A Kazdan-Warner type identity for the σ_k curvature

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The Weyl-Schouten tensor A_g of a metric g is defined to be

$$A_g = \frac{1}{n-2}\{Ric_g - \frac{Scal_g}{2(n-1)}g\}.$$

The σ_k curvature of g is defined to be the k-th elementary symmetric function of the eigenvalues of the 1-1 tensor $g^{-1} \circ A_g$. σ_1 of g is simply a dimensional constant multiple of the scalar curvature of g. Since the thesis of Viaclovsky [V1] and the fundamental work of A. Chang, M. Gursky, and P. Yang [CGY1], there has been very intensive research and progress on an extensive list of geometrical and PDE problems involving the σ_k curvature of a metric for $k > 1$, mostly involving a conformal change of metric, see the bibliography for an incomplete list of recent work in this area. Since the Weyl-Schouten tensor transforms in the following way under a conformal change of metric $g = e^{2w(x)}g_0$,

$$A_g = A_{g_0} - \left[\nabla^2 w - dw \otimes dw + \frac{1}{2}|\nabla w|^2 g_0\right],$$

thet σ_k curvature of g, when $k \geq 2$, is then expressed as a fully nonlinear expression involving w and its derivatives up to order 2. Almost all analytical work involving the σ_k curvature restricts attention to the so called admissible metrics, for which the σ_k curvature, regarded as a differential operator on w is, *elliptic*. And for this reason,

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it is natural to consider $\sigma_k^{1/k}$, not σ_k, to be the analytical object of study, as $\sigma_k^{1/k}$, regarded as a differential operator on w, is concave on the second derivatives of w, and the concavity property is crucial for applying the Evans-Krylov regularity theory. For this reason in the PDE analysis of solvability results involving the σ_k curvature one is often led to imposing conditions on $\sigma_k^{1/k}$. However, as this note indicates, global geometric obstruction conditions are naturally in terms of σ_k, not $\sigma_k^{1/k}$.

For $k = 1$, Kazdan and Warner first noticed a global geometric obstruction for a function $K(x)$ on the round sphere $\mathbb{S}^n$ to be the scalar curvature of a conformal metric, expressed as

$$\int_{\mathbb{S}^n} \langle \nabla x_j, \nabla K \rangle d\,vol_g = 0, \qquad \text{for } j = 1, \cdots n+1,$$

where x_j are the coordinate functions on $\mathbb{S}^n$ from the standard embedding. Later this obstruction was extended to a general manifold involving general conformal killing vector field by Bourguignon and Ezin—note that ∇x_j generates conformal killing vector fields on $\mathbb{S}^n$. Schoen also derived local versions and used them in obtaining a priori estimates for metrics of constant scalar curvature. Here we obtain a natural generalization of these obstructions for the σ_k curvatures on a compact Riemannian manifold.

Theorem 1. *Let (M^n, g) be a compact Riemannian manifold of dimension n, $\sigma_k(g^{-1} \circ A_g)$ be the σ_k curvatures of g, for $k = 1, \ldots, n$, and X be a conformal Killing vectorfield on (M^n, g). When $k > 2$, also assume that (M^n, g) is locally conformally flat. Then*

$$\int_M \langle X, \nabla \sigma_k(g^{-1} \circ A_g) \rangle \, d\,vol_g = 0. \tag{1}$$

When $(M^n, g) = (\mathbb{S}^n, g_{\text{can}})$, the obstruction can be obtained by a more elementary variational means, as was used in our work [CHY1]. In [B] Bourguinon uses the construction of a closed 1-form on the infinite dimensional manifold consisting of metrics conformal to (M^n, g), which is also invariant under the action of conformal diffeomorphisms of (M^n, g), to prove a generalized Kazdan-Warner type identity involving the differential of the scalar curvature of g by a conformal killing vector field. He also sketches a way to obtain generalized integral identities involving the higher degree Pfaffian polynomials of the curvature of g. That method in fact can be adapted to prove (1). However, the most direct and elementary proof for (1) is to adapt the argument of Bourguinon-Ezin [BE], Schoen [S1], [S2], using the following elementary algebraic and analytic properties of the σ_k curvature.

Proposition 1 ([Rei]). *Defining the k-th Newton transform of a* 1-1 *tensor* Λ *by*

$$T_k(\Lambda) = \sum_{j=0}^{n} (-1)^j \sigma_{k-j}(\Lambda)\Lambda^j,$$

we have

(i) $(k+1)\sigma_{k+1}(\Lambda) = T_k(\Lambda)^a_b \Lambda^b_a$.

(ii) $(n-k)\sigma_k(\Lambda) = T_k(\Lambda)^a_a$.

(iii) $T_k(\Lambda)^a_b = \sigma_k(\Lambda)\delta^a_b - T_{k-1}(\Lambda)^a_c \Lambda^c_b$.

(iv) $T_k(\Lambda)^a_b = \frac{1}{k!}\sum \delta \begin{pmatrix} a_1 \cdots a_k a \\ b_1 \cdots b_k b \end{pmatrix} \Lambda^{b_1}_{a_1} \cdots \Lambda^{b_k}_{a_k}$, *where* $\delta \begin{pmatrix} a_1 \cdots a_k a \\ b_1 \cdots b_k b \end{pmatrix}$ *is the Kronecker symbol, which has a value* $+1$ *(respectively,* -1*) if* $a_1 \cdots a_k a$ *are distinct and* $b_1 \cdots b_k b$ *is an even (respectively, odd) permutation of* $a_1 \cdots a_k a$*; and a value* 0 *otherwise.*

In the following, we will interchangably use the ∇ operator with indices or indices after a comma sign, whichever is more convenient, to denote covariant differentiation in g. $T_k(g^{-1} \circ A_g)$ as a 1-1 *tensor field* on M enjoys the following analytic property.

Proposition 2 ([Rei], [V1]). *If g is locally conformally flat, or if $k = 1$, then* $\nabla_a T_k(g^{-1} \circ A_g)^a_b = 0$.

Remark. *Viaclovsky's proof of Proposition 2 in [V1] is imbedded in the middle of his computations for his Proposition 6, using Cartan's formalism. For ease of reference, we point out a straightforward tensor calculus argument adapting that of Reilly in [Rei]. The case of $k = 1$ is a direct consequence of Bianchi's identity, as* $T_1(g^{-1} \circ A_g)^a_b = \frac{1}{n-2}\left(\frac{R}{2}\delta^a_b - R^a_b\right)$. *For the case $k \geq 2$, note that in a locally conformally flat metric g, we have the following classic, yet not widely used property:*

$$A_{ab,c} = A_{ac,b}. \tag{2}$$

(2) *means that $A_{ab,c}$ as a $(0,3)$ tensor is totally symmetric.*

Using (2) *we can follow Reilly to conclude*

$$\nabla_a T^a_b = \frac{1}{k\,!} \sum_{j=1}^{k} \sum_{a=1}^{n} \sum \delta \begin{pmatrix} a_1 \cdots a_k a \\ b_1 \cdots b_k b \end{pmatrix} A^{b_1}_{a_1} \cdots \nabla_a A^{b_j}_{a_j} \cdots A^{b_k}_{a_k}.$$

The summation turns out to be 0*, because* $\nabla_a A_{a_j}^{b_j}$ *is symmetric in* a_j *and* a *by* (2)*, while the Kronecker symbol is antisymmetric in* a_j *and* a*.*

(2) *can be deduced from the definition and property of the Cotton tensor. More directly, we start from the once contracted Bianchi identity,*

$$R_{ab,c} - R_{ac,b} = -R^d{}_{abc\,,\,d}.$$

Substituting the terms in the left hand side in terms of A_{ab}*, and the right hand side in terms of the representation*

$$R_{abcd} = W_{abcd} + A_{ac}g_{bd} - A_{ad}g_{bc} + g_{ac}A_{bd} - g_{ad}A_{bc},$$

we arrive at

$$(n-3)\left[A_{ab,c} - A_{ac,b}\right] = -W^d{}_{abc\,,\,d},$$

which proves (2) *in the case* $n > 3$*. The case* $n = 3$ *requires a separate argument using the conformal flatness in dimension* 3*.*

Proof of Theorom 1. Let ϕ_t denote the local one-parameter family of conformal diffeomorphisms of (M, g) generated by X. Thus for some function w_t we have

$$\phi_t^*(g) = e^{2w_t} g =: g_t.$$

We have the following properties:

$$\sigma_k(g^{-1} \circ A_g) \circ \phi_t = \sigma_k(g_t^{-1} \circ A_{g_t}), \tag{3}$$

$$\dot{w} := \frac{d}{dt}\Big|_{t=0} w_t = div\, X/n = \nabla_a X^a/n, \tag{4}$$

$$\frac{d}{dt}\Big|_{t=0} \left(g_t^{-1} \circ A_{g_t}\right)_b^a = -\nabla_b^a \dot{w} - 2\dot{w} A_b^a, \tag{5}$$

We next point out the following useful

Fact.

$$\left(1 - \frac{2k}{n}\right) \langle X, \nabla\sigma_k\rangle = -\nabla_a \left[T_b^a \nabla^b\Big(\frac{div\, X}{n}\Big) + \frac{2k}{n}\sigma_k X^a\right], \tag{6}$$

where we use T_b^a *to denote the components of* T_{k-1} *and have dropped the dependence of* σ_k *on* g*. We will do the same in the following, whenever there is no possibility of confusion.*

Proof of (6). Using (i), (3), (4), (5) and Proposition 2, we have

$$\begin{aligned}
\langle X, \nabla\sigma_k\rangle &= T_a^b\left[-\nabla_b^a\dot{w} - 2\dot{w}A_b^a\right] \\
&= -T_a^b\nabla_b^a\dot{w} - 2k\sigma_k\dot{w} \\
&= -T_a^b\nabla_b^a\dot{w} - \frac{2k}{n}\sigma_k\nabla_b X^b \\
&= -T_a^b\nabla_b^a\dot{w} + \frac{2k}{n}\langle X, \nabla\sigma_k\rangle - \frac{2k}{n}\nabla_b\left(\sigma_k X^b\right) \\
&= -\nabla_b\left[T_a^b\nabla^a\dot{w} + \frac{2k}{n}\sigma_k X^b\right] + \frac{2k}{n}\langle X, \nabla\sigma_k\rangle
\end{aligned} \tag{7}$$

(6) now follows directly. □

We now continue our proof of Theorem 1. When $2k \neq n$, it follows directly from integrating (6). When $2k = n$, we first prove that

$$\int_M \langle X, \nabla\sigma_k(g^{-1}\circ A_g)\rangle\, d\,vol_g = -\int_M \sigma_k(g^{-1}\circ A_g)div_g X\, dvol_g$$

is independent of g within its conformal class. Let $g_t = e^{2t\eta}g$ denote any conformal variation of g. Then, noting that

$$\frac{d}{dt}\Big|_{t=0} div_{g_t}X = n\langle X, \nabla\eta\rangle,$$

we have, using again (4), (5), (6), and Proposition 2,

$$
\begin{aligned}
&\frac{d}{dt}\Big|_{t=0} \int_M \sigma_k(g_t^{-1} \circ A_{g_t}) div_{g_t} X \, dvol_{g_t} \\
&= \int_M \left\{ -T^a_b \cdot \left(\nabla^b_a \eta\right) div_g X + n\sigma_k \langle X, \nabla \eta \rangle \ \right\} dvol_g \\
&= \int_M \left\{ -T^a_b \nabla^b_a (div_g X) - n\nabla_a(\sigma_k X^a) \right\} \eta \, dvol_g \\
&=0,
\end{aligned}
\tag{8}
$$

where, in the last line, we have used (6) for the case $2k = n$.

Now we can complete the proof of Theorem 1 in the case $2k = n$ by following the argument of Bourguignon and Ezin: Either the connected component of the identity of the conformal group $C_0(M, g)$ is compact, then there is a metric $\hat{g}$ conformal to g admitting $C_0(M, g)$ as a group of isometries, from which it follows that $div_{\hat{g}} X \equiv 0$ and (1) therefore holds; or, $C_0(M, g)$ is non-compact, then by a theorem of Obata-Ferrand, (M, g) is conformal to the standard sphere, in which case we can pick the canonical metric to compute the integral on the left of (1) and conclude that it is 0. □

Another property in the case of $k = 1$ or $k < n$ and (M, g) locally conformally flat is

$$
\langle X, \nabla \sigma_k \rangle = -\frac{n}{n-k} \nabla_b \left[(T_k)^b_a X^a \right] + \nabla_b \left[\sigma_k X^b \right],
$$

where $(T_k)^b_a$ stand for the components of T_k. This can also be used to prove Theorem 1 in such cases.

This is proved in the case that (M, g) is locally conformally flat by starting with

$$
\nabla_a \sigma_k = T^b_c \nabla_a A^c_b = T^b_c \nabla_b A^c_a = \nabla_b \left[T^b_c A^c_a \right], \quad \text{using (2) and Proposition 2.}
$$

Set $H^b_a = T^b_c A^c_a$. Then $H^a_a = k\sigma_k$, and $H^b_a = \sigma_k \delta^b_a - (T_k)^b_a$ by (iii) of Proposition 1. Define

$\overset{\circ}{H}{}^b_a = H^b_a - \frac{H^c_c}{n} \delta^b_a = H^b_a - \frac{k\sigma_k}{n} \delta^b_a$. Then $\overset{\circ}{H}{}^a_a = 0$, and

$$
\nabla_b \overset{\circ}{H}{}^b_a = \nabla_b H^b_a - \frac{k}{n} \nabla_a \sigma_k = \frac{n-k}{n} \nabla_a \sigma_k.
$$

This last property can also be checked using Bianchi identity in the case $k = 1$ without the locally conformally flat condition. Thus

$$
\begin{aligned}
\langle X, \nabla\sigma_k\rangle &= \frac{n}{n-k} X^a \nabla_b \overset{\circ}{H}{}^b_a \\
&= \frac{n}{n-k} \left\{ \nabla_b \left[X^a \overset{\circ}{H}{}^b_a \right] - \nabla_b X^a \overset{\circ}{H}{}^b_a \right\} \\
&= \frac{n}{n-k} \left\{ \nabla_b \left[X^a \overset{\circ}{H}{}^b_a \right] - \frac{1}{2} \left(\nabla_b X^a + \nabla_a X^b \right) \overset{\circ}{H}{}^b_a \right\} \\
&= \frac{n}{n-k} \nabla_b \left[X^a \overset{\circ}{H}{}^b_a \right] \\
&= \frac{n}{n-k} \nabla_b \left[X^a \left(\frac{n-k}{n} \sigma_k \delta^b_a - (T_k)^b_a \right) \right] \\
&= -\frac{n}{n-k} \nabla_b \left[X^a (T_k)^b_a \right] + \nabla_b \left[\sigma_k X^b \right].
\end{aligned}
$$

References

[B] J. P. Bourguignon, *Invariants intgraux fonctionnels pour des quations aux drives partielles d'origine gomtrique*, Differential geometry, Pescola 1985, 100–108, Lecture Notes in Math., **1209**, Springer, Berlin, 1986.

[BE] J. P. Bourguignon and J. P. Ezin, *Scalar curvature functions in a conformal class of metrics and conformal transformations*, Trans. Amer. Math. Soc. **301** (1987), no. 2, 723–736.

[BV] S. Brendle and J. Viaclovsky, *A variational characterization for $\sigma_{n/2}$*, Calc. Var. PDE. **20**(2004), no.4, 399-402.

[CY3] S.-Y. Chang and P. Yang, *The Inequality of Moser and Trudinger and applications to conformal geometry*, Comm. Pure and Applied Math., Vol LVI, no. 8, August 2003, pp 1135-1150. Special issue dedicated to the memory of Jurgen K. Moser.

[CGY1] S.-Y. Chang, M. Gursky and P. Yang, *An equation of Monge-Ampère type in conformal geometry, and four-manifolds of positive Ricci curvature*, Annals of Math., **155**(2002), 709-787.

[CGY2] S.-Y. Chang, M. Gursky and P. Yang, *An a priori estimate for a fully nonlinear equation on four-manifolds*, Dedicated to the memory of Thomas H. Wolff. J. Anal. Math. **87** (2002), 151–186.

[CGY3] S.-Y. Chang, M. Gursky and P. Yang, *Entire solutions of a fully nonlinear equation*, "Lectures in Partial Differential Equations in honor of Louis Nirenberg's 75th birthday", chapter 3. International Press, 2003.

[CHY1] S.-Y. A. Chang, Z. Han, and P. Yang, *A priori estimates for solutions of the prescribed σ_2 curvature equation on S^4*, work in preparation.

[CHY2] S.-Y. A. Chang, Z. Han, and P. Yang, *Classification of singular radial solutions of the σ_k-Yamabe equation on annular domains in $\mathbb{R}^n$*, submitted. Available at http://www.math.rutgers.edu/~zchan.

[CHgY] S.-Y. A. Chang, F. Hang, and P. Yang, *On a class of locally conformally flat manifolds*, IMRN, **No.4**(2004), 185-209.

[ChW] K.S. Chou and X.-J. Wang, *Variational theory of the Hessian equations*, Comm. Pure Appl. Math., **54**(2001), 1029-1064.

[F] H. Fang, *Conformally invariant curvatures on LCF manifolds*, preprint.

[G1] M. González, *Singular sets of a class of fully non-linear equations in conformal geometry*, Ph.D thesis, Princeton University, 2004.

[G2] M. González, *Singular sets of a class of locally conformally flat manifolds*, preprint, 2004.

[G3] M. González, *Removability of singularities for a class of fully non-linear equations*, preprint, 2004.

[GLW] P. Guan, C.S. Lin and G. Wang, *Application of The Method of Moving Planes to Conformally Invariant Equations*, Mathematische Zeitschrift **247** (2004), no.1, 1–19.

[GVW] P. Guan, J. Viaclovsky and G. Wang, *Some properties of the Schouten tensor and applications to conformal geometry*, Transactions of American Math. Society, **355** (2003), 925-933.

[GW1] P. Guan and G. Wang, *Local estimates for a class of fully nonlinear equations arising from conformal geometry*, International Mathematics Research Notices, V. 2003, Issue 26(2003), 1413-1432.

[GW2] Pengfei Guan and Guofang Wang, *A fully nonlinear conformal flow on locally conformally flat manifolds*, Journal fur die reine und angewandte Mathematatik, **557** (2003), 219-238.

[GV1] M. Gursky and J. Viaclovsky, *A new variational characterization of three-dimensional space forms*, Invent. Math. **145** (2001), 251–278.

[GV2] M. Gursky and J. Viaclovsky, *Fully nonlinear equations on Riemannian manifolds with negative curvature*, Indiana Univ. Math. J. **52** (2003), no. 2, 399-419.

[GV3] M. Gursky and J. Viaclovsky, *A fully nonlinear equation on four-manifolds with positive scalar curvature*, Journal of Differential Geometry **63**(2003), no.1, 131–154.

[LL1] Aobing Li and YanYan Li, *On some conformally invariant fully nonlinear equations*, Comm. Pure Appl. Math. **56** (2003), 1416-1464.

[LL2] Aobing Li and YanYan Li, *A fully nonlinear version of the Yamabe problem and a Harnack type inequality*, C.R. Math. Acad. Sci. Paris **336** (2003), no.4, 319–324.

[LL3] Aobing Li and YanYan Li, *A Liouville type theorem for some conformally invariant fully nonlinear equations*, arXiv:math.AP/0212376 v1 30 Dec 2002, 10 pages.

[LL4] Aobing Li and YanYan Li, *A general Liouville type theorem for some conformally invariant fully nonlinear equations*, arXiv:math.AP/0301239 v1 21 Jan 2003, 16 pages.

[LL5] Aobing Li and YanYan Li, *Further results on Liouville type theorems for some conformally invariant fully nonlinear equations*, arXiv:math.AP/0301254 v1 22 Jan 2003, 9 pages.

[LL6] Aobing Li and YanYan Li, *On some conformally invariant fully nonlinear equations, Part II: Liouville, Harnack and Yamabe*, preprint, 2004.

[Rei] R. Reilly, *Applications of the Hessian operator in a Riemannian manifold.* Indiana Univ. Math. J. **26** (1977), no. 3, 459–472.

[S1] R. Schoen, *Variational theory for the total scalar curvature functional for Riemannian metrics and related topics.* Topics in calculus of variations (Montecatini Terme, 1987), 120–154, Lecture Notes in Math., 1365, Springer, Berlin, 1989.

[S2] R. Schoen, *On the number of constant scalar curvature metrics in a conformal class.* Differential geometry, 311–320, Pitman Monogr. Surveys Pure Appl. Math., 52, Longman Sci. Tech., Harlow, 1991.

[TW1] N. Trudinger and X. J. Wang, *Hessian measures. I*, Dedicated to Olga Ladyzhenskaya. Topol. Methods Nonlinear Anal. **10** (1997), no. 2, 225–239.

[TW2] N. Trudinger and X. J. Wang, *Hessian measures. II*, Ann. of Math. (2) **150** (1999), no. 2, 579–604.

[TW3] N. Trudinger and X. J. Wang, *Hessian measures. III*, J. Funct. Anal. **193** (2002), no. 1, 1–23.

[V1] J. Viaclovsky, *Conformal geometry, contact geometry, and the calculus of variations*, Duke Math. J. **101, No. 2** (2000), 283-316.

[V2] J. Viaclovsky, *Estimates and existence results for some fully nonlinear elliptic equations on Riemannian manifolds*, Communications in Analysis and Geometry **10** (2002), no.4, 815-846.

[V3] J. Viaclovsky, *Conformally Invariant Monge-Ampère Equations: Golbal Solutions*, Trans. Amer. Math. Soc. **352** (2000), no. 9, 4371–4379.