Two Kazdan-Warner type identities for the renormalized volume coefficients and the the Gauss-Bonnet curvatures of a Riemannian metric

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Abstract

In this note, we prove two Kazdan-Warner type identities involving $v^{(2k)}$, the renormalized volume coefficients of a Riemannian manifold (M^n, g), and G_{2r}, the so-called Gauss-Bonnet curvature, and a conformal Killing vector field on (M^n, g). In the case when the Riemannian manifold is locally conformally flat, $v^{(2k)} = (-2)^{-k}\sigma_k$, $G_{2r}(g) = \frac{4^r(n-r)!r!}{(n-2r)!}\sigma_r$ and our results reduce to earlier ones established by Viaclovsky in [V2] and the second author in [H].

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1 Introduction

In [V2] and [H], the following result was proved

Theorem A ([V2], [H]) *Let (M, g) be a compact Riemannian manifold of dimension $n \geq 3$, $\sigma_k(g^{-1} \circ A_g)$ be the σ_k curvature of g, and X be a conformal Killing vector field on (M, g). When $k \geq 3$, we also assume that (M, g) is locally conformally flat, then*

$$\int_M \langle X, \nabla\sigma_k(g^{-1} \circ A_g)\rangle dv_g = 0. \tag{1.1}$$

Recall that on an n-dimensional Riemannian manifold (M, g), $n \geq 3$, the full Riemannian curvature tensor Rm decomposes as

$$Rm = W_g \oplus (A_g \odot g) \tag{1.2}$$

where W_g denotes the Weyl tensor of g,

$$A_g = \frac{1}{n-2}(\mathrm{Ric_g} - \frac{R_g}{2(n-1)}g) \tag{1.3}$$

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denotes the Schouten tensor, and $\odot$ is the Kulkarni-Nomizu wedge product. Under a conformal change of metrics $g_w = e^{2w}g$, where w is a smooth function over the manifold, the Weyl curvature changes pointwise as $W_{g_w} = e^{2w}W_g$. Thus, essential information of the Riemannian curvature tensor under a conformal change of metrics is reflected by the change of the Schouten tensor. One often tries to study the Schouten tensor through studying the elementary symmetric functions $\sigma_k(g^{-1} \circ A_g)$ (which we later denote as $\sigma_k(g)$) of the eigenvalues of the Schouten tensor, called the σ_k curvatures of g, and studying how they deform under conformal change of metrics.

The following question is natural in relation to Theorem A:

Question. *Can we generalize Theorem A without the condition "locally conformally flat" for all $k \geq 1$?*

In this note, we give an affirmative answer to the above question. Renormalized volume coefficients, $v^{(2k)}(g)$, of a Riemannian metric g, were introduced in the physics literature in the late 1990's in the context of AdS/CFT correspondence—see [G] for a mathematical discussion, and were shown in [GJ] to be equal to $\sigma_k(g^{-1}A_g)$, up to a scaling constant, when (M, g) is locally conformally flat. In fact, in the normalization we are going to adopt,

$$v^{(2)}(g) = -\frac{1}{2}\sigma_1(g), \qquad v^{(4)}(g) = \frac{1}{4}\sigma_2(g). \tag{1.4}$$

For $k = 3$, Graham and Juhl ([GJ], page 5) have aslo listed the following formula for $v^{(6)}(g)$:

$$v^{(6)}(g) = -\frac{1}{8}[\sigma_3(g) + \frac{1}{3(n-4)}(A_g)^{ij}(B_g)_{ij}], \tag{1.5}$$

where

$$(B_g)_{ij} := \frac{1}{n-3}\nabla^k\nabla^l W_{likj} + \frac{1}{n-2}R^{kl}W_{likj} \tag{1.6}$$

is the *Bach* tensor of the metric. Just as $\int_M \sigma_k(g^{-1} \circ A_g)\, dv_g$ is conformally invariant when $2k = n$ and (M, g) is locally conformally flat, Graham showed in [G] that $\int_M v^{(2k)}(g)\, dv_g$ is also conformally invariant on a general manifold when $2k = n$. Chang and Fang showed in [CF] that, for $n \neq 2k$, the Euler-Lagrange equations for the functional $\int_M v^{(2k)}(g)\, dv_g$ under conformal variations subject to the constraint $Vol_g(M) = 1$ satisfies $v^{(2k)}(g) = \text{const.}$, which is a generalized characterization for the curvatures $\sigma_k(g^{-1} \circ A_g)$ when (M, g) is locally conformally flat, as given by Viaclovsky [V1].

In this note, we will first show that the curvatures $v^{(2k)}(g)$ will play the role of $\sigma_k(g^{-1} \circ A_g)$ in (1.1) for a general manifold. We note that Graham [G] also gives an explicit expression of $v^{(8)}(g)$, but the explicit expression of $v^{(2k)}(g)$ for general k is not known because they are algebraically complicated (see page 1958 of [G]). Thus the study of the $v^{(2k)}(g)$ curvatures involves significant challenges not shared by that of $\sigma_k(g)$: firstly, for $k \geq 3$, $v^{(2k)}(g)$ depends on derivatives of curvature of g— in fact, for $k \geq 3$, $v^{(2k)}(g)$ depends on derivatives of curvatures of order up to $2k - 4$; secondly, the $v^{(2k)}(g)$ are defined via an indirect highly nonlinear inductive algorithm (see [G]). Despite these difficulties, we can use some properties of these $v^{(2k)}(g)$ curvatures to prove the following

Theorem 1. *Let (M, g) be a compact Riemannian manifold of dimension $n \geq 3$, X be a conformal Killing vector field on (M^n, g). For $k \geq 1$, we have*

$$\int_M \langle X, \nabla v^{(2k)}(g)\rangle dv_g = 0. \tag{1.7}$$

Remark 1. From (1.4), we know that Theorem 1 is equivalent to Theorem A when $k = 1, 2$, or when (M^n, g) is locally conformally flat for $k \geq 3$.

One main reason for interest in identities such as (1.1) and (1.7) is that they play crucial roles in analyzing potentially blowing up conformal metrics with a prescribed curvature function— a prescribed $v^{(2k)}(g)$ in this case here, although little is known about this problem at this stage; Theorem 1 establishes one ingredient for attacking this problem.

Our second result involves the Gauss-Bonnet curvatures $G_{2r}(2r \leq n)$, introduced by H. Weyl in 1939, which is defined by (also see [La])

$$G_{2r}(g) = \delta^{j_1 j_2 \ldots j_{2r-1} j_{2r}}_{i_1 i_2 \ldots i_{2r-1} i_{2r}} R^{i_1 i_2}{}_{j_1 j_2} \ldots R^{i_{2r-1} i_{2r}}{}_{j_{2r-1} j_{2r}}, \tag{1.8}$$

where $\delta^{j_1 j_2 \ldots j_{2r-1} j_{2r}}_{i_1 i_2 \ldots i_{2r-1} i_{2r}}$ is the generalized Kronecker symbol. Note that $G_2 = 2R$, R the scalar curvature. We can prove that

Theorem 2. *Let (M^n, g) be a compact Riemannian manifold, and X be a conformal Killing vector field. Then for the Gauss-Bonnet curvatures defined above, we have*

$$\int_M \langle X, G_{2r}(g)\rangle dv_g = 0. \tag{1.9}$$

Remark 2. When (M, g) is locally conformally flat, we see that the Gauss curvature $G_{2r}(g) = \frac{4^r (n-r)! r!}{(n-2r)!} \sigma_r$, so Theorem 2 reduces to Theorem A.

Remark 3. M. Labbi ([La]) proved that the first variation of the functional $\int_M G_{2r} dv_g$ within metrics with constant volume gave the so-called generalized Einstein metric, and this functional has the variational property for $2r < n$ and is a topological invariant for $2r = n$. In fact, if $n = 2r$, this functional is the Gauss-Bonnet integrand up to a constant ([C]).

In the next section, we first provide a general proof for Theorem 1 by adapting an ingredient in a preprint version of [H], and making use of a variation formula for $v^{(2k)}(g)$ established in [G] and [CF]. And because of the explicit expression for $v^{(6)}(g)$ and potential applications to other related problems in low dimensions, we provide a self-contained proof for Theorem 1 in the case $k = 3$ in section 3. We will give a proof of Theorem 2 in section 4.

2 Proof of Theorem 1

We will need the following variation formula for $v^{(2k)}(g)$, see [G].

Proposition 1. *Under the conformal transformation $g_t = e^{2t\eta} g$, the variation of $v^{(2k)}(g_t)$ is given by*

$$\frac{\partial}{\partial t}\Big|_{t=0} v^{(2k)}(g_t) = -2k\eta v^{(2k)} + \nabla_i (L^{ij}_{(k)} \eta_j), \tag{2.1}$$

where $L^{ij}_{(k)}$ *is defined as in [G] by*

$$L^{ij}_{(k)} = -\sum_{l=1}^{k} \frac{1}{l!} v^{(2k-2l)}(g)\partial_\rho^{l-1} g^{ij}(\rho)\Big|_{\rho=0},$$

with $g_{ij}(\rho)$ *denoting the extension of* g *such that*

$$g_+ = \frac{(d\rho)^2 - 2\rho g(\rho)}{4\rho^2}$$

is an asymptotic solution to $Ric(g_+) = -ng_+$ *near* $\rho = 0$.

An integral version of (2.1) first appeared in [CF]:

$$\int_M \Big\{ \frac{d}{dt}\Big|_{t=0} [v^{(2k)}(g_t)] + 2k\eta v^{(2k)}(g) \Big\} dv_g = 0. \tag{2.2}$$

Proof of Theorem 1 in the case $n \neq 2k$. Let X be a conformal vector field on M. Let ϕ_t denote the local one-parameter family of conformal diffeomorphisms of (M, g) generated by X. Thus for some smooth function ω_t on M, we have

$$\phi_t^*(g) = e^{2\omega_t} g =: g_t. \tag{2.3}$$

We have the following properties

$$\phi_t^* v^{(2k)}(g) = v^{(2k)}(\phi_t^* g) = v^{(2k)}(e^{2\omega_t} g), \tag{2.4}$$

$$\dot\omega := \frac{d}{dt}\Big|_{t=0} \omega_t = \frac{\mathrm{div}X}{n}, \tag{2.5}$$

$$\frac{d}{dt}\Big|_{t=0} \left(g_t^{-1} \circ A(g_t) \right) = -\nabla^2 \dot\omega - 2\dot\omega g^{-1} \circ A(g). \tag{2.6}$$

$$\frac{\partial}{\partial t}\Big|_{t=0} \mathrm{div}_{g_t} X = nX\eta = n\langle X, \nabla\eta\rangle. \tag{2.7}$$

Using (2.4), (2.5), and (2.1), we have

$$\begin{aligned}
\langle X, \nabla v^{(2k)}(g)\rangle &= \frac{d}{dt}\Big|_{t=0} [v^{(2k)}(g_t)] \\
&= -2k\dot\omega v^{(2k)} + \nabla_i (L^{ij}_{(k)} \dot\omega_j) \\
&= -\frac{2k}{n} (\mathrm{div}X) v^{(2k)} + \nabla_i (L^{ij}_{(k)} \dot\omega_j) \\
&= -\frac{2k}{n} \mathrm{div}(v^{(2k)} X) + \frac{2k}{n} \langle X, \nabla v^{(2k)}(g)\rangle + \frac{1}{n} \nabla_i (L^{ij}_{(k)} (\mathrm{div}X)_j),
\end{aligned}$$

from which it follows that

$$\left(1 - \frac{2k}{n}\right) \langle X, \nabla v^{(2k)}(g)\rangle = -\frac{2k}{n} \mathrm{div}(v^{(2k)} X) + \frac{1}{n} \nabla_i (L^{ij}_{(k)} (\mathrm{div}X)_j). \tag{2.8}$$

Theorem 1 in the case $2k \neq n$ now follows directly by integrating (2.8) over M. □

Proof of Theorem 1 in the case $2k = n$. As in [H], we will prove that for any conformal metric $g_1 = e^{2\eta} g$ of g,

$$\int_M \langle X, v^{(2k)}(g_1)\rangle dv_{g_1} = \int_M \langle X, v^{(2k)}(g)\rangle dv_g = -\int_M \mathrm{div}_g X v^{(2k)}(g) dv_g, \tag{2.9}$$

i.e. $\int_M \langle X, v^{(2k)}(g)\rangle dv_g$ is independent of the particular choice of metrics in the conformal class. To this end, we only have to prove that for $g_t = e^{2t\eta} g$,

$$\frac{\partial}{\partial t}\Big|_{t=0} \int_M \mathrm{div}_{g_t} X v^{(2k)}(g_t) dv_{g_t} = 0. \tag{2.10}$$

We prove (2.10) by direct computations using Proposition 1. Indeed,

$$\begin{aligned}
&\frac{\partial}{\partial t}\Big|_{t=0} \int_M \mathrm{div}_{g_t} X v^{(2k)}(g_t) dv_{g_t} \\
&= \int_M \Big[n\langle X, \nabla\eta\rangle v^{(2k)} + \mathrm{div} X\big(-2k\eta v^{(2k)} + \nabla_i (L^{ij}_{(k)}\eta_j)\big) + n\eta \mathrm{div} X v^{(2k)}\Big] dv_g \\
&= \int_M \Big[n\langle X, \nabla\eta\rangle v^{(2k)} + \mathrm{div} X \nabla_i (L^{ij}_{(k)}\eta_j)\Big] dv_g \\
&= \int_M \Big[\langle n v^{(2k)} X, \nabla\eta\rangle - L^{ij}_{(k)} (\mathrm{div} X)_i \eta_j)\Big] dv_g \\
&= \int_M \Big[-\mathrm{div}(n v^{(2k)} X) + \nabla_j \left(L^{ij}_{(k)} (\mathrm{div} X)_i \right) \Big] \eta dv_g = 0
\end{aligned} \tag{2.11}$$

in the case $n = 2k$ by (2.8).

The remaining argument is an adaptation of an argument of Bourguignon and Ezin ([BE]): either the connected component of the identity of the conformal group $C_0(M, g)$ is compact, then there is a metric $\hat{g}$ conformal to g admitting $C_0(M, g)$ as a group of isometries, from which it follows that $\mathrm{div}_{\hat{g}} X \equiv 0$ and (1.7) therefore holds; or, $C_0(M, g)$ is non-compact, then by a theorem of Obata-Ferrand, (M, g) is conformal to the standard sphere, in which case we can pick the canonical metric to compute the integral on the left hand side of (1.7) and conclude that it is zero. □

3 Self-contained proof of Theorem 1 in the case $k = 3$

We aim to give a direct, self-contained derivation for a more explicit version of (2.1), more precisely, under conformal change of metrics $g_t = e^{2t\eta} g$, we have

$$\frac{\partial}{\partial t}\Big|_{t=0} v^{(6)}(g_t) = -6 v^{(6)}(g)\eta + \nabla^j \Big[\Big(\frac{T^{(2)}_{ij}(g)}{8} + \frac{B_{ij}(g)}{24(n-4)} \Big) \nabla^i \eta \Big], \tag{3.1}$$

where $T^{(2)}_{ij}(g)$ is the Newton tensor associated with A_g, as defined in Reilly [R]:

Definition. For an integer $k \geq 0$, k-th Newton tensor is

$$T^{(k)}_{ij} = \frac{1}{k!} \sum \delta^{j_1 \cdots j_k j}_{i_1 \cdots i_k i} A_{i_1 j_1} \cdots A_{i_k j_k}$$

where $\delta^{j_1 \cdots j_k j}_{i_1 \cdots i_k i}$ is the generalized Kronecker symbol.

With (3.1) we can repeat the proof in the last section to prove Theorem 1 in the case $k = 3$.

First we recall the transformation laws for the tensors B_{ij} and A_{ij} under conformal change of metrics $g_t = e^{2t\eta}g$—see [CF]:

$$A_{ij}(g_t) = A_{ij} - t\nabla^2_{ij}\eta + t^2\nabla_i\eta\nabla_j\eta - t^2\frac{|\nabla\eta|^2_g}{2}g_{ij};$$

$$B_{ij}(g_t) = e^{-2t\eta}\Big(B_{ij} + (n-4)t(C_{ijk} + C_{jik})\nabla^k\eta + (n-4)t^2W_{ikjl}\nabla^k\eta\nabla^l\eta\Big).$$

where C_{ijk} are the components of the *Cotton* tensor defined by

$$C_{ijk} = A_{ij,k} - A_{ik,j}$$

with $A_{ij,k}$ being the components of the covariant derivative of the Schouten tensor A_{ij}.

Thus

$$\frac{\partial}{\partial t}\Big|_{t=0}A^{ij}(g_t) = -\nabla^{ij}\eta - 4A^{ij}(g)\eta, \quad \text{and} \quad \frac{\partial}{\partial t}\Big|_{t=0}B_{ij}(g_t) = (n-4)(C_{ijk} + C_{jik})\nabla^k\eta - 2\eta B_{ij}.$$

We recall some properties to be used.

Proposition 2. *([V1],[H],[HL]). We have*

(i) $k\sigma_k(g) = \sum_{i,j} T^{(k-1)}_{ij}A_{ij}$

(ii) $\sum_i T^{(k)}_{ii} = (n-k)\sigma_k(g)$.

(iii) $\sum_l \nabla^l W_{lijk} = -(n-3)C_{ijk}$.

Using the relation between $v^{(6)}$ and $\sigma_3(g)$, $A^{ij}B_{ij}$ as in (1.5), we find

$$\begin{aligned}
&-8\frac{\partial}{\partial t}\Big|_{t=0}v^{(6)}(g_t)\\
=&T^{(2)}_{ij}(g)\Big(-\nabla^{ij}\eta - 2\eta A^{ij}(g)\Big) + \frac{1}{3(n-4)}\Big[-B_{ij}(g)\nabla^{ij}\eta + (n-4)A^{ij}(g)(C_{ijk} + C_{jik})\nabla^k\eta - 6\eta A^{ij}B_{ij}\Big]\\
=&-6\Big(\sigma_3(g) + \frac{1}{3(n-4)}A^{ij}B_{ij}\Big)\eta - \Big[T^{(2)}_{ij}(g) + \frac{B_{ij}(g)}{3(n-4)}\Big]\nabla^{ij}\eta + \frac{2}{3}A^{ij}(g)C_{ijk}\nabla^k\eta\\
=&48v^{(6)}(g)\eta - \nabla^j\Big[\Big(T^{(2)}_{ij}(g) + \frac{B_{ij}(g)}{3(n-4)}\Big)\nabla^i\eta\Big] + \Big[\sum_j\Big(T^{(2)}_{ij,j}(g) + \frac{B_{ij,j}(g)}{3(n-4)}\Big) + \frac{2}{3}A^{kl}C_{kli}\Big]\nabla^i\eta,
\end{aligned}$$

where we used (1.5) and (i) of Proposition 2. In the following we will verify that

$$\sum_j\Big(T^{(2)}_{ij,j}(g) + \frac{B_{ij,j}(g)}{3(n-4)}\Big) + \frac{2}{3}A^{kl}C_{kli} = 0,$$

thus establishing (3.1). The above property would follow from the following

Lemma 1. *(i)* $\sum_j T^{(2)}_{ij,j} = -A^{pq}C_{pqi}$;

(ii) $\sum_j B_{ij,j} = (n-4)A^{kl}C_{kli}$.

Proof of (i). We have the following calculation in normal coordinate,

$$\begin{aligned}\sum_j T^{(2)}_{ij,j} &= \sum (\frac{1}{2!}\sum \delta^{j_1j_2j}_{i_1i_2i} A_{i_1j_1}A_{i_2j_2})_j \\ &= \sum \delta^{j_1j_2j}_{i_1i_2i} A_{i_1j_1}A_{i_2j_2,j} \\ &= -A^{pq}C_{pqi},\end{aligned}$$

where we used

$$\delta^{j_1j_2j}_{i_1i_2i} = \begin{vmatrix} \delta_{i_1j_1} & \delta_{i_1j_2} & \delta_{i_1j} \\ \delta_{i_2j_1} & \delta_{i_2j_2} & \delta_{i_2j} \\ \delta_{ij_1} & \delta_{ij_2} & \delta_{ij} \end{vmatrix}$$

and $\sum_i A_{ii,j} = \sum_i A_{ij,i}$, which itself is a consequence of the second Bianchi identity. □

Proof of (ii). First, using (iii) of Proposition 2 and substituting R_{ij} in terms of A_{ij} in the definition of the Bach tensor B_{ij}, we obtain

$$\begin{aligned}B_{ij} &= -\sum_k C_{ikj,k} + \sum_{k,l} A_{kl}W_{likj} \\ &= -\sum_k (A_{ik,jk} - A_{ij,kk}) + \sum_{k,l} A_{kl}W_{likj}.\end{aligned}$$

Thus

$$\begin{aligned}&\sum_j B_{ij,j} \\ &= -\sum_{j,k}(A_{ik,jkj} - A_{ij,kkj}) + \sum_{k,l,j}(A_{kl,j}W_{likj} + A_{kl}W_{likj,j}) \\ &= -\sum_{j,k}(A_{ik,jkj} - A_{ik,jjk}) + \sum_{k,l,j} A_{kl,j}W_{likj} - (n-3)\sum_{k,l} A_{kl}C_{kil} \\ &= -\sum_{j,k,m}(A_{ik,m}R_{mjkj} + A_{im,j}R_{mkkj} + A_{mk,j}R_{mikj}) + \sum_{k,l,j} A_{kl,j}W_{likj} + (n-3)\sum_{k,l} A_{kl}C_{kli} \\ &= \sum_{j,k,m}(-A_{mk,j}R_{mikj} + A_{km,j}W_{mikj}) + (n-3)\sum_{k,l} A_{kl}C_{kil} \\ &= \sum_{j,k,m} A_{mk,j}(-A_{mk}g_{ij} + A_{mj}g_{ik} - g_{mk}A_{ij} + g_{mj}A_{ik}) + (n-3)\sum_{k,l} A_{kl}C_{kli} \\ &= \sum_{m,k}(-A_{mk,i}A_{mk} + A_{mi,k}A_{mk} - A_{mk,j}g_{mk}A_{ij} + A_{mj,k}g_{mk}A_{ij}) + (n-3)\sum_{k,l} A_{kl}C_{kli}\end{aligned}$$

$$
\begin{aligned}
&=\sum_{m,k} A_{mk}\left(A_{mi,k}-A_{mk,i}\right)+(n-3)\sum_{k,l} A_{kl}C_{kli}\\
&=\sum_{m,k} A_{mk}C_{mik}+(n-3)\sum_{k,l} A_{kl}C_{kli}\\
&=(n-4)\sum_{k,l} A_{kl}C_{kli},
\end{aligned}
$$

where we have used

$$R_{mikj}=W_{mikj}+A_{mk}g_{ij}-A_{mj}g_{ik}+g_{mk}A_{ij}-g_{mj}A_{ik}.$$

□

Proof of Theorem 1 of the special case $k=3$. We use the notations of section 2, let ϕ_t be the local one-parameter family of conformal diffeomorphisms of (M,g) generated by X. For $g_t=\phi_t^*(g)=e^{2\omega_t}g$, similar to (3.1) we have

$$\langle X, v^{(6)}\rangle=\frac{\partial}{\partial t}\Big|_{t=0} v^{(6)}(g_t)=-6v^{(6)}(g)\dot{\omega}+\sum_{i,j}\nabla^j\left[\Big(\frac{T_{ij}^{(2)}(g)}{8}+\frac{B_{ij}(g)}{24(n-4)}\Big)\nabla^i\dot{\omega}\right], \tag{3.2}$$

if $n\neq 2k$ then integrating (3.2) we can get Theorem 1.

While if $n=2k$, then by use of (3.1) and (3.2), we can prove that $\int_M\langle X, v^{(6)}(g)\rangle dv_g$ is independent of the particular choice of the metric within the conformal class. The remaining of the proof is verbatim the same as that of section 2. □

4 Proof of Theorem 2

In this section, we will prove Theorem 2 using a similar method as in section 2. Let (M^n,g) be a compact Riemannian manifold, and we denote by R_{ijkl} the Riemann curvature tensor in local coordinates. Define a tensor $P_r(2r\leq n)$ by

$${P_r}_i^{\ j}=\delta^{jj_1j_2\ldots j_{2r-1}j_{2r}}_{ii_1i_2\ldots i_{2r-1}i_{2r}}R^{i_1i_2}_{j_1j_2}\ldots R^{i_{2r-1}i_{2r}}_{\phantom{i_{2r-1}i_{2r}}j_{2r-1}j_{2r}},$$

where $\delta^{jj_1j_2\ldots j_{2r-1}j_{2r}}_{ii_1i_2\ldots i_{2r-1}i_{2r}}$ is the generalized Kronecker symbol. First we give the following lemma.

Lemma 2. *The tensor P_r is divergence free, i.e.*

$${P_r}_{i,j}^{\ j}=0,\ \textit{for any } i.$$

This property was present in [La] and [Lo], although with different notations and formalism. Since we define the tensor P_r explicitly as above, and the property in Lemma 2 for P_r is a direct consequence of Bianchi's identity, we include a proof here.

Proof. We have the following direct computations.

$$\begin{aligned}
{P_r}_{i,j}^{j} &= r\delta^{jj_1j_2\ldots j_{2r-1}j_{2r}}_{ii_1i_2\ldots i_{2r-1}i_{2r}} R^{i_1i_2}{}_{j_1j_2,j}\cdots R^{i_{2r-1}i_{2r}}{}_{j_{2r-1}j_{2r}} \\
&= -r\delta^{jj_1j_2\ldots j_{2r-1}j_{2r}}_{ii_1i_2\ldots i_{2r-1}i_{2r}} R^{i_1i_2}{}_{j_2j,j_1}\cdots R^{i_{2r-1}i_{2r}}{}_{j_{2r-1}j_{2r}} \\
&\quad - r\delta^{jj_1j_2\ldots j_{2r-1}j_{2r}}_{ii_1i_2\ldots i_{2r-1}i_{2r}} R^{i_1i_2}{}_{jj_1,j_2}\cdots R^{i_{2r-1}i_{2r}}{}_{j_{2r-1}j_{2r}} \\
&= -2r\delta^{jj_1j_2\ldots j_{2r-1}j_{2r}}_{ii_1i_2\ldots i_{2r-1}i_{2r}} R^{i_1i_2}{}_{j_1j_2,j}\cdots R^{i_{2r-1}i_{2r}}{}_{j_{2r-1}j_{2r}} \\
&= -2{P_r}_{i,j}^{j},
\end{aligned}$$

where we have used the second Bianchi identity. It then follows that ${P_r}_{i,j}^{j} = 0$. □

We need the following algebraic lemma.

Lemma 3. *The generalized Kronecker symbol satisfies*

$$\sum_{i,j=1}^{n} \delta^i_j \delta^{jj_1\ldots j_r}_{ii_1\ldots i_r} = (n-r)\delta^{j_1\ldots j_r}_{i_1\ldots i_r},$$

for any $1 \le i_1, \ldots, j_r \le n$*, and* $r \le n$.

The proof of Lemma 3 is a direct calculation by use of the definition and we omit it here.

Let X be a conformal vector field, denoted by ϕ_t be the one-parameter subgroup of diffeomorphism generated by X. Then there exists a family of functions ω_t such that $g_t = \phi_t^* g = e^{2\omega_t} g$. We have (2.5), $\omega_0 = 0$, and

$$G_{2r}(g_t) = \phi_t^* G_{2r}(g). \tag{4.1}$$

Under conformal change of metrics $g_t = e^{2\omega_t} g$, we have the following formula (see e.g. [CLN]),

$$R^{ij}{}_{kl}(g_t) = e^{-2\omega_t}\Big(R^{ij}{}_{kl} - (\alpha\odot g)^{ij}{}_{kl}\Big), \tag{4.2}$$

where we denote $\alpha_{ij} = (\omega_t)_{ij} - (\omega_t)_i(\omega_t)_j + \frac{|\nabla\omega_t|^2}{2} g_{ij}$ for convenience (note that $(\omega_t)_{ij}$ is the covariant derivative with respect to the fixed metric g.) and $\odot$ is the Kulkani-Nomizu product, defined by

$$(\alpha\odot g)_{ijkl} = \alpha_{ik}g_{jl} + \alpha_{jl}g_{ik} - \alpha_{il}g_{jk} - \alpha_{jk}g_{il}.$$

From (4.2) we see that

$$G_{2r}(g_t) = e^{-2r\omega_t}\delta^{j_1j_2\ldots j_{2r-1}j_{2r}}_{i_1i_2\ldots i_{2r-1}i_{2r}}\Big(R^{i_1i_2}{}_{j_1j_2} - (\alpha\odot g)^{i_1i_2}{}_{j_1j_2}\Big)\cdots\Big(R^{i_{2r-1}i_{2r}}{}_{j_{2r-1}j_{2r}} - (\alpha\odot g)^{i_{2r-1}i_{2r}}{}_{j_{2r-1}j_{2r}}\Big). \tag{4.3}$$

Taking derivative with respect to t on both sides of (4.1) and using (4.3), we see by use of (2.5)

$$\begin{aligned}
\langle X, G_{2r}(g)\rangle &= \frac{\partial}{\partial t}\Big|_{t=0} G_{2r}(g_t) \\
&= -2r\dot\omega G_{2r}(g) - r\delta^{j_1j_2\ldots j_{2r-1}j_{2r}}_{i_1i_2\ldots i_{2r-1}i_{2r}}\big(\frac{\partial\alpha}{\partial t}\Big|_{t=0}\odot g\big)^{i_1i_2}{}_{j_1j_2} R^{i_3i_4}{}_{j_3j_4}\cdots R^{i_{2r-1}i_{2r}}{}_{j_{2r-1}j_{2r}} \\
&= -2r\dot\omega G_{2r}(g) - 4r(n-2r+1){P_{r-1}}_i^j \dot\omega^i_j \qquad (4.4) \\
&= -2r\frac{\mathrm{div}X}{n} G_{2r}(g) - \frac{4r(n-2r+1)}{n}{P_{r-1}}_i^j (\mathrm{div}X)^i{}_j \\
&= -2r\frac{\mathrm{div}X}{n} G_{2r}(g) - \frac{4r(n-2r+1)}{n}\nabla_j\Big({P_{r-1}}_i^j(\mathrm{div}X)^i\Big).
\end{aligned}$$

where we have used Lemma 3 in the third equality and Lemma 2 in the last equality. Integrating (4.4) over M and using the divergence theorem, we see that

$$\int_M \langle X, G_{2r}(g)\rangle dv = -2r\int_M \frac{\mathrm{div}X}{n} G_{2r}(g)dv = \frac{2r}{n}\int_M \langle X, G_{2r}(g)\rangle dv, \tag{4.5}$$

Hence, if $n > 2r$, it follows from (4.5) that $\int_M \langle X, G_{2r}(g)\rangle dv = 0$. If $n = 2r$, we follow similar ideas as in section 2, i.e. we need to prove that the integral

$$\int_M G_{2r}(g)\mathrm{div}_g X dv_g,$$

is independent of a particular choice of metrics within a conformal class. Let $g_1 = e^{2\eta}g(\eta \in C^\infty(M))$ be any metric in the conformal class $[g]$. Considering a family of metrics $g_t = e^{2t\eta}g$ connecting g and g_1, we need to prove that

$$\frac{d}{dt}\Big|_{t=0}\int_M G_{2r}(g_t)\mathrm{div}_{g_t} X dv_{g_t} = 0.$$

By a direct computation, we have

$$\begin{aligned}
&\frac{d}{dt}\Big|_{t=0}\int_M G_{2r}(g_t)\mathrm{div}_{g_t} X dv_{g_t}\\
=&\int_M \Big[\frac{\partial}{\partial t}\Big|_{t=0} G_{2r}(g_t)\mathrm{div}X + G_{2r}(g)\frac{\partial}{\partial t}\Big|_{t=0}\mathrm{div}_{g_t}X + n\eta G_{2r}(g)\mathrm{div}X\Big]dv_g\\
=&\int_M \Big[-2r\eta G_{2r}(g)\mathrm{div}X - 4r(n-2r+1){P_{r-1}}_i^j\eta^i_j\mathrm{div}X + nG_{2r}(g)\langle\nabla\eta, X\rangle + nG_{2r}(g)\mathrm{div}X\eta\Big]dv_g\\
=&\int_M \Big[-2r\eta G_{2r}(g)\mathrm{div}X - 4\eta r(n-2r+1){P_{r-1}}_i^j(\mathrm{div}X)^i_j - n\eta\langle\nabla G_{2r}(g), X\rangle\Big]dv_g\\
=&0,
\end{aligned}$$

where we have used (2.7) in the second equality, the divergence theorem in the third equality and (4.4) in the last equality. The remaining proof follows the idea of [BE] as in section 2. Hence we complete the proof of Theorem 2.

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