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Partial Differential Equations

A Kazdan–Warner type identity for the σ_k curvature

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Abstract

We prove a Kazdan–Warner type identity involving the σ_k curvature and a conformal Killing vector field on a compact manifold. Our method also works to provide a unified proof for the necessary conditions in the Christoffel–Minkowski problem. ***To cite this article: Z.-C. Han, C. R. Acad. Sci. Paris, Ser. I 342 (2006).***

Résumé

Une identité de type Kazdan–Warner pour la σ_k-courbure. Nous prouvons une identité de type Kazdan–Warner reliant la σ_k-courbure et un champ de vecteurs conforme sur une variété compacte. Notre méthode permet aussi de fournir une preuve unifiée pour les conditions nécessaires dans le problème de Christoffel–Minkowski. ***Pour citer cet article : Z.-C. Han, C. R. Acad. Sci. Paris, Ser. I 342 (2006).***

1. Introduction and statement of results

The Schouten tensor A_g of a metric g is defined to be

$$A_g = \frac{1}{n-2}\left\{Ric_g - \frac{Scal_g}{2(n-1)}g\right\}.$$

The σ_k curvature of g is defined to be the kth elementary symmetric function of the eigenvalues of the 1-1 tensor $g^{-1} \circ A_g$. σ_1 of g is simply a dimensional constant multiple of the scalar curvature of g. Since the first systematic study of the σ_k curvature in the thesis of Viaclovsky [19] there has been very intensive research and progress on an extensive list of geometrical and PDE problems involving the σ_k curvature of a metric for $k > 1$, mostly involving

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a conformal change of metric—more than 40 publications have appeared in the last few years, one can begin with [4,8,9,15] for recent work in this area and further references. Since the Schouten tensor transforms as

$$A_g = A_{g_0} - \left[\nabla^2 w - \mathrm{d}w \otimes \mathrm{d}w + \frac{1}{2}|\nabla w|^2 g_0\right],$$

under a conformal change of metric $g = \mathrm{e}^{2w(x)} g_0$, the σ_k curvature of g, when $k \geqslant 2$, is then expressed as a fully non-linear expression involving w and its derivatives up to order 2. Almost all analytical work involving the σ_k curvature restricts attention to the so called *admissible* metrics, for which the σ_k curvature, regarded as a differential operator on w is, *elliptic*. For this reason, it is natural to consider $\sigma_k^{1/k}$, not σ_k, to be the analytical object of study, as $\sigma_k^{1/k}$, regarded as a differential operator on w, is concave on the second derivatives of w, and the concavity property is crucial for applying the Evans–Krylov regularity theory. Also for this reason in the PDE analysis of solvability results involving the σ_k curvature one is often led to imposing conditions on $\sigma_k^{1/k}$. However, as this note indicates, global geometric obstruction conditions are naturally in terms of σ_k, not $\sigma_k^{1/k}$.

For $k = 1$, Kazdan and Warner [11,12] first noticed a global geometric obstruction for a function $K(x)$ on the round sphere $\mathbb{S}^n$ to be the scalar curvature of a conformal metric g, expressed as

$$\int_{\mathbb{S}^n} \langle \nabla x_j, \nabla K \rangle \, \mathrm{d}vol_g = 0, \quad \text{for } j = 1, \ldots, n+1,$$

where x_j are the coordinate functions on $\mathbb{S}^n$ from the standard embedding. Later these obstructions were extended to manifolds involving a general conformal Killing vector field by Bourguignon [1], Bourguignon and Ezin [2]—note that ∇x_j generates conformal Killing vector fields on $\mathbb{S}^n$. Schoen also derived local versions [18,13] and used them in the construction and a priori estimates for metrics of constant scalar curvature. Here we obtain a natural generalization of these obstructions for the σ_k curvatures.

Theorem 1. *Let (M^n, g) be a compact Riemannian manifold of dimension $n \geqslant 3$, $\sigma_k(g^{-1} \circ A_g)$ be the σ_k curvatures of g, and X be a conformal Killing vector field on (M^n, g). When $k > 2$, also assume that (M^n, g) is locally conformally flat. Then*

$$\int_M \left\langle X, \nabla \sigma_k\left(g^{-1} \circ A_g\right)\right\rangle \mathrm{d}vol_g = 0. \tag{1}$$

These obstructions can be obtained by a variational means, as was done in [10,5,6], and play important roles in proving a priori estimates for metrics in terms of their σ_k curvatures. The method in [1] uses the construction of a closed 1-form on the infinite dimensional manifold consisting of metrics conformal to (M^n, g), invariant under the action of conformal diffeomorphisms of (M^n, g). In [1] Bourguignon also sketches a way to obtain generalized integral identities involving the higher degree Pfaffian polynomials of the curvature of g. That method in fact can also be adapted to prove (1) using the information in [3,5,19]. However, both proofs in [1] and [2] need to appeal to the Lelong–Ferrand–Obata theorem [14,16]. A more direct and elementary proof for (1), which also produces *local* balancing identities useful for proving a priori bounds, is with tensor calculus using the following elementary algebraic and analytic properties of the σ_k curvature—this proof can be thought of as adaptions of the arguments in [2] and [18].

Proposition 2. [17,19] *Define $T_k(\Lambda) = \sum_{j=0}^{k} (-1)^j \sigma_{k-j}(\Lambda)\Lambda^j$. Then we have*

(i) $(k+1)\sigma_{k+1}(\Lambda) = T_k(\Lambda)_b^a \Lambda_a^b$.
(ii) $\nabla_c A_{ab} = \nabla_b A_{ac}$, *if g is locally conformally flat.*
(iii) $\nabla_a T_k(g^{-1} \circ A_g)_b^a = 0$, *if either $k = 1$ or g is locally conformally flat.*

Remark 3. The conclusion in (iii) follows from that in (ii) as in [17]. From the proof below, the following is evident: for any symmetric $(0, 2)$ tensor A satisfying the conclusion in (ii), (1) would hold for $k < n$. This unifies the proof for the necessary conditions in the Christoffel–Minkowski problem [7] with the case here for the σ_k curvatures when $k < n$. A peculiar feature is that the case $k = n$ needs be handled separately, while known properties of the σ_k curvatures, e.g. see (3) below, and the proof in [2] suggest that $2k = n$ may need to be handled separately.

2. Proof of Theorem 1

Our proof is based on the following properties

$$\frac{n-k}{n}\nabla_a\sigma_k = \nabla_b \mathring{H}_a^b, \tag{2}$$

$$(n-2k)\langle X, \nabla\sigma_k\rangle = -\nabla_a\big[T_b^a\nabla^b(\operatorname{div} X) + 2k\sigma_k X^a\big], \tag{3}$$

where $\mathring{H}_a^b = H_a^b - \frac{H_c^c}{n}\delta_a^b$, $H_a^b = T_c^b A_a^c$, and T_b^a denote the components of T_{k-1}.

Assuming (2) and (3), we can conclude our proof of Theorem 1 as follows. Based on (2), we have, for any conformal Killing vector field X^a,

$$\frac{n-k}{n}\langle X, \sigma_k\rangle = \nabla_b\big(X^a\mathring{H}_a^b\big) - \nabla_b X^a \mathring{H}_a^b = \nabla_b\big(X^a\mathring{H}_a^b\big), \tag{4}$$

where we have used $\nabla_b X^a + \nabla_a X^b = \frac{2\operatorname{div} X}{n}\delta_a^b$, $\mathring{H}_a^a = 0$, and $\mathring{H}_{ac} := g_{bc}\mathring{H}_a^b$ is symmetric in a and c. Theorem 1 follows from integrating (4) over M when $k \neq n$, or integrating (3) over M when $k = n$.

Proof of (2). When (ii) in Proposition 2 holds, (iii) also holds, and we have

$$\nabla_a\sigma_k = T_c^b\nabla_a A_b^c = T_c^b\nabla_b A_a^c = \nabla_b\big[T_c^b A_a^c\big] = \nabla_b H_a^b.$$

This also holds for $k = 1$ without knowing (ii) by Bianchi identities. Then using $H_a^a = k\sigma_k$, which follows from (i) of Proposition 2, we conclude

$$\nabla_b\mathring{H}_a^b = \nabla_b H_a^b - \frac{k}{n}\nabla_a\sigma_k = \frac{n-k}{n}\nabla_a\sigma_k.$$

Proof of (3). Let ϕ_t denote the local one-parameter family of conformal diffeomorphisms of (M, g) generated by X. Thus for some function w_t we have $\phi_t^*(g) = e^{2w_t}g =: g_t$. We have the following properties:

$$\sigma_k\big(g^{-1}\circ A_g\big)\circ\phi_t = \sigma_k\big(g_t^{-1}\circ A_{g_t}\big), \tag{5}$$

$$\dot{w} := \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} w_t = \operatorname{div} X/n = \nabla_a X^a/n, \tag{6}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0}\big(g_t^{-1}\circ A_{g_t}\big)_b^a = -\nabla_b^a\dot{w} - 2\dot{w}A_b^a. \tag{7}$$

Using (5)–(7) and Proposition 2, we conclude (3) by

$$\begin{aligned}
\langle X, \nabla\sigma_k\rangle &= T_a^b\big[-\nabla_b^a\dot{w} - 2\dot{w}A_b^a\big] \\
&= -T_a^b\nabla_b^a\dot{w} - 2k\sigma_k\dot{w} \\
&= -T_a^b\nabla_b^a\dot{w} - \frac{2k}{n}\sigma_k\nabla_b X^b \\
&= -T_a^b\nabla_b^a\dot{w} + \frac{2k}{n}\langle X, \nabla\sigma_k\rangle - \frac{2k}{n}\nabla_b\big(\sigma_k X^b\big) \\
&= -\nabla_b\left[T_a^b\nabla^a\dot{w} + \frac{2k}{n}\sigma_k X^b\right] + \frac{2k}{n}\langle X, \nabla\sigma_k\rangle.
\end{aligned} \tag{8}$$

Remark 4. (2) and (4) depend only on (ii), while (3) depends also on the conformal transformation laws for A. For the Christoffel–Minkowski problem, one looks for a convex hypersurface whose kth Weingarten curvature (the kth elementary symmetric function of the principal curvatures) at its point with normal vector ν is $W_k(\nu)$. Let $u(\nu)$ denote the support function of the surface, then $W_k^{-1}(\nu) = \sigma_k(u_{ab}(\nu) + u(\nu)\delta_{ab})$, and $A_{ab} := u_{ab}(\nu) + u(\nu)\delta_{ab}$ satisfy the conclusion in (ii). Thus it follows from (4) that

$$\int_{\mathbf{S}^n}\frac{\nu_i}{W_k(\nu)}\,\mathrm{d}vol_{\mathbf{S}^n}(\nu) = \int_{\mathbf{S}^n}\nu_i\sigma_k\big(u_{ab}(\nu) + u(\nu)\delta_{ab}\big)\,\mathrm{d}vol_{\mathbf{S}^n}(\nu) = -\frac{1}{n}\int_{\mathbf{S}^n}\langle\nabla\nu_i, \nabla\sigma_k\rangle\,\mathrm{d}vol_{\mathbf{S}^n}(\nu) = 0,$$

for $k < n$. The case $k = n$ follows from a direct integration by parts using $\sigma_k(u_{ab}(\nu) + u(\nu)\delta_{ab}) = T_{ab}(u_{ab}(\nu) + u(\nu)\delta_{ab})$, $T_{ab,b} = 0$, and $\nabla_{ab}\nu_i = -\nu_i\delta_{ab}$.

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