Local pointwise estimates for solutions of the σ_2 curvature equation on 4 manifolds

Zheng-Chao Han *
Department of Mathematics
Rutgers University
110 Frelinghuysen Road
Piscataway, NJ 08854
zchan@math.rutgers.edu

Abstract

The study of the k-th elementary symmetric function of the Weyl-Schouten curvature tensor of a Riemannian metric, the so called σ_k curvature, has produced many fruitful results in conformal geometry in recent years, especially when the dimension of the underlying manifold is 3 or 4. In these studies in conformal geometry, the deforming conformal factor is considered to be a solution of a fully nonlinear elliptic PDE. Important advances have been made in recent years in the understanding of the analytic behavior of solutions of the PDE, including the adaptation of Bernstein type estimates in integral form, global and local derivative estimates, classification of entire solutions and analysis of blowing up solutoins. Most of these results require derivative bounds on the σ_k curvature. The derivative estimates also require an a priori L^∞ bound on the solution. This work provides local L^∞ and Harnack estimates for solutions of the σ_2 curvature equation on 4 manifolds, under only L^p bounds on the σ_2 curvature, and the natural assumption of small volume(or total σ_2 curvature).

1 Introduction and Statements of the results

This paper addresses local L^∞ and Harnack estimates for *admissible* solutions w to either

$$\sigma_2(g^{-1} \circ A_g) = K(x), \tag{1}$$

*The author acknowledges the support on this work by NSF through grant DMS-0103888, and the encouragement of Professors H. Brezis, S.-Y. A. Chang and P. Yang.

or

$$\sigma_2(g_0^{-1} \circ A_g) = f(x), \tag{2}$$

where g_0 is a fixed background metric on a 4-manifold M^4 and $g = e^{2w(x)}g_0$ is a metric conformal to g_0, A_g is the the Weyl-Schouten tensor of the metric g,

$$\begin{aligned} A_g &= \frac{1}{n-2}\{Ric - \frac{R}{2(n-1)}g\} \\ &= A_{g_0} - \left[\nabla^2 w - dw \otimes dw + \frac{1}{2}||\nabla w||^2 g_0\right], \end{aligned} \tag{3}$$

and $\sigma_k(\Lambda)$, for any $1-1$ tensor Λ on an $n-$dimensional vector space and $k \in \mathbb{N}$, $0 \le k \le n$, is the k-th elementary symmetric function of the eigenvalues of Λ; $K(x)$ and $f(x)$ are two nonnegative functions with only some *integrability* assumptions, and an *admissible* solution is defined to be a $w \in C^2(M^4)$ such that for all $x \in M^4$, $A_g(x) \in \Gamma_2^+$ (see next paragraph for the definition of Γ_2^+) and (1) or (2) is satisfied. Note that, since

$$\sigma_2(g^{-1} \circ A_g) = e^{-4w}\sigma_2(g_0^{-1} \circ A_g),$$

so a solution of (1) is a solution of (2) with $f(x) = K(x)e^{4w}$.

It is natural to restrict to metrics whose Weyl-Schouten tensor is in the Γ_k^+ class, *i.e.*, those metrics such that $\sigma_j(g^{-1} \circ A_g) > 0$ for $1 \le j \le k$, because, *for a metric in such a class with* $k > 1$, (i) $\sigma_k(g^{-1} \circ A_g)$ places a much stronger control on the curvature tensor: Chang, Gursky and Yang [CGY1] observed that if $\sigma_1(g^{-1} \circ A_g), \sigma_2(g^{-1} \circ A_g) > 0$ at a point on a 4-dimensional manifold, then the Ricci tensor of g is positive definite at that point; this algebraic relation has been generalized to higher dimensions by Guan, Viaclovsky, and Wang [GVW]; (ii) the expression $\sigma_k(g^{-1} \circ A_g)$ is a *fully nonlinear* PDO in w that becomes *elliptic*. Another reason the study of σ_k curvature has attracted strong interest in recent years is its appearance in conforml invariant. Viaclovsky establishes in [V1] that when $2k = n$ on a closed manifold M^n, and M is locally conformally flat if $k > 2$, the integral $\int_M \sigma_k(g^{-1} \circ A_g) d\, vol_g$ is conformally invariant. In fact, in dimension 4, the σ_2 curvature comes into play in the Chern-Gauss-Bonnet formula. The first important application of the σ_k-curvature to conformal geometry is the main theorem in [CGY1], where the authors prove that if (a) $\int_{M^4} \sigma_2(A_g) d\, vol_g$, which is conformally invariant on M^4, is positive; and (b) the Yamabe class of (M^4, g) is positive, then there is a conformal metric $\tilde{g} = e^{2w}g$ on M^4 such that $A_{\tilde{g}} \in \Gamma_2^+$. Note that $\sigma_1(A_g)$ is simply a constant multiple of the scalar curvature of g, so A_g in the Γ_k^+ class is a generalization of the notion that the scalar curvature R_g of g having a fixed $+$ sign. Further applications of the σ_k curvature in geometry appear in [CGY2, CGY3, GV1, GV2, GV3].

Note that $\sigma_k(g^{-1} \circ A_g)$ has a divergence structure as given by

Proposition 1. *If $g = e^{2w}|dx|^2$ is locally conformally flat, then*

$$k\sigma_k(g^{-1} \circ A_g) = (n-2k)\sum_{j=1}^{k} \frac{\sigma_{k-j}(g^{-1} \circ A_g)}{2^j}||\nabla w||_g^{2j} - \nabla_a X^a, \tag{4}$$

where

$$X^a = \left[\sum_{j=1}^{k} \frac{T_{k-j}(g^{-1} \circ A_g)^a_b ||\nabla w||_g^{2(j-1)}}{2^{j-1}}\right] \nabla^b w,$$

$T_{k-j}(g^{-1} \circ A_g)^a_b$ is the $(k-j)$-th Newton transform of $g^{-1} \circ A_g$:

$$T_{k-j}(g^{-1} \circ A_g) = \sum_{i=0}^{k-j}(-1)^i \sigma_{k-j-i}(g^{-1} \circ A_g)(g^{-1} \circ A_g)^i.$$

In this theorem all norms and differentiation are instrinsic with g.

We remind the reader that $T_{k-j}(g^{-1} \circ A_g)$ is positive definite for $1 \leq j \leq k$ when $A_g \in \Gamma_k^+$, see [V2]. Also note that when $2k = n$, $\sigma_k(g^{-1} \circ A_g)$ is a pure divergence, confirming the results of [V1] as mentioned above. In her thesis [G1] Gonzalez also exploits the divergence strucuture of $\sigma_k(g^{-1} \circ A_g)$. See also [G2, G3]. Her work deals with the case $2k < n$ as opposed to our case where $2k = n$. The case she deals with exhibits somewhat different analytical behavior, as hinted by the extra terms in (4) with definite signs after $(n-2k)$. She does not give an expression such as (4), instead, uses an inductive relation between $\sigma_k(g^{-1} \circ A_g)$ and $\sigma_{k-1}(g^{-1} \circ A_g)$. (4) appears to be based on the same analytical structure as exploited in [G1].

Since most of the geometric applications involving σ_k deal with the case of $k = 2$ in dimension 4, we limit ourselves to this case in this paper. For simplicity of presentation, we will take the background metric $g_0 = |dx|^2$ to be the flat metric in a ball in $\mathbb{R}^4$. For a general g_0, only minor computational changes are needed. In our case, the left hand sides of (2) can be written as divergence in the background metric g_0:

$$2\sigma_2(g_0^{-1} \circ A_g) = -\partial_a(M^a_b \partial^b w), \tag{5}$$

where

$$M^a_b = T_1(g_0^{-1} \circ A_g)^a_b + \frac{|\nabla w|^2}{2}\delta^a_b.$$

Here and in the remaining of the paper, ∇w denotes the gradient of w in the background metric g_0. (5), which follows from (4), is already exploited in [CGY2].

It turns out to make sense to discuss a weak notion of admissible solution (subsolution, supersolution) of (1) or (2). An admissible $W^{2,2}$ solution (subsolution, supersoluton) of (1) or (2) is a $w \in W^{2,2}(M^4)$ such that, for *a.e.* x, $A_g(x) \in \Gamma_2^+$, and the left hand side of (1) or (2) $= (\leq, \geq)$ the right hand side. In Propositions 2 and 3 we will consider pointwise upper (lower) bound and weak Harnack inequality for a $W^{2,2}$ admissible subsolution (supersolution) of (2), respectively. In Theorems 2 and 3 we provide Harnack inequality for $W^{2,2}$ admissible solutions of (2). For geometric applications involving solutions of (1), the following estimate under small volume is probably most useful.

Theorem 1. *Assume* $0 \leq \inf_{B_{2R}} K \leq \sup_{B_{2R}} K < \infty$. *There exist absolute constant* $\epsilon_0 > 0$ *small and* $C^* > 0$ *depending on* $(\sup_{B_{2R}} K) \int_{B_{2R}} e^{4w}\, dvol_{g_0}$ *such that for any admissible solution* w *of* (1) *on* $B_{2R} \subset \mathbb{R}^4$, *if*

$$\int_{B_{2R}} K(x) e^{4w}\, dvol_{g_0} < \epsilon_0, \tag{6}$$

then

$$\sup_{B_R} e^w \leq C^* \left(\frac{1}{R^4} \int_{B_{2R}} e^{4w}\, dvol_{g_0} \right)^{\frac{1}{4}},$$

and

$$\sup_{B_R} e^w \leq C^* \inf_{B_R} e^w.$$

Remark. *From the proof it will be clear that the condition* $\sup_{B_{2R}} K < \infty$ *may be relaxed to* $||K||_p < \infty$ *for some* $p > 1$. *Then* ϵ_0 *depends on* p *and* C^* *also depends on* p *as well as on* $R^{-4/p}||K||_{L^p(B_{2R})} \int_{B_{2R}} e^{4w}\, dvol_{g_0}$. *If one is willing to assume* $\sup_{B_{2R}} K < \infty$ *and the smallness of* $(\sup_{B_{2R}} K) \int_{B_{2R}} e^{4w}\, dvol_{g_0}$, *then the following slightly different, less geometric version of Theorem 1, is much easier to prove.*

Theorem $1'$**.** *Assume* $0 \leq \inf_{B_{2R}} K \leq \sup_{B_{2R}} K < \infty$. *There exist absolute constants* $\epsilon_1 > 0$ *small and* $C_* > 0$ *such that for any admissible solution* w *of* (1) *on* $B_{4R} \subset \mathbb{R}^4$, *if*

$$(\sup_{B_{2R}} K) \int_{B_{2R}} e^{4w}\, dvol_{g_0} < \epsilon_1,$$

then

$$\sup_{B_R} e^w \le C_* \left(\frac{1}{R^4} \int_{B_{2R}} e^{4w} \, dvol_{g_0} \right)^{\frac{1}{4}},$$

and

$$\sup_{B_R} e^w \le C^* \inf_{B_R} e^w.$$

Theorems 1 and 1′ are consequences of the following propositions and theorems, which provide pointwise and Harnack estimates for solutions/subsolutions/supersolutions of the fully nonlinear equation (2), and are of independent interest.

Proposition 2. *Assume w is an admissible $W^{2,2}$ subsolution of (2) on $B_{2R} \subset \mathbb{R}^4$, and $||f||_{L^p(B_{2R})} < \infty$ for some $p > 1$. Then w is bounded from above on B_R and for any $\beta > 0$, there exists*

$$C = C(p, R^{4(1-\frac{1}{p})} ||f||_{L^p(B_{2R})}, \beta) > 0$$

such that

$$\sup_{B_R} e^w \le C \left(\frac{1}{R^4} \int_{B_{2R}} e^{\beta w} d\, vol_{g_0} \right)^{\frac{1}{\beta}}, \tag{7}$$

and there exists $C^* = C^*(p, \beta) > 0$ such that, if $m \le w \le M$ on B_{2R}, and $\gamma = \max(1, R^{\frac{4}{3}(1-\frac{1}{p})} ||f||^{\frac{1}{3}}_{L^p(B_{2R})})$, then

$$\sup_{B_R} (\gamma + w - m) \le C^* \left(\frac{1}{R^4} \int_{B_{2R}} (\gamma + w - m)^\beta \right)^{\frac{1}{\beta}}, \tag{8}$$

$$\left(\frac{1}{R^4} \int_{B_{2R}} (\gamma + M - w)^\beta \right)^{\frac{1}{\beta}} \le C^* \inf_{B_R} (\gamma + M - w). \tag{9}$$

Here and in the following, the integrals are taken with respect to the measure generated by g_0.

Proposition 3. *Assume w is an admissible $W^{2,2}$ supersolution of (2) on $B_{2R} \subset \mathbb{R}^4$, and $||f||_{L^p(B_{2R})} < \infty$ for some $p > 1$. Then w is bounded from below on B_R and for any $\beta > 0$, there exists*

$$C = C(p, R^{4(1-\frac{1}{p})}||f||_{L^p(B_{2R})}, \beta) > 0$$

such that

$$\inf_{B_R} e^w \geq C \left(\frac{1}{R^4} \int_{B_{2R}} e^{-\beta w} d\, vol_{g_0} \right)^{\frac{1}{-\beta}}, \tag{10}$$

and there exists $C^* = C^*(p, \beta) > 0$ such that, if $m \leq w \leq M$ on B_{2R}, and $\gamma = \max(1, R^{\frac{4}{3}(1-\frac{1}{p})}||f||^{\frac{1}{3}}_{L^p(B_{2R})})$, then

$$\sup_{B_R}(\gamma + M - w) \leq C^* \left(\frac{1}{R^4} \int_{B_{2R}} (\gamma + M - w)^\beta \right)^{\frac{1}{\beta}}, \tag{11}$$

$$\left(\frac{1}{R^4} \int_{B_{2R}} (\gamma + w - m)^\beta \right)^{\frac{1}{\beta}} \leq C^* \inf_{B_R}(\gamma + w - m). \tag{12}$$

Remark. *As will be seen later in the proofs, the estimates of Proposition 3 follow from the same scheme of proof as for Proposition 2. However, slightly different formulations of the estimates of Proposition 3 follow easily by the superharmonicity of w, which is a consequence of $A_{e^{2w}g_0} \in \Gamma_2^+$. We formulate them in conjuction with those of Proposition 2, because, together, they give the following*

Theorem 2. *Let w be an admissible $W^{2,2}$ solution of (2) on $B_{2R} \subset \mathbb{R}^4$, and $||f||_{L^p(B_{2R})} < \infty$ for some $p > 1$. Set $\gamma = \max(1, R^{\frac{4}{3}(1-\frac{1}{p})}||f||^{\frac{1}{3}}_{L^p(B_{2R})})$. Then there exists $C^* = C^*(p) > 0$ such that,*

(i) if M is an upper bound of w on B_{2R}, then we have

$$\sup_{B_R}(\gamma + M - w) \leq C^* \inf_{B_R}(\gamma + M - w), \tag{13}$$

(ii) if m is a lower bound of w on B_{2R}, then we have

$$\sup_{B_R}(\gamma + w - m) \leq C^* \inf_{B_R}(\gamma + w - m). \tag{14}$$

A different formulation of Harnack estimate in terms of e^w is given as

Theorem 3. *If w is an admissible solution of* (2) *on $B_{2R} \subset \mathbb{R}^4$, and $||f||_{L^p(B_{2R})} < \infty$ for some $p > 1$, then there exists $C = C(p, R^{4(1-\frac{1}{p})}||f||_{L^p(B_{2R})})$ such that*

$$\sup_{B_R} e^w \le C \inf_{B_R} e^w. \tag{15}$$

Let us put our results into perspective. Of the many important, recent analytic contributions on related problems, this work is more directly related to [V2, GW1], although it is also closely related to [CGY1]-[CGY2] and [G1]. For equations of the same type as (1), but with general k and n, Viaclovsky establishes in [V2] *global* C^1 and C^2 estimates for C^4 admissible solutions, *assuming* C^0 *estimate on the solution.* In [CGY1]-[CGY2], Chang, Gursky and Yang develop important integral estimates for related equations, some for a singularly perturbed fourth order equation. Later Guan and Wang establish in [GW1] *local* C^1 and C^2 estimates for C^4 admissible solutions of equations similar to those in [V2], *assuming a one-sided* C^0 *estimate on the solution.* All these results require *derivative bounds of* $K(x)$, of course, also provide stronger estimates, namely derivative estimates. Similar results were then proved for a more general class of fully nonlinear equations by A. Li and YanYan Li in [LL1]-[LL6], where they also establish Liouville type theorems, Harnack type theorems in the sense of Schoen, compactness and existence results. All these are very important and useful results. In fact, if C^1 bounds on K are allowed, then the conclusion of Theorem 1 is covered by [GW1]. However, for some applications in blowing up analysis, the derivative bounds of $K(x)$ in the assumptions of [V2] and [GW1] are absent, and only some L^p bounds are under control. Our Theorem 1 provides a partial substitute.

Using our results, we establish the following theorem in a joint work with S.-Y. A. Chang and P. Yang [CHY1].

Theorem 4. *Consider a family of admissible conformal metrics $g_j = e^{2w_j} g_c$ on $\mathbb{S}^4$ with $\sigma_2(g_j^{-1} \circ A_{g_j}) = K(x)$, where g_c denotes the canonical round metric on $\mathbb{S}^4$ and $K(x)$ denotes a fixed L^∞ function on $\mathbb{S}^4$ with a positive lower bound.*

1. *There exists at most one isolated simple blow up point in the sense that, if* $\max w_j = w_j(P_j) \to \infty$, *then there exists conformal automorphism φ_j of $\mathbb{S}^4$ such that, if we define $v_j(P) = w_j \circ \varphi_j(P) + \ln|\det(d\varphi_j)|$, we have*

$$v_j(P) - \frac{1}{4} \ln \frac{6}{K(P_j)} \to 0 \quad \textit{in} \quad L^\infty \quad \textit{and} \quad \int_{\mathbb{S}^4} |\nabla v_j|^4 \to 0.$$

2. If, furthermore, $K(x)$ is C^2 and satisfies a non-degeneracy condition

$$\Delta K(P) \neq 0 \quad \textit{whenever} \quad \nabla K(P) = 0.$$

Then there exists apriori $C^{2,\alpha}$ estimate on w_j depending on $\max K$, $\min K$, the C^2 norm of K and the modulus of continuity of $\nabla^2 K$.

Propositions 2, 3, Theorems 2 and 3 are proved by a Moser iteration scheme. However, the fully nonlinear equations (1) or (2), when regarded as an elliptic equation in w in divergence form, is *not* uniformly elliptic. Moser iteration procedures have been successfully employed to deal with some non-uniformly elliptic quasilinear equations, see the classic paper [S], and, for general reference, also the monographs [GT, LU]. But they do not directly apply to our situation. We establish the Moser iteration scheme by exploiting the special divergence structure in the equation through the following Lemma.

Main Lemma. *Let $G(w)$ be a nonnegative Lipschitz function of w. If w is an admissible subsolution of (2), we will require $G'(w) \geq 0$; and if w is an admissible supersolution of (2), we will require $G'(w) \leq 0$. Let $\eta \in C_0^2(B_2)$ be a non-negative cut-off function on B_2 satisfying $\eta \equiv 1$ on B_1 and $\eta|\nabla^2\eta| \lesssim |\nabla\eta|^2$. Then*

$$\int_{B_2} \eta^4 |G'(w)||\nabla w|^4 \lesssim \int_{B_2} \eta^2 |G(w)| \left[|\nabla w|^2 |\nabla \eta|^2 + |\nabla w|^3 \eta |\nabla \eta| \right] + \int_{B_2} \eta^4 |G(w)||f|, \quad (16)$$

here and in the following, the integrations are all done with respect to the background metric g_0, and we write $X \lesssim Y$ when there is an absolute constant $c > 0$, depending perhaps only on dimension, such that $X \leq cY$.

It turns out that [CGY2] already used an integral estimate like the one in the Main Lemma for a specific $f \equiv 0$ and $G(w) = w - \bar{w}$, where $\bar{w}$ is the average of w on B_{2R}, although they only used that as a step in the classification of entire solutions and did not pursue the iteration of such integral estimates as done here. Most of the results here were obtained in 2002; some were in slightly different formulations and had different proofs. In particular, my orginal formulation and proof of Theorem 1, using harmonic approximation, requied some boundary information of the solution. I wish to thank Professors Chang and Yang for their questioning of my earlier proof, which prompted me to find the current better proof. I would also like to call attention to [G1, G2, G3], where, as mentioned earlier, M. Gonzalez also exploits the divergence structure of σ_k and adapts the Moser iteration scheme. However, other than these two similarities on methods, which were developed independently — we didn't learn of each other's work until after we both completed our work and began to report on them, there is no overlap between our work. Our work originated from different motivations and address different situations: this work was motived mainly for applications in apriori estimates such as in

Theorem 4; while Gonzalez's work was mostly for studying the size of the singular set of solutions to the σ_k equations of the type similar to (1) with $2k < n$ and their removability. In fact, one of Gonzalez's theorems says that, when $2k < n$, an isolated singularity of the σ_k Yamabe equation with finite volume is removable. The corresponding statement in our case does not hold, despite our local L^∞ estimates under small volume. This can be seen from solutions in my joint work [CHY2] with S.-Y. A. Chang and P. Yang.

2 Sketch of proofs

We will first indicate how Theorem $1'$ follows from Proposition 2 and Theorem 3. Then we will provide a proof for the Main Lemma, which is the basis for all the iteration procedures. Finally we will describe the proofs for Propositions 2, 3, Theorems 2, 3, and 1. The proof for Proposition 1 is in fact quite routine, making use of the transformation formulas such as (3) and the fact that $\nabla_a T_i(g^{-1} \circ A_g)^a_b = 0$ in our situation. Since it is not used essentially in this work, it will be omitted here and will be provided in a future work.

Proof of Theorem $1'$. Here we can take R to be $1/2$. The general case follows from rescaling. (1) has translation covaraince: if we set $\widetilde{w} = w + \frac{1}{4}\ln ||K||_\infty$, then $\widetilde{w}$ satisfies

$$\sigma_2(g_0^{-1} \circ A_{\widetilde{w}}) = \widetilde{K}(x) e^{4\widetilde{w}},$$

where $\widetilde{K}(x) = K(x)/||K||_\infty$. It is easier to work with $u(x) = e^{\widetilde{w}(x)}$. For the local upper bound, we only need to bound $\sup_{|x|\le 1}(1-|x|)u(x)$ in terms of $\int_{B_1} u^4(x)dx$. Note that $\sup_{|x|\le 1}(1-|x|)u(x) < \infty$ by Proposition 2. Let x_0 with $|x_0| < 1$ satisfy

$$\sup_{|x|\le 1}(1-|x|)u(x) \le 2(1-|x_0|)u(x_0).$$

Set $1 - |x_0| = 2r_0$ and $v(z) = \rho u(x_0 + \rho z)$, with $\rho > 0$ chosen so that $v(0) = 1$. When $|z| \le r_0/\rho$, we have $1 - |x_0 + \rho z| \ge r_0$, so that

$$r_0 u(x_0 + \rho z) \le (1 - |x_0 + \rho z|)u(x_0 + \rho z) \le 4 r_0 u(x_0).$$

Thus $v(z) \le 4$ for $|z| \le r_0/\rho$. Note that $u^2(x)|dx|^2 = v^2(z)|dz|^2$, so that

$$\sigma_2(g_0^{-1} \circ A_v) = \widetilde{K}(x_0 + \rho z) v^4,$$

where A_v is the Weyl-Schouten tensor of $v^2(z)|dz|^2$. If $r_0/\rho \geq 1$, then $v(z) \leq 4$ on $|z| \leq 1$.

The conditions for Proposition 2 are satisfied on $|z| \leq 1$. Noting that $||\widetilde{K}||_\infty = 1$. So we apply Proposition 2 with $R = 1/2$ and $p = \beta = 4$ to obtain an absolute constant $C_* > 0$ such that

$$1 = v(0) \leq C_* \left(\int_{|z|\leq 1} v^4(z)dz \right)^{\frac{1}{4}} = C_* \left(\int_{B(x_0,r_0)} u^4(x)dx \right)^{\frac{1}{4}} \leq C_* \left(||K||_\infty \int_{B_1} e^{4w(x)}dx \right)^{\frac{1}{4}}.$$

This can't happen if $||K||_\infty \int_{B_1} e^{4w(x)}dx \leq \epsilon_1$ and $\epsilon_1 C_*^4 < 1$. Choose and fix such an ϵ_1. Then we must have $r_0/\rho < 1$. Again we can apply Proposition 2 on $|z| \leq r_0/\rho$ to obtain

$$1 \leq C_*^4 \left(\frac{\rho}{r_0} \right)^4 \int_{|z|\leq \frac{r_0}{\rho}} v^4(z)dz \leq C_*^4 \left(\frac{\rho}{r_0} \right)^4 \int_{|x|\leq 1} u^4(x)dx.$$

Recall that $\rho u(x_0) = 1$. So we have

$$r_0^4 u(x_0)^4 \leq C_*^4 \int_{|x|\leq 1} u^4(x)dx,$$

from which it follows that

$$\sup_{|x|\leq 1} (1 - |x|)u(x) \leq 4r_0 u(x_0) \leq 4C_* \left(\int_{|x|\leq 1} u^4(x)dx \right)^{\frac{1}{4}}.$$

For the Harnack estimate, we will apply Theorem 3 with $f = K(x)e^{4w}$. For that purpose we will need to have an upper bound on $R^{4(1-\frac{1}{p})}||f||_{L^p(B_{2R})}$. An almost identical verification is carried out in the proof of Theorem 1 later. Please refer to that part of the proof. □

Proof of the Main Lemma. Recall that for $g = e^{2w}g_0 = e^{2w}|dx|^2$,

$$2\sigma_2(g_0^{-1} \circ A_g) = -\partial_a(M_b^a \partial^b w),$$

where

$$M^a_b = T_1(g_0^{-1} \circ A_g)^a_b + \frac{|\nabla w|^2}{2}\delta^a_b.$$

We obtain

$$\int_{B_2} 2\sigma_2(g_0^{-1} \circ A_g)\eta^4 G(w) = \int_{B_2} M^a_b \partial_a w \partial^b \left[\eta^4 G(w)\right]$$

$$= \int_{B_2} \eta^4 G'(w) M^a_b \partial_a w \partial^b w + \int_{B_2} 4\eta^3 G(w) M^a_b \partial_a w \partial^b \eta$$

$$\begin{cases} \geq \frac{1}{2}\int_{B_2} |\nabla w|^4 \eta^4 G'(w) + \int_{B_2} 4\eta^3 G(w) M^a_b \partial_a w \partial^b \eta, & \text{if } G' \geq 0; \\ \leq \frac{1}{2}\int_{B_2} |\nabla w|^4 \eta^4 G'(w) + \int_{B_2} 4\eta^3 G(w) M^a_b \partial_a w \partial^b \eta, & \text{if } G' \leq 0. \end{cases}$$

Here and in the following of the proof, all the integration by parts used can be justified for $W^{2,2}$ admissible solutions. Thus, in all cases, we have

$$\int_{B_2} |\nabla w|^4 \eta^4 |G'(w)| \leq 8 \left| \int_{B_2} \eta^3 G(w) M^a_b \partial_a w \partial^b \eta \right| + 4 \int_{B_2} |f(x)| \eta^4 |G(w)|.$$

M^a_b is an expression involving the first and second derivatives of w, so, apriori, we have *no upper bound* on the eigenvalues of M^a_b. However, it follows from (3) and the definition of M^a_b that

$$M^a_b = w^a_b - (\Delta w)\delta^a_b - w^a w_b,$$

and

$$\begin{aligned} 2M^a_b \partial_a w &= \partial_b(|\nabla w|^2) - 2(\Delta w)\partial_b w - 2|\nabla w|^2 \partial_b w \\ &= 2\partial_b(|\nabla w|^2) - \partial_a(2\partial^a w \partial_b w) - 2|\nabla w|^2 \partial_b w, \end{aligned}$$

We can integrate by parts to estimate the integral

$$\int_{B_2} \eta^3 G(w) M^a_b \partial_a w \partial^b \eta = \int_{B_2} \left(\partial^a w \partial_b w - |\nabla w|^2 \delta^a_b\right) \partial_a \left(\eta^3 \partial^b \eta G(w)\right) - \eta^3 G(w) |\nabla w|^2 \partial_b w \partial^b \eta$$

$$= \int_{B_2} \left(\partial^a w \partial_b w - |\nabla w|^2 \delta^a_b\right) \left\{ \left(3\eta^2 \partial_a \eta \partial^b \eta + \eta^3 \partial^b_a \eta\right) G(w) + \right.$$

$$\left. + G'(w) \eta^3 \partial_a w \partial^b \eta \right\} - \int_{B_2} \eta^3 G(w) |\nabla w|^2 \partial_b w \partial^b \eta$$

$$= \int_{B_2} \left(\partial^a w \partial_b w - |\nabla w|^2 \delta^a_b\right) \left(3\eta^2 \partial_a \eta \partial^b \eta + \eta^3 \partial^b_a \eta\right) G(w)$$

$$- \int_{B_2} \eta^3 G(w) |\nabla w|^2 \partial_b w \partial^b \eta.$$

Using $|\eta \partial^b_a \eta| \lesssim |\nabla \eta|^2$, we conclude the proof of the Main Lemma. □

Proof of Propositions 2 and 3. [S] serves as a useful guide in the adaptation of Moser's iteration procedure to our situation. The proof of (7) and (10) is done by plugging in (16) $G(w) = e^{4\beta w}$, or more strcitly speaking, truncations of $e^{4\beta w}$ at large $|w|$. For simplicity, we will not do the truncation in the test function; Instead, we will demonstrate the estimates apriori by simply plugging $G(w) = e^{4\beta w}$ in (16). Then

$$\begin{aligned} \int_{B_2} \eta^4 |G'(w)| |\nabla w|^4 &= 4|\beta| \int_{B_2} \eta^4 e^{4\beta w} |\nabla w|^4 \\ &= 4|\beta|^{-3} \int_{B_2} \eta^4 |\nabla e^{\beta w}|^4, \end{aligned} \tag{17}$$

$$\int_{B_2} \eta^2 |G(w)||\nabla w|^2 |\nabla \eta|^2 = \beta^{-2} \int_{B_2} \eta^2 |\nabla e^{\beta w}|^2 e^{2\beta w} |\nabla \eta|^2$$

$$\leq \beta^{-2} \left(\int_{B_2} \eta^4 |\nabla e^{\beta w}|^4 \right)^{\frac{1}{2}} \left(\int_{B_2} e^{4\beta w} |\nabla \eta|^4 \right)^{\frac{1}{2}} \tag{18}$$

$$\leq \frac{1}{2|\beta|^3} \int_{B_2} \eta^4 |\nabla e^{\beta w}|^4 + \frac{1}{2|\beta|} \int_{B_2} e^{4\beta w} |\nabla \eta|^4,$$

and

$$\int_{B_2} \eta^3 |G(w)||\nabla w|^3 |\nabla \eta| = |\beta|^{-3} \int_{B_2} \eta^3 |\nabla e^{\beta w}|^3 e^{\beta w} |\nabla \eta|$$

$$\leq |\beta|^{-3} \left(\int_{B_2} \eta^4 |\nabla e^{\beta w}|^4 \right)^{\frac{3}{4}} \left(\int_{B_2} e^{4\beta w} |\nabla \eta|^4 \right)^{\frac{1}{4}} \tag{19}$$

$$\leq \frac{3}{4|\beta|^3} \int_{B_2} \eta^4 |\nabla e^{\beta w}|^4 + \frac{1}{4|\beta|^3} \int_{B_2} e^{4\beta w} |\nabla \eta|^4.$$

Combining the above, we have

$$\int_{B_2} |\nabla(\eta e^{\beta w})|^4 \lesssim (1+\beta^2) \int_{B_2} e^{4\beta w} (|\nabla \eta|^4 + \eta^4) + |\beta|^3 ||f||_p ||\eta e^{\beta w}||_{4p'}^4, \tag{20}$$

where $p' = \frac{p}{p-1}$ is the Hölder conjugate of p. Set $q = 8p'$ and $\theta = (2p'-1)^{-1}$. Then $0 < \theta < 1$ and

$$\frac{1}{4p'} = \frac{1-\theta}{q} + \frac{\theta}{4}.$$

We use interpolation to estimate the last term in (20), $||\eta e^{\beta w}||_{4p'}$, in terms of $||\eta e^{\beta w}||_4$ and $||\eta e^{\beta w}||_q$:

$$||\eta e^{\beta w}||_{4p'}^4 \leq ||\eta e^{\beta w}||_4^{4\theta} ||\eta e^{\beta w}||_q^{4(1-\theta)}, \tag{21}$$

and use the Sobolev inequality to estimate $||\eta e^{\beta w}||_q$. Putting these in (20), we have

$$||\nabla \left(\eta e^{\beta w}\right)||_4^4 \lesssim (1+\beta^2)||(|\nabla\eta|+\eta)e^{\beta w}||_4^4 + q^{3(1-\theta)}|\beta|^3||f||_p||\eta e^{\beta w}||_4^{4\theta}||\nabla \left(\eta e^{\beta w}\right)||_4^{4(1-\theta)}$$

$$\lesssim (1+\beta^2)||(|\nabla\eta|+\eta)e^{\beta w}||_4^4 + \theta\left(|\beta|q^{1-\theta}||f||_p^{\frac{1}{3}}\right)^{\frac{3}{\theta}}||\eta e^{\beta w}||_4^4 + (1-\theta)||\nabla \left(\eta e^{\beta w}\right)||_4^4.$$

Thus

$$||\nabla \left(\eta e^{\beta w}\right)||_4^4 \lesssim \left(|\beta|q^{1-\theta}||f||_p^{\frac{1}{3}}\right)^{\frac{3}{\theta}}||\eta e^{\beta w}||_4^4 + \theta^{-1}(1+\beta^2)||(|\nabla\eta|+\eta)e^{\beta w}||_4^4,$$

and

$$||\eta e^{\beta w}||_q^4 \lesssim (|\beta|q)^{3/\theta}||f||_p^{1/\theta}||\eta e^{\beta w}||_4^4 + \theta^{-1}q^3(1+\beta^2)||(|\nabla\eta|+\eta)e^{\beta w}||_4^4.$$

Since $\frac{3}{\theta} = 3(2p'-1) > 2$, we can write the above estimate as

$$||\eta e^{\beta w}||_q \le M(1+|\beta|)^{\frac{3}{4\theta}}||(|\nabla\eta|+\eta)e^{\beta w}||_4, \tag{22}$$

where M is a constant depending on $||f||_p$ and p:

$$M \sim [p'||f||_p]^{(2p'-1)/4} + p'.$$

From (22), one can invoke the Moser iteration procedure to prove (7) and (10).

To prove (9) and (11), for instance, we first plug $G(w) = (\gamma + M - w)^{4\beta-3}, \beta \neq 0, \frac{3}{4}$ into (16). Then

$$\eta^4|G'(w)||\nabla w|^4 = |4\beta-3|\beta^{-4}\eta^4|\nabla(\gamma+M-w)^\beta|^4,$$

$$\eta^2|G(w)||\nabla w|^2|\nabla\eta|^2 = \beta^{-2}\eta^2|\nabla(\gamma+M-w)^\beta|^2(\gamma+M-w)^{2\beta-1}|\nabla\eta|^2$$

$$\le \frac{\epsilon^2}{2\beta^4}\eta^4|\nabla(\gamma+M-w)^\beta|^4 + \frac{1}{2\epsilon^2}(\gamma+M-w)^{4\beta-2}|\nabla\eta|^4$$

$$\le \frac{\epsilon^2}{2\beta^4}\eta^4|\nabla(\gamma+M-w)^\beta|^4 + \frac{\gamma^{-2}}{2\epsilon^2}(\gamma+M-w)^{4\beta}|\nabla\eta|^4,$$

$$\eta^3|\nabla w|^3|G(w)||\nabla\eta| = |\beta|^{-3}\eta^3|\nabla(\gamma+M-w)^\beta|^3(\gamma+M-w)^\beta|\nabla\eta|$$

$$\leq \frac{3\epsilon^{4/3}}{4\beta^4}\eta^4|\nabla(\gamma+M-w)^\beta|^4 + \frac{1}{4\epsilon^4}(\gamma+M-w)^{4\beta}|\nabla\eta|^4,$$

and

$$\eta^4|G(w)||f| = \eta^4(\gamma+M-w)^{4\beta-3}|f| \leq \gamma^{-3}\eta^4|f|(\gamma+M-w)^{4\beta}.$$

Choosing $\epsilon > 0$ small (its size can be independent of β as long as $4\beta-3$ stays away from 0), and treating $\int_{B_2}\gamma^{-3}\eta^4|f|(\gamma+M-w)^{4\beta}$ as in (20) and (21)—noting that $\gamma = \max(1, ||f||_p^{1/3})$, we have

$$|4\beta-3|\int_{B_2}\eta^4|\nabla(\gamma+M-w)^\beta|^4 \lesssim \beta^4\int_{B_2}(\gamma+M-w)^{4\beta}(|\nabla\eta|^4+\eta^4). \tag{23}$$

To complete the proof of (11), we take $\beta > 3/4$. Then $G'(w) \leq 0$, so we have (23) and can use it to carry out the Moser iteration to obtain (11) with the restriction $\beta > 3$ there. But this restriction can be dropped via a device such as Lemma 5.1 in [Gia].

Lemma 5.1 in [Gia]. *Let $E(t)$ be a nonnegative bounded function on $0 \leq T_0 \leq T_1$. Suppose that for $T_0 \leq t < s \leq T_1$ we have*

$$E(t) \leq \theta E(s) + A(s-t)^{-\alpha} + B$$

with $\alpha > 0$, $0 \leq \theta < 1$ and A, B nonnegative constants. Then there exists a constant $c = c(\alpha,\theta)$, such that for all $T_0 \leq \rho < R \leq T_1$ we have

$$E(\rho) \leq c\left[A(R-\rho)^{-\alpha} + B\right]$$

To complete the proof of (9), we can use (23) with $\beta < 3/4, \neq 0$ and need to appeal to the John-Nirenberg Theorem. For this purpose, we plug $G(w) = (\gamma+M-w)^{-3}$ into (16). Similar computations as above lead to

$$\int_{B_{2R}}\eta^4|\nabla\log(\gamma+M-w)|^4 \leq \int_{B_{2R}}|\nabla\eta|^4 + \gamma^{-3}\int_{B_{2R}}|f|. \tag{24}$$

The right hand side of (24) has an absolute upper bound, so by the John-Nirenberg Theorem, there exist absolute constants $\beta_1, C_* > 0$ such that

$$\left(\frac{1}{R^4}\int_{B_{2R}}(\gamma+M-w)^{\beta_1}\right)\left(\frac{1}{R^4}\int_{B_{2R}}(\gamma+M-w)^{-\beta_1}\right) \leq C_*. \tag{25}$$

Then we can use (23) for $\beta = -\beta_1 < 0$ and carry out the Moser iteration to obtain

$$\left(\frac{1}{R^4} \int_{B_{2R}} (\gamma + M - w)^{-\beta_1} \right)^{-\frac{1}{\beta_1}} \leq \widetilde{C}^* \inf_{B_R} (\gamma + M - w). \tag{26}$$

(26) and (25) imply (9) for $\beta = \beta_1$. The general case follows from the case for $\beta = \beta_1$, (23) and the Sobolev inequality. The proof for (12) and (8) is done similarly. □

Proof of Theorem 2. (13) follows from (11) and (9). (14) follows from (12) and (8). □

Proof of Theorem 3. This will be completed if we can bridge the gap betwen (7) and (10). Note that since $A_{e^{2w}g_0} \in \Gamma_2^+$, $u = e^w$ satisfies $-6\Delta u = Ru^3 > 0$, which implies a BMO estimate for $w = \ln u$ by

$$0 < \int -\Delta u (u^{-1}\eta^2) = \int |\nabla \ln u|^2 \eta^2 + 2\eta \nabla \ln u \cdot \nabla \eta,$$

where η is a standard cut-off funtion. So

$$\int_{B_R} |\nabla w|^2 \leq 4 \int_{B_R} |\nabla \eta|^2 \lesssim R^2. \tag{27}$$

This can also be done as in (2.17) of [CGY2]. Then by the John-Nirenberg Theorem on BMO functions, there exist absolute constants $\beta_* > 0$ and $C_* > 0$ such that

$$\left(\frac{1}{R^4} \int_{B_{2R}} e^{\beta_* w} \right) \left(\frac{1}{R^4} \int_{B_{2R}} e^{-\beta_* w} \right) \leq C_*.$$

This estimate, together with (7) and (10), conclude the proof of Theorem 3. □

Proof of Theorem 1. The proof consists of two elements:

i. There exists absolute constant $\epsilon_0 > 0$ such that if w is a solution of (1) and $\int_{B_R} K(x) e^{4w} \leq \epsilon_0$, then

$$\int_{B_{R/2}} |\nabla w|^4 \leq C, \tag{28}$$

where C depends on $(\sup_{B_R} K) \int_{B_R} e^{4w}$.

ii. Now treat $K(x)e^{4w}$ as $f(x)$. We can verify that, for $p > 1$, $R^{4(1-\frac{1}{p})}||f||_{L^p(B_{R/2})}$ has a bound from above depending only on $(\sup_{B_R} K) \int_{B_R} e^{4w}$, so that by (7) and (15), we conclude Theorem 1.

To prove (28), we multiply both sides of (1) by $\eta^4(w - \bar{w})$, where $\bar{w}$ is the average of w over B_R and η is a standard cut-off function supported in B_R such that for $\rho < R$, $\eta \equiv 1$ on B_ρ and $|\nabla \eta| \leq 2(R-\rho)^{-1}$. As before, we have

$$\begin{aligned}
\int_{B_R} \eta^4 |\nabla w|^4 &\leq \int_{B_R} \eta^2 |w - \bar{w}| \left[|\nabla w|^2 |\nabla \eta|^2 + |\nabla w|^3 \eta |\nabla \eta| \right] + \eta^4 K(x) e^{4w} (w - \bar{w}) \\
&\leq \left(\int_{B_R} \eta^4 |\nabla w|^4 \right)^{1/2} \left(\int_{B_R} |w - \bar{w}|^2 |\nabla \eta|^4 \right)^{1/2} + \\
&+ \left(\int_{B_R} \eta^4 |\nabla w|^4 \right)^{3/4} \left(\int_{B_R} |w - \bar{w}|^4 |\nabla \eta|^4 \right)^{1/4} + \int_{B_R} \eta^4 K(x) e^{4w} (w - \bar{w}).
\end{aligned} \tag{29}$$

By the BMO estimate (27),

$$\int_{B_R} |w - \bar{w}|^2 |\nabla \eta|^4 \lesssim R^4 (R-\rho)^{-4},$$

and

$$\int_{B_R} |w - \bar{w}|^4 |\nabla \eta|^4 \lesssim R^4 (R-\rho)^{-4}.$$

The last term in (29) is estimated by Jensen's inequality as

$$\int_{B_R} \eta^4 K(x) e^{4w}(w-\bar{w}) \le \left(\int_{B_R} K(x)e^{4w}\right)\left[\ln\left(\int_{B_R} K(x)e^{5w-\bar{w}}\right) - \ln\left(\int_{B_R} K(x)e^{4w}\right)\right]$$

$$\le \left(\int_{B_R} K(x)e^{4w}\right)\left[\ln\left(\int_{B_R} K(x)e^{5(w-\bar{w})}\right) + 4\bar{w} - \ln\left(\int_{B_R} K(x)e^{4w}\right)\right].$$

By the Moser-Trudinger inequality,

$$\ln\left(\int_{B_R} K(x)e^{5(w-\bar{w})}\right) \le c_1 \int_{B_R} |\nabla w|^4 + c_2 + \ln(\sup K) + \ln|B_R|, \tag{30}$$

where $c_1, c_2 > 0$ are absolute constants. By Jensen's inequality again, we have

$$e^{4\bar{w}} \le \frac{1}{|B_R|}\int_{B_R} e^{4w},$$

which implies

$$4\bar{w} + \ln|B_R| \le \ln\left(\int_{B_R} e^{4w}\right).$$

Substituting all these estimates back into (29), we have

$$\int_{B_\rho} |\nabla w|^4 \le \left(\int_{B_R} K(x)e^{4w}\right)\left\{c_1\int_{B_R} |\nabla w|^4 + c_2 + \ln\left[\sup K\left(\int_{B_R} e^{4w}\right) / \left(\int_{B_R} K(x)e^{4w}\right)\right]\right\} +$$

$$+ CR^4(R-\rho)^{-4}$$

$$\le \epsilon_0 c_1 \int_{B_R} |\nabla w|^4 + \epsilon_0\left[c_2 + \ln\left(\sup K \int_{B_R} e^{4w}\right) - \ln\epsilon_0\right] + CR^4(R-\rho)^{-4},$$

if $\int_{B_R} K(x)e^{4w} \le \epsilon_0$. Suppose $\epsilon_0 > 0$ is chosen so that $\epsilon_0 c_1 < 1$, then from the previously cited Lemma 5.1 in [Gia], we conclude that, for some absolute constant $\delta > 0$,

$$\int_{B_\rho} |\nabla w|^4 \le \delta \left\{ \epsilon_0 \left[c_2 + \ln \left(\sup K \int_{B_R} e^{4w} \right) - \ln \epsilon_0 \right] + CR^4 (R-\rho)^{-4} \right\}.$$

By taking $\rho = R/2$, we obtain (28).

We next use (28) to estimate $R^{4(1-\frac{1}{p})}||f||_{L^p(B_{R/2})}$ with $f = K(x)e^{4w}$ and $p > 1$, say, $p = 5/4$. By the John-Nirenberg theorem and (28), we have

$$\int_{B_{R/2}} e^{5(w-\bar{w})} \le C|B_{R/2}|,$$

so that

$$\int_{B_{R/2}} e^{5w} \le C|B_{R/2}|e^{5\bar{w}} \le C|B_{R/2}| \left(\frac{1}{|B_{R/2}|} \int_{B_{R/2}} e^{4w} \right)^{5/4},$$

and

$$R^{4(1-\frac{1}{p})}||f||_{L^p(B_{R/2})} \le CR^{4/5} \cdot |B_{R/2}|^{-1/5} \cdot (\sup K) \int_{B_{R/2}} e^{4w} \le C(\sup K) \int_{B_R} e^{4w}. \quad (31)$$

Now we can use (7) and (15) to conclude Theorem 1. When only $||K||_p < \infty$, for some $p > 1$, is assumed, the only changes are in the estimates (30) and (31). Both can be handled by first applying the Hölder inequality, then applying the Moser-Trudinger inequality. For instance,

$$\ln\left(\int_{B_R} K(x)e^{5(w-\bar{w})}\right) \leq \ln\left(||K||_p||e^{5(w-\bar{w})}||_{p'}\right)$$

$$\leq \frac{1}{p'}\ln\left(\int_{B_R} e^{5p'(w-\bar{w})}\right) + \ln||K||_p$$

$$\leq c_1(p')^3\int_{B_R}|\nabla w|^4 + \frac{1}{p'}\left(c_2 + \ln|B_R|\right) + \ln||K||_p.$$

Now in choosing ϵ_0, we have to require $\epsilon_0 c_1(p')^3 < 1$. The rest of the modifications are straightforward. □

References

[BV] S. Brendle and J. Viaclovsky, *A variational characterization for $\sigma_{n/2}$*, to appear in Calc. Var. PDE.

[CY3] S.-Y. Chang and P. Yang, *The Inequality of Moser and Trudinger and applications to conformal geometry*, Comm. Pure and Applied Math., Vol LVI, no. 8, August 2003, pp 1135-1150. Special issue dedicated to the memory of Jurgen K. Moser.

[CGY1] S.-Y. Chang, M. Gursky and P. Yang, *An equation of Monge-Ampère type in conformal geometry, and four-manifolds of positive Ricci curvature*, Annals of Math., **155**(2002), 709-787.

[CGY2] S.-Y. Chang, M. Gursky and P. Yang, *An a priori estimate for a fully nonlinear equation on four-manifolds*, Dedicated to the memory of Thomas H. Wolff. J. Anal. Math. **87** (2002), 151–186.

[CGY3] S.-Y. Chang, M. Gursky and P. Yang, *Entire solutions of a fully nonlinear equation*, "Lectures in Partial Differential Equations in honor of Louis Nirenberg's 75th birthday", chapter 3. International Press, 2003.

[CHY1] S.-Y. A. Chang, Z. Han, and P. Yang, *Apriori estimates for solutions of the prescribed σ_2 curvature equation on S^4*, work in preparation.

[CHY2] S.-Y. A. Chang, Z. Han, and P. Yang, *Classification of singular radial solutions of the σ_k-Yamabe equation on annular domains in $\mathbb{R}^n$*, preprint. Available at http://www.math.rutgers.edu/~zchan.

[CHgY] S.-Y. A. Chang, F. Hang, and P. Yang, *On a class of locally conformally flat manifolds*, IMRN, **No.4**(2004), 185-209.

[ChW] K.S. Chou and X.-J. Wang, *Variational theory for Hessian equations*, Comm. Pure Appl. Math., **54**(2001), 1029-1064.

[dG] E. De Giorgi, *Sulla differenzabilità e l'analiticità delle estremali degli integrali multipli regolari*, Mem. Accad. Sci. Torino Cl. Sci. Fis. Mat. Natur. (3) **3**(1957), 25-43.

[F] H. Fang, *Conformally invariant curvatures on LCF manifolds*, preprint.

[Gia] M. Giaquinta, *Introduction to Regularity Theory for Nonlinear Elliptic Systems*, Birkhäuser, 1993.

[GT] D. Gilbarg and N. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Second Edition, Springer-Verlag, 1983.

[G1] M. Gonzalez, Ph.D thesis, Princeton University, 2004.

[G2] M. Gonzalez, *Singular sets of a class of locally conformaly flat manifolds*, preprint, 2004.

[G3] M. Gonzalez, *Removability of singularities for a class of fully non-linear equations*, preprint, 2004.

[GLW] P. Guan, C.S. Lin and G. Wang, *Application of The Method of Moving Planes to Conformally Invariant Equations*, to appear in Mathematische Zeitschrift.

[GVW] P. Guan, J. Viaclovsky and G. Wang, *Some properties of the Schouten tensor and applications to conformal geometry*, Transactions of American Math. Society, **355** (2003), 925-933.

[GW1] P. Guan and G. Wang, *Local estimates for a class of fully nonlinear equations arising from conformal geometry*, International Mathematics Research Notices, V. 2003, Issue 26(2003), 1413-1432.

[GW2] Pengfei Guan and Guofang Wang, *A fully nonlinear conformal flow on locally conformally flat manifolds*, Journal fur die reine und angewandte Mathematik, **557** (2003), 219-238.

[GV1] M. Gursky and J. Viacolvsky, *A new variational characterization of three-dimensional space forms*, Invent. Math. **145** (2001), 251–278.

[GV2] M. Gursky and J. Viacolvsky, *Fully nonlinear equations on Riemannian manifolds with negative curvature*, Indiana Univ. Math. J. **52** (2003), no. 2, 399-420.

[GV3] M. Gursky and J. Viacolvsky, *A fully nonlinear equation on four-manifolds with positive scalar curvature*, to appear in Journal of Differential Geometry.

[LU] O. Ladyzhenskaya and N. Ural'tseva, *Linear and Quasilinear Elliptic Equations*, Academic Press, 1968.

[LL1] Aobing Li and YanYan Li, *On some conformally invariant fully nonlinear equations*, Comm. Pure Appl. Math. **56** (2003), 1414-1464.

[LL2] Aobing Li and YanYan Li, *A fully nonlinear version of the Yamabe problem and a Harnack type inequality*, arXiv:math.AP/0212031 v1 2 Dec 2002, 11 pages.

[LL3] Aobing Li and YanYan Li, *A Liouville type theorem for some conformally invariant fully nonlinear equations*, arXiv:math.AP/0212376 v1 30 Dec 2002, 10 pages.

[LL4] Aobing Li and YanYan Li, *A general Liouville type theorem for some conformally invariant fully nonlinear equations*, arXiv:math.AP/0301239 v1 21 Jan 2003, 16 pages.

[LL5] Aobing Li and YanYan Li, *Further results on Liouville type theorems for some conformally invariant fully nonlinear equations*, arXiv:math.AP/0301254 v1 22 Jan 2003, 9 pages.

[LL6] Aobing Li and YanYan Li, *On some conformally invariant fully nonlinear equations, Part II: Liouville, Harnack and Yamabe*, preprint, 65 pages.

[M1] J. Moser, *A new proof of de Giorgi's theorem concerning the regularity problem for elliptic differential equations*, Comm. Pure Appl. Math. **13**(1960), 457-468.

[M2] J. Moser, *On Harnack's theorem for elliptic differential equations*, Comm. Pure Appl. Math. **14**(1961), 577-591.

[S] J. Serrin, *Local behavior of solutions of quasilinear equations*, Acta Math., **111** (1964), 247-302.

[TW1] N. Trudinger and X. J. Wang, *Hessian measures. I*, Dedicated to Olga Ladyzhenskaya. Topol. Methods Nonlinear Anal. **10** (1997), no. 2, 225–239.

[TW2] N. Trudinger and X. J. Wang, *Hessian measures. II*, Ann. of Math. (2) **150** (1999), no. 2, 579–604.

[TW3] N. Trudinger and X. J. Wang, *Hessian measures. III*, J. Funct. Anal. **193** (2002), no. 1, 1–23.

[V1] J. Viaclovsky, *Conformal geometry, contact geometry, and the calculus of variations*, Duke Math. J. **101, No. 2** (2000), 283-316.

[V2] J. Viaclovsky, *Estimates and existence results for some fully nonlinear elliptic equations on Riemannian manifolds*, Communications in Analysis and Geometry **10** (2002), no.4, 815-846.

[V3] J. Viaclovsky, *Conformally Invariant Monge-Ampère Partial Differential Equations: Golbal Solutions*, Trans. Amer. Math. Soc. **352** (2000), no. 9, 4371–4379.