i \le t \le t_i \tag{96}$$

where $C, C' > 0$ is independent of i.

Marques obtained a cruder version of these estimates in [45], replacing $\frac{2}{n-2}$ on the left inequality in (95) by $\frac{2}{n-2} - ce^{-\tau t_i}$ with $\tau = 2$ and some $c > 0$, and similarly, replacing $\frac{2}{n-2}$ on the left inequality in (96) by $\frac{2}{n-2} - ce^{-\tau\bar{t}_i}$, and $\frac{2}{n-2}$ on the right inequality in (96) by $\frac{2}{n-2} + ce^{-\tau\bar{t}_i}$. His proof was based on that of Chen-Lin in [15], which estimates the $e^{-\tau t}$ factor in (93) by its value at the left end of the interval and estimates the resulting differential inequality as one with constant coefficients. His estimates are adequate to handle his cases of $3 \le n \le 5$, but are not adequate for our cases. We will provide our proof for (95) and (96) in the next section. The key is to tackle the $e^{-\tau t}w$ term directly. The behavior exhibited is somewhat analogous to the behavior as in [31]—see also Theorem 8.1 (and Problems 29–31) in Chapter 3 of [16], but our proof does not rely on the method for treating linear systems as in [31], instead relies on some comparison principles.

We denote $r = |x|$,

$$\bar{r}_i = e^{-\bar{t}_i}, \quad r_i = e^{-t_i}, \quad r_i^* = e^{-t_i^*}.$$

Thus $\bar{r}_i > r_i > r_i^*$.

First, we can compute $P(r,u)$ in terms of $v(t,\theta) := e^{-\frac{n-2}{2}t}u(e^{-t}\theta)$ as

$$P(r,u) = \frac{|\mathbb{S}^{n-1}|}{2}\int_{\mathbb{S}^{n-1}}\left[v_t^2(t,\theta) - |\nabla_\theta v(t,\theta)|^2 - (n-2)^2\left(\frac{v(t,\theta)^2}{4} - v(t,\theta)^{\frac{2n}{n-2}}\right)\right]d\theta,$$

and using the Harnack and gradient estimates on $u(x)$ as given by Lemma 5.1 we see that, in terms of $v(t,\theta)$, we have

$$|\nabla v(t,\theta)| = O(1)w(t),$$

uniformly for $\theta \in \mathbb{S}^{n-1}$, so it follows that

$$\int_{\mathbb{S}^{n-1}}\left[v_t^2(t_i,\theta) - |\nabla_\theta v(t_i,\theta)|^2\right]d\theta \to 0 \quad \text{as } i \to \infty,$$

and

$$P(u) = \lim_{i\to\infty} P(r_i, u) = 0. \tag{97}$$

Next we claim a more precise estimate for $|\nabla v(t_i, \theta)|$ at $t = t_i$, which implies an equivalent estimate for $u(x)$ on $|x| = r_i$.

$$|\nabla v(t_i, \theta)| = o(1)w(t_i) \quad \text{uniformly for } \theta \in \mathbb{S}^{n-1}. \tag{98}$$

Indeed, let $\zeta_i(y) = \frac{u(r_i y)}{u(r_i e_1)}$, where $e_1 = (1, 0, \ldots, 0)$. We have

$$-L_{g_i}\zeta_i = n(n-2)\left(r_i^{\frac{n-2}{2}} u(r_i e_1)\right)^{\frac{4}{n-2}} \zeta_i^{\frac{n+2}{n-2}} \quad \text{in } B_{1/r_i} \setminus \{0\},$$

where $(g_i)_{kl} = g_{kl}(r_i y)$. By Lemma 5.1, ζ_i is locally uniformly bounded in $\mathbb{R}^n \setminus \{0\}$. By the choice of r_i, $r_i^{\frac{n-2}{2}} u(r_i e_1) \to 0$ as $i \to \infty$. Hence, $\zeta_i \to \zeta$ in $C^2_{loc}(\mathbb{R}^n \setminus \{0\})$ for some ζ satisfying

$$-\Delta\zeta = 0 \quad \text{in } \mathbb{R}^n \setminus \{0\}, \quad \zeta \geq 0,$$

$\zeta(e_1) = 1$ and $\partial_r \left(\int_{\mathbb{S}^{n-1}} \zeta(r\theta) r^{\frac{n-2}{2}}\, d\theta\right) = 0$ at $r = 1$, which is based on the choice of r_i in the definition for $\zeta_i(y)$. By the Bôcher theorem and the two normalizing conditions above, $\zeta(y) = a|y|^{2-n} + b$ with $a = b = \frac{1}{2}$. In terms of $v(t, \theta)$, this is implying that $v(t_i + \tau, \theta)/v(t_i, e_1)$ converges to $\cosh(\tau)$ uniformly on any compact interval of $\tau \times \mathbb{S}^{n-1}$, and its derivatives converge to the respective ones of $\cosh(\tau)$. This in particular implies that $\nabla_\theta v(t_i + \tau, \theta)/v(t_i, e_1) \to 0$ uniformly at $\{\tau\} \times \mathbb{S}^{n-1}$ at any τ, and $\nabla_t v(t_i, \theta)/v(t_i, e_1) \to 0$ uniformly for $\theta \in \mathbb{S}^{n-1}$. Hence, (98) follows.

Making use of (98), we have $|v(t_i, \theta) - w(t_i)| = o(1)w(t_i)$ and $|\nabla_{t,\theta} v(t_i, \theta)| = o(1)w(t_i)$ uniformly for $\theta \in \mathbb{S}^{n-1}$, so

$$\begin{aligned} P(r_i, u) =& |\mathbb{S}^{n-1}| \Big[-\frac{1}{2}\left(\frac{n-2}{2}\right)^2 w^2(t_i)(1 + o(1)) \\ &+ \frac{(n-2)^2}{2} w^{\frac{2n}{n-2}}(t_i)(1 + o(1))\Big]. \end{aligned}$$

Hence for sufficiently large i

$$w^2(t_i) \leq c_n |P(r_i, u)|. \tag{99}$$

It follows from the Pohozaev identity (91) and (97) that

$$\begin{aligned} |P(r_i, u)| \leq& \int_{B_{r_i} \setminus B_{r_i^*}} |\mathcal{A}(u)| \mathrm{d}x + \int_{B_{r_i^*}} |\mathcal{A}(u)| \mathrm{d}x \\ =:& I_1 + I_2, \end{aligned}$$

where

$$\mathcal{A}(u) = \left(x^k \partial_k u + \frac{n-2}{2} u\right)(L_g u - \Delta u).$$

By Lemma 5.1, we have

$$|\mathcal{A}(u)| \le C|x|^{\tau-n}.$$

Hence,

$$I_2 \le C(r_i^*)^\tau = Ce^{-\tau t_i^*}.$$

By the first inequality in (95), we have

$$w(t) \le Cw(t_i)\exp\left(\left(\frac{n-2}{2}\right)(t-t_i)\right), \quad t_i \le t \le t_i^*,$$

which implies

$$u(x) \le Cw(t_i)e^{-\tau t_i}|x|^{2-n} \quad \text{for } r_i^* \le |x| \le r_i.$$

By Lemma 5.1, we also have

$$|\mathcal{A}(u)(x)| \le Cu(x)^2|x|^{\tau-2}.$$

Hence,

$$I_1 \le Cw(t_i)^2e^{-(n-2)t_i}\int_{r_i^*\le|x|\le r_i}|x|^{2+\tau-2n}\,\mathrm{d}x \le cw(t_i)^2e^{(2-n)t_i}(r_i^*)^{1-\frac{n}{2}}.$$

By (95) and (96), we see that

$$t_i^* - t_i \le \left(\frac{2}{n-2} + Ce^{-\tau t_i}\right)\ln\frac{\varepsilon_0}{w(t_i)} + C, \quad \frac{2}{n-2}\ln\frac{\varepsilon_0}{w(t_i)} \le t_i - \bar{t}_i + C.$$

Hence,

$$t_i^* - t_i \le \left(\frac{2}{n-2} + Ce^{-\tau t_i}\right)\left(t_i - \bar{t}_i + C\right)\frac{n-2}{2} + C \le t_i - \bar{t}_i + C'', \tag{100}$$

for some constant $C'' > 0$ independent of i, where we have used $e^{-\tau t_i}(t_i - \bar{t}_i) \le e^{-\tau t_i}t_i$ is bounded above independent of i. Using (100) we can estimate I_1 more precisely:

$$\begin{aligned} I_1 &\le Cw(t_i)^2e^{-(n-2)t_i}(r_i^*)^{1-\frac{n}{2}} \\ &= Cw(t_i)^2e^{-(n-2)t_i+(\frac{n-2}{2})t_i^*} \\ &\le Cw(t_i)^2e^{-\tau\bar{t}_i}. \end{aligned}$$

Combining the estimates of I_1 and I_2, we have,

$$|P(r_i,u)| \le Cw(t_i)^2e^{-\tau\bar{t}_i} + Ce^{-\tau t_i^*}. \tag{101}$$

Using (99) and (101), we can combine terms to obtain $w(t_i)^2 \le Ce^{-\tau t_i^*}$, which is

$$\log\frac{1}{w(t_i)} \ge \frac{n-2}{4}t_i^* - C. \tag{102}$$

From the first inequality of (96) and the first inequality of (95), we have

$$t_i - \bar{t}_i \geq \frac{2}{n-2} \ln \frac{\varepsilon_0}{w(t_i)} - C$$

and

$$t_i^* - t_i \geq \frac{2}{n-2} \ln \frac{\varepsilon_0}{w(t_i)} - C.$$

Adding them up and using (102) and (100), we have

$$t_i^* - \bar{t}_i \geq \frac{4}{n-2} \ln \frac{1}{w(t_i)} - C \geq t_i^* - C,$$

which implies $\bar{t}_i \leq C$. This contradicts $\bar{t}_i \to \infty$. Therefore, (94) holds.

Based on (94) we clearly have $w'(t) < 0$ for $t > T_1$, where T_i is sufficiently large. Equation (93) now implies

$$w_{tt} - (\frac{n-2}{2} - \delta)^2 w \geq 0 \quad \text{for } t \geq T_1,$$

where $\delta > 0$ is some small constant. Thus for $t \geq T_1$, $w_t^2 - (\frac{n-2}{2} - \delta)^2 w^2$ is non-increasing, non-negative due to $w(t) \to 0$ as $t \to \infty$, and the integration of this quantity leads to

$$w(t) \leq w(T_1) \exp(-(\frac{n-2}{2} - \delta)(t - T_1)), \quad t > T_1,$$

whose equivalent form is

$$u(x) \leq C(\delta)|x|^{-\delta}.$$

Then standard elliptic estimate immediately implies that u has a removable singularity at the origin.

Therefore, we complete the proof of Proposition 5.3. □

Proof of Theorem 1.3. It follows from Lemma 5.2 and Proposition 5.3.

□

Proof of Theorem 1.4. By Lemma 5.1, we have

$$\begin{aligned} -u^{-\frac{n+2}{n-2}} \Delta u &= -u^{-\frac{n+2}{n-2}} L_g u - u^{-\frac{n+2}{n-2}} (\Delta - L_g) u \\ &= n(n-2) + O(|x|^{\frac{n+2}{2}} \cdot |x|^{\tau - 2 - \frac{n-2}{2}}) \\ &= n(n-2) + O(|x|^{\tau}) \quad \text{as } x \to 0. \end{aligned}$$

Since 0 is not a removable singularity, (9) holds. Then the theorem follows immediately from Theorem 1 of Taliaferro-Zhang [58]—one step in its proof, (2.25), relies on a statement from [11], for which one can also appeal to Theorem 3 of Han-Li-Teixeira [28]. □

Proof of Theorem 1.1 and Theorem 1.2. By the discussion in Section 2, the study of asymptotical behavior at the infinity of solution of the Yamabe equation (3) with asymptotically flat metric g satisfying (4) is equivalent to the study of isolated singularity of solutions of (6) with g satisfying (7), after taking a Kelvin transform. Hence, Theorem 1.1 and Theorem 1.2 follow from Theorem 1.3 and Theorem 1.4.

□

6 Details of improved ODE estimates (95) and (96)

First, we recall a comparison principle.

Lemma 6.1. *Suppose $\eta(t)$ and $b(t)$ are C^2 functions defined on $[t_1, t_2]$, and $b(t) \geq 0$. Suppose that*

$$\eta''(t) - b(t)\eta(t) \geq 0$$

If

$$\eta(t_1) \geq 0 \quad \textit{and} \quad \eta'(t_1) \geq 0,$$

then $\eta(t) \geq \eta(t_1)$ and $\eta'(t) \geq 0$ on $[t_1, t_2]$.

If

$$\eta(t_2) \geq 0 \quad \textit{and} \quad \eta'(t_2) \leq 0,$$

then $\eta(t) \geq \eta(t_2)$ and $\eta'(t) \leq 0$ on $[t_1, t_2]$.

Proof. Suppose that $\eta(t_1) \geq 0$ and $\eta'(t_1) \geq 0$, but $\eta(\bar{t}) < \eta(t_1)$ for some $\bar{t} \in (t_1, t_2]$. Then there must be some point $t_1 \leq \hat{t} < \bar{t}$ such that $\eta(\hat{t}) = \eta(t_1)$ and $\eta(t) < \eta(t_1)$ for $t \in (\hat{t}, \bar{t})$. By Hopf lemma, we have $\eta'(\hat{t}) < 0$. Let

$$t^* = \inf\{t_1 \leq t < \hat{t} : \eta(s) > \eta(\hat{t}) \geq 0, \eta'(s) < 0 \quad \text{for } t < s < \hat{t}\}.$$

Then either $t^* = t_1$ or $\eta'(t^*) = 0$. In either case Hopf Lemma would apply at t^* and imply that $\eta'(t) < 0$, which would be a contradiction. We conclude that $\eta(t) \geq \eta(t_1)$. Now it follows that $\eta''(t) \geq b(t)\eta(t) \geq 0$, so η' is non-decreasing on $[t_1, t_2]$, which implies $\eta'(t) \geq \eta'(t_1) \geq 0$ on $[t_1, t_2]$.

If $\eta(t_2) \geq 0$ and $\eta'(t_2) \leq 0$, the proof is similar. Therefore, Lemma (6.1) is proved. □

The following lemma will be used to prove the left side inequalities in (95) and (96).

Lemma 6.2. *Suppose that a, b, t_1, t_2 and τ are positive numbers, and $1 < t_1 < t_2$. Suppose that η is a positive C^2 function defined on $[t_1, t_2]$ and satisfies*

$$\eta''(t) - (a^2 + be^{-\tau t})\eta(t) \leq 0 \quad \textit{for } t_1 \leq t \leq t_2. \tag{103}$$

(i) If $\eta'(t_2) \geq 0$, then

$$\eta(t) \leq \eta(t_2) \frac{\cosh(a(t - t_2) - f(t))}{\cosh(f(t_2))}, \quad t_1 \leq t \leq t_2, \tag{104}$$

where $f(t) = \frac{b}{2a\tau} e^{-\tau t}$. Consequently,

$$t_2 - t \geq \frac{1}{a} \ln \frac{\eta(t)}{\eta(t_2)} - C,$$

where $C > 0$ depends only on a, b and τ.

(ii) If $\eta'(t_1) \le 0$, then

$$\eta(t) \le \eta(t_1)\frac{\cosh(a(t-t_1)+f(t))}{\cosh(f(t_1))}, \quad t_1 \le t \le t_2, \tag{105}$$

and

$$t - t_1 \ge \frac{1}{a}\ln\frac{\eta(t)}{\eta(t_1)} - C,$$

where $C > 0$ depends only on a, b and τ.

Proof. We remark that the left hand side of (103) can be transformed into a Bessel type equation, after the change of variable $x = e^{-t}$, so we can apply Lemma 6.1 to $\eta(t)$ with a solution of this Bessel type equation. But here we give an explicit comparison function.

Let

$$\zeta(t) := \frac{\eta(t_2)}{\cosh(f(t_2))}\cosh(a(t-t_2)-f(t)).$$

Differentiating ζ and using $f'(t) = -\tau f(t)$, we find that

$$\zeta'(t) = \frac{\eta(t_2)}{\cosh(f(t_2))}(a+\tau f(t))\sinh(a(t-t_2)-f(t)),$$

$$\zeta''(t) = \frac{\eta(t_2)}{\cosh(f(t_2))}\left(-\tau^2 f(t)\sinh(a(t-t_2)-f(t)) + (a+\tau f(t))^2\cosh(a(t-t_2)-f(t))\right).$$

Hence,

$$\begin{aligned}
&\zeta''(t) - (a^2 + be^{-\tau t})\zeta(t) \\
&= \frac{\eta(t_2)}{\cosh(f(t_2))}\Big(((a+\tau f(t))^2 - (a^2+be^{-\tau t}))\cosh(a(t-t_2)-f(t)) \\
&\quad - \tau^2 f(t)\sinh(a(t-t_2)-f(t))\Big) \\
&\ge \frac{\eta(t_2)}{\cosh(f(t_2))}(2a\tau\frac{b}{2a\tau} - b)e^{-\tau t}\cosh(a(t-t_2)-f(t)) = 0,
\end{aligned}$$

where we have used $\sinh(a(t-t_2)-f(t)) \le 0$ if $t \le t_2$. Moreover,

$$\zeta(t_2) = \eta(t_2) \quad \text{and} \quad \zeta'(t_2) \le 0.$$

By Lemma 6.1, we have $\eta(t) \le \zeta(t)$ for $t \in [t_1, t_2]$. Therefore, (104) is proved.

The proof of (105) is the same. Therefore, Lemma 6.2 is proved. □

Proof of lower bounds in (95) *and* (96). For the lower bound in (95), we apply (ii) of Lemma 6.2, identifying $t_1 = t_i$ and $t_2 = t_i^*$. For the lower bound in (96), we apply (i) of Lemma 6.2, identifying $t_1 = \bar{t}_i$ and $t_2 = t_i$. □

Lemma 6.3. *Suppose that b, c, n, t_1, t_2 and τ are positive numbers, $t^* \leq t_1 < t_2$ and $n > 2$. Suppose that w is a positive C^2 function defined on $[t_1, t_2]$ and satisfies*

$$w''(t) - (\frac{n-2}{2})^2(1 - be^{-\tau t})w(t) + \frac{n(n-2)c}{4}w(t)^{\frac{n+2}{n-2}} \geq 0, \varepsilon_0 \geq w(t), \quad \textit{for } t_1 \leq t \leq t_2. \tag{106}$$

Then there exist positive constants t^*, ε_0, C_1 and C_2, depending only on b, c, n and τ, such that:

(i) If $w'(t_2) \leq 0$, then there holds

$$t_2 - t \leq (\frac{2}{n-2} + C_1 e^{-\tau t_1}) \ln \frac{w(t)}{w(t_2)} + C_2 \quad \textit{for } t_1 \leq t \leq t_2. \tag{107}$$

(ii) If $w'(t_1) \geq 0$, then there holds

$$t_2 - t \leq (\frac{2}{n-2} + C_1 e^{-\tau t_1}) \ln \frac{w(t)}{w(t_1)} + C_2 \quad \textit{for } t_1 \leq t \leq t_2. \tag{108}$$

Proof. The estimates in this lemma were proved by Marques [45] for $\tau = 2$, which in turn is based on the argument of Chen-Lin [15]. These upper bounds are also adequate for our purposes, so we only sketch a different proof following the previous proof of Lemma 6.2. The upper bound in our (96) is stronger than that in (107), where our set up also gives us $w'(t) < 0$ for $t_1 < t < t_2$. Although the stronger version is not needed for our proof of the main theorems of this paper, we will describe how to prove it at the end.

Let

$$\zeta(t) = B \cosh^{\frac{2-n}{2}}(a(t - \bar{t})),$$

where $a = \sqrt{1 - Ae^{-\tau t_1}}$ for some $A > 0$ to be fixed, $B^{\frac{4}{n-2}}c = 1$, and $\bar{t} \geq t_2$ such that $\zeta(t_2) = w(t_2)$. Then by a direct calculation similar to the one in the proof for the previous Lemma, we have

$$\begin{aligned}
&\zeta''(t) - ((\frac{n-2}{2})^2 - be^{-\tau t})\zeta(t) + \frac{n(n-2)c}{4}\zeta^{\frac{n+2}{n-2}} \\
&= \frac{(n-2)^2 B}{4}\left\{-\frac{n}{n-2}\frac{a^2-1}{\cosh^2(a(t-\bar{t}))} + (a^2 - 1 + \frac{4}{(n-2)^2}be^{-\tau t})\right\}\cosh^{\frac{2-n}{2}}(a(t-\bar{t})) \\
&\leq \frac{(n-2)^2 B}{4}\Big(\frac{n}{n-2}Ae^{-\tau t_1}C\varepsilon_0^{\frac{n-2}{2}} - Ae^{-\tau t_1} + \frac{4}{(n-2)^2}be^{-\tau t}\Big)\cosh^{\frac{2-n}{2}}(a(t-\bar{t})) \\
&\leq 0,
\end{aligned}$$

if we take ε_0 to be sufficiently small so that $\frac{n}{n-2}C\varepsilon_0^{\frac{n-2}{2}} < 0.1$, say; and take A such that $0.9A > b$. Moreover,

$$\zeta'(t_2) \geq 0,$$

since $\bar{t} \geq t_2$.

Let $z = w - \zeta$. It follows that

$$z''(t) - (\frac{n-2}{2})^2(1 - be^{-\tau t})z(t) + \frac{n(n+2)}{4}\xi(t)^{\frac{4}{n-2}}z(t) \geq 0,$$

$$z(t_2) = 0, \quad z'(t_2) \leq 0,$$

where

$$n(n+2)\xi(t)^{\frac{4}{n-2}} = \begin{cases} n(n-2)\dfrac{w(t)^{\frac{n+2}{n-2}} - \zeta(t)^{\frac{n+2}{n-2}}}{w - \zeta}, & \text{if} \quad w(t) \neq \zeta(t), \\ n(n+2)w(t)^{\frac{4}{n-2}}, \quad \text{if} \quad w(t) = \zeta(t). \end{cases}$$

Note that $\zeta(t), w(t) \leq \varepsilon_0$ for $t_1 \leq t \leq t_2$, so by taking ε_0 small and t^* large, we then have

$$(\frac{n-2}{2})^2(1 - be^{-\tau t}) - \frac{n(n+2)}{4}\xi(t)^{\frac{4}{n-2}} \geq 0.$$

By Lemma 6.1, we have $w - \zeta \geq 0$ in $[t_1, t_2]$. Hence, (107) follows immediately.

The proof of (108) is the same.

Our proof for the upper bound in (96) is a modification of the proof in [45]. We use the set up in (107), identifying $\bar{t}_i = t_1$ and $t_i = t_2$. Multiplying both sides of (106) by $2w'(t) < 0$, we have

$$\begin{aligned} 0 &\geq \left[w'(t)^2 - (\frac{n-2}{2})^2(1 - be^{-\tau t})w(t)^2 + c(\frac{n-2}{2})^2 w(t)^{\frac{2n}{n-2}}\right]' + (\frac{n-2}{2})^2 b\tau w(t)^2 \\ &\geq \left[w'(t)^2 - (\frac{n-2}{2})^2(1 - be^{-\tau t})w(t)^2 + c(\frac{n-2}{2})^2 w(t)^{\frac{2n}{n-2}}\right]' \end{aligned}$$

so $w'(t)^2 - (\frac{n-2}{2})^2(1 - be^{-\tau t})w(t)^2 + c(\frac{n-2}{2})^2 w(t)^{\frac{2n}{n-2}}$ is non-increasing in t, and

$$\begin{aligned} &w'(t)^2 - (\frac{n-2}{2})^2(1 - be^{-\tau t})w(t)^2 + c(\frac{n-2}{2})^2 w(t)^{\frac{2n}{n-2}} \\ \geq &- (\frac{n-2}{2})^2(1 - be^{-\tau t_2})w(t_2)^2 + c(\frac{n-2}{2})^2 w(t_2)^{\frac{2n}{n-2}}, \end{aligned}$$

from which we obtain

$$\begin{aligned} w'(t)^2 \geq &(\frac{n-2}{2})^2(1 - be^{-\tau t})\left[w(t)^2 - w(t_2)^2\right] - c(\frac{n-2}{2})^2\left[w(t)^{\frac{2n}{n-2}} - w(t_2)^{\frac{2n}{n-2}}\right] \\ &+ b(\frac{n-2}{2})^2(e^{-\tau t_2} - e^{-\tau t})w(t_2)^2. \end{aligned}$$

When $w(t)/w(t_2) \leq 2$, then for t^* sufficiently large, $\varepsilon_0 > 0$ sufficiently small, there exists $0 < \delta < 1$ such that for $t < s < t_2$, we have $w''(s) - (\frac{n-2}{2})^2(1-\delta)^2 w(s) \geq 0$, so by Lemma 6.1, $w(s) \geq w(t_2)\cosh[\frac{n-2}{2}(1-\delta)(s-t_2)]$, from which we get $2 \geq \cosh[\frac{n-2}{2}(1-\delta)(s-t_2)]$.

This gives an absolute upper bound for $t_2 - t'$, where t' is defined by $t < t' < t_2$ such that $w(t')/w(t_2) = 2$.

For $t_1 < t < t'$, set $\eta(t) = w(t)/w(t_2)$, then $\eta(t) \geq 2$, and

$$\eta'(t)^2 \geq (\frac{n-2}{2})^2 \left\{(1 - be^{-\tau t}) \left[\eta(t)^2 - 1\right] - cw(t_2)^{\frac{4}{n-2}} \left[\eta(t)^{\frac{2n}{n-2}} - 1\right] + b(e^{-\tau t_2} - e^{-\tau t})\right\}.$$

For t^* sufficiently large, we have $b(e^{-\tau t_2} - e^{-\tau t}) \geq -0.1(1 - be^{-\tau t})$ and $1 - be^{-\tau t} \geq 1/2$, so we get

$$\eta'(t)^2 \geq (\frac{n-2}{2})^2 (1 - be^{-\tau t}) \left\{\eta(t)^2 - 1.1 - 2cw(t_2)^{\frac{4}{n-2}} \left[\eta(t)^{\frac{2n}{n-2}} - 1\right]\right\}.$$

Now if we introduce $s = \frac{n-2}{2} \int_t^{t_2} \sqrt{1 - be^{-\tau \hat{t}}} d\hat{t}$, then we get

$$\left|\frac{d\eta}{ds}\right|^2 \geq \eta^2 - 1.1 - 2cw(t_2)^{\frac{4}{n-2}} \left[\eta^{\frac{2n}{n-2}} - 1\right].$$

Set $s' = \frac{n-2}{2} \int_{t'}^{t_2} \sqrt{1 - be^{-\tau \hat{t}}} d\hat{t}$. Then, on the one hand,

$$s - s' \leq \int_2^{w(t)/w(t_2)} \frac{d\eta}{\sqrt{\eta^2 - 1.1 - 2cw(t_2)^{\frac{4}{n-2}} \left[\eta^{\frac{2n}{n-2}} - 1\right]}} \leq \log \frac{w(t)}{w(t_2)} + C$$

for some constant C, where the integral on the right is estimated in a similar way as in [15] and [45]. On the other hand, $s - s' = \frac{n-2}{2} \int_t^{t'} \sqrt{1 - be^{-\tau \hat{t}}} d\hat{t} = \frac{n-2}{2} [t' - t + O(1)]$. The upper bound in (96) now follows. □

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